S. M. NIKOLSKY

A Course of Mathematical Analysis

Volume

2

MIR PUBLISHERS MOSCOW

Academician S.M. NIKOLSKY, a State Prize winner, the author of more than 130 scientific papers and several monographs including *Quadrature Formulas, Approximation of Functions of Several Variables and Embedding Theorems* and *Integral Representation of Functions and Embedding Theorems* (co-author).

He contributed many fundamental results to the theory of approximation of functions, variational methods for solving boundary-value problems and functional analysis. For his monograph *Approximation of Functions of Several Variables and Embedding Theorems* he was awarded the Chebyshev Prize of the USSR Academy of Sciences.

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2

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С. М. НИКОЛЬСКИЙ

КУРС МАТЕМАТИЧЕСКОГО АНАЛИЗА

ТОМ 2

ИЗДАТЕЛЬСТВО «НАУКА»
МОСКВА

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A Course of Mathematical Analysis

Volume 2

Translated from the Russian
by
V. M. VOLOSOV, D. Sc.

MIR PUBLISHERS
MOSCOW

First published 1977
Second printing 1981
Third printing 1985
Fourth printing 1987

TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is: USSR, 129820, Moscow I-110, GSP
Pervy Rizhsky Pereulok, 2
MIR PUBLISHERS

На английском языке

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The major part of this two-volume textbook stems from the course in mathematical analysis given by the author for many years at the Moscow Physico-technical Institute.

The first volume consisting of eleven chapters includes an introduction (Chapter 1) which treats of fundamental notions of mathematical analysis using an intuitive concept of a limit. With the aid of visual interpretation and some considerations of a physical character it establishes the relationship between the derivative and the integral and gives some elements of differentiation and integration techniques necessary to those readers who are simultaneously studying physics.

The notion of a real number is interpreted in the first volume (Chapter 2) on the basis of its representation as an infinite decimal.

Chapters 3–11 contain the following topics: Limit of Sequence, Limit of Function, Functions of One Variable, Functions of Several Variables, Indefinite Integral, Definite Integral, Some Applications of Integrals, Series.

CHAPTER 12

Multiple Integrals

§ 12.1. Introduction

Let us consider a continuous surface, lying in the three-dimensional space with rectangular coordinates (x, y, z), which is determined by an equation

$$z = f(Q) = f(x, y) \qquad (Q = (x, y) \in \Omega)$$

where Ω is a bounded (two-dimensional) set possessing area (two-dimensional measure*). For instance, Ω can be a circle, a rectangle, an ellipse, etc. We shall suppose that the function $f(x, y)$ is positive. Let us state the following problem: it is required to find the volume of the solid bounded above by the given surface and below by the plane $z = 0$, its lateral boundary being the cylindrical surface with generators parallel to the z-axis and passing through the boundary curve γ of the set Ω.

To determine the sought-for volume we resort to the following natural procedure.

The set Ω is divided into a finite number N of parts (subdomains)

$$\Omega_1, \ldots, \Omega_N \tag{1}$$

any two of which either do not intersect or intersect only along some parts of their boundaries. Let these subdomains be such that they possess areas (two-dimensional measures) which we shall denote as $m\Omega_1, \ldots, m\Omega_N$ respectively.

Let us introduce the notion of the diameter of a set: if A is a set in the plane its *diameter $d(A)$* is defined as

$$d(A) = \sup_{P', P'' \in A} |P' - P''|$$

where the supremum is taken over all the pairs of points P', P'' belonging to A.

Now we choose an arbitrary point $Q_j = (\xi_j, \eta_j)$ $(j = 1, \ldots, N)$ in each part Ω_j and form the sum

$$V_N = \sum_{j=1}^{N} f(Q_j) m\Omega_j \tag{2}$$

---

* See § 12.2.

which can be regarded as an approximation to the sought-for volume V. We can naturally suppose that the smaller the diameters $d(\Omega_j)$ of the subdomains Ω_j are, the higher is the accuracy of the approximation $V \approx V_N$. Therefore the *volume* V of the solid in question can be defined as the limit

$$V = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} f(Q_j) m\Omega_j \tag{3}$$

to which sum (2) tends when the maximum diameter of the subdomains of partitions (1) are made to tend to zero provided that this limit exists and is independent of the way in which the sequence of partitions (1) is chosen.

Now we can abstract from the problem of finding the volume of a solid and regard expression (3) as the result of an operation performed on the given function f defined in Ω. It is called the *Riemann double integral of the function f over the domain Ω* and is denoted

$$V = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} f(Q_j) m\Omega_j = \iint_{\Omega} f(x, y)\, dx\, dy = \int_{\Omega} f(Q)\, dQ = \int_{\Omega} f\, d\Omega$$

Let us consider a problem leading to the notion of the triple integral. Suppose that there is a physical body occupying a domain (set) Ω in the three-dimensional space with rectangular coordinates (x, y, z) and that the mass of the body is distributed (nonuniformly, in the general case) over Ω with volume density $\mu(x, y, z) = \mu(Q)$ $(Q = (x, y, z) \in \Omega)$. It is required to determine the total mass of the body Ω.

To solve this problem it is natural to partition Ω into N parts $\Omega_1, \ldots, \Omega_N$ whose volumes (three-dimensional measures) are $m\Omega_1, \ldots, m\Omega_N$ (on condition that these volumes exist), to choose an arbitrary point $Q_j = (x_j, y_j, z_j) \in \Omega_j$ $(j = 1, \ldots, N)$ in each of the parts and to define the sought-for mass as the limit

$$M = \lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} \mu(Q_j) m\Omega_j \tag{4}$$

Expression (4) can again be regarded as the result of an operation performed on the function μ defined in the three-dimensional set Ω. It is called the *Riemann triple integral of f on Ω* and is denoted as

$$M = \lim_{\max d(\Omega_j)\to 0} \sum \mu(Q_j) m\Omega_j = \int_{\Omega} \mu(Q)\, dQ = \iiint_{\Omega} \mu(x, y, z)\, dx\, dy\, dz$$

The *Riemann n-fold multiple integral* is defined in the same way.

We shall see that the theory of (Riemann) multiple integration which includes existence theorems and theorems on the additive properties of the integral can be presented for the n-dimensional case in exactly the same manner as in the case of dimension 1. However, the theory of multiple integrals involves some specific difficulties which were not encountered in the theory of one-fold integration.

The matter is that the (Riemann) one-fold integral was defined for an extremely simple set, namely, for a closed interval $[a, b]$ which was partitioned into parts which were also closed intervals. Therefore we had no difficulties in defining the lengths *(one-dimensional measures)* of the intervals. But in the case of a double integral or, generally, n-fold integral, the domain of integration Ω can be split into parts with curvilinear boundaries, which makes it necessary to define the notion of the area or, generally, of the n-dimensional measure of such a part. A similar question would also appear in the case $n = 1$ if we defined the one-fold Riemann integral for a set of a more complex structure than that of a closed interval.

In this connection we must state a strict definition of the notion of measure of a set and investigate the properties of the measure. Therefore we begin this chapter with the theory of the Jordan* measure closely related to the theory of the Riemann integral. This theory forms the basis for the representation of the theory of the Riemann multiple integral. The latter theory provides an important method for evaluation of n-fold multiple integrals by reducing them to the so-called *iterated (repeated) integrals* involving n one-fold integrations with respect to each of the variables; in many important cases this procedure admits of the application of the Newton-Leibniz theorem established for one-fold integrals.

§ 12.2. Jordan Squarable Sets

Let us consider the plane $R = R_2$ with a definitely chosen rectangular coordinates (x, y); this coordinate system will also be denoted by the same letter R.

If some other coordinate system (ξ, η) is taken in the same plane we shall denote the plane (and the new coordinate system) by R'.

A rectangle Δ in the plane R will be regarded as the simplest set. It can be defined analytically by assuming that there is a system of rectangular coordinates R' in which Δ is representable as a set of points (ξ, η) satisfying inequalities of the form

$$a_1 \leqslant \xi \leqslant a_2, \qquad b_1 \leqslant \eta \leqslant b_2 \tag{1}$$

where a_1, a_2, b_1 and b_2 are some numbers such that $a_1 < a_2$ and $b_1 < b_2$. The coordinate system R' possesses the property that the sides of Δ are parallel to its coordinate axes. To stress that the sides of Δ are parallel to the coordinate axes of the system R' we shall write $\Delta = \Delta_{R'}$. The rectangles of the type of Δ are understood here as closed sets (closed rectangles including their boundaries).

Now we define the notion of an *elementary figure* σ: a set $\sigma \subset R$ will be called an elementary figure if it is representable as a (set-theoretic) sum of a finite number of rectangles $\Delta \subset R$ any two of which either do not intersect or intersect only along some parts of their boundaries. The area $|\sigma|$ of a two-dimensional elementary figure σ is defined as the sum of the areas of the rectangles Δ of which σ is composed.

---

* C. Jordan (1838-1922), a French mathematician.

A given figure σ can be represented as a finite sum (union) of rectangles Δ in infinitely many ways but the area $|\sigma|$ is independent of the representation. This assertion can readily be proved using the means of elementary geometry, and we do not dwell on it here.

An empty set is also regarded as a figure and its measure (area) is understood as being zero.

In inequalities (1) defining a rectangle Δ we assumed that $a_1 < a_2$ and $b_1 < b_2$. Therefore separate points and line segments will not be regarded as rectangles; our representation of the theory of measure will not involve such "degenerate" rectangles.

If an elementary figure σ is representable as a sum of rectangles Δ whose sides are parallel to the axes of the coordinate system R we shall write $\sigma = \sigma_R$.

Enumerated below are some simple properties of elementary figures σ. Their proofs are quite simple and we do not dwell on them here.

(a) *If* $\sigma_1 \subset \sigma_2$ *then* $|\sigma_1| \leqslant |\sigma_2|$.

(b) *The* (set-theoretic) *sum of figures* σ'_R *and* σ''_R *is a figure* σ_R *and there holds the inequality*

$$|\sigma'_R+\sigma''_R| \leqslant |\sigma'_R|+|\sigma''_R|$$

It becomes an equality if σ'_R *and* σ''_R *either do not intersect each other or intersect only along some parts of their boundaries.*

(c) *The difference of two figures* σ'_R *and* σ''_R *is not necessarily a closed set and therefore it may not be an elementary figure. It can only be a figure (possibly empty) if* $\sigma'_R \subset \sigma''_R$ *or if* σ'_R *and* σ''_R *do not intersect. However, the closure* $\overline{\sigma'_R-\sigma''_R}$ *of this difference is always a figure and there holds the inequality*

$$|\overline{\sigma'_R-\sigma''_R}| \geqslant |\sigma'_R|-|\sigma''_R|$$

It turns into an equality if $\sigma''_R \subset \sigma'_R$.

(d) *If a figure* σ_R *is divided into two parts by a line parallel to one of the coordinate axes of the system* R *these parts are two figures* σ'_R *and* σ''_R.

To these properties we shall add two more; one of them is connected with the notion of a network.

Let us take an arbitrary natural number N and construct two families of straight lines: $x = kh$ and $y = lh$ ($h = 2^{-N}$; $k, l = 0, \pm1, \pm2, \ldots$). These families determine the rectangular network S_N dividing R into the squares Δ_h with sides of length h parallel to the axes of R. When we pass from a network S_N to S_{N+1} each of the squares of S_N splits into four congruent squares.

Let $G \subset R$ be an arbitrary bounded nonempty set. Let the symbol $\omega_N(G) = \omega_N$ denote the figure consisting of all the squares Δ_h of the network S_N which are entirely contained in G and let $\tilde{\omega}_N(G) = \tilde{\omega}_N$ be the figure consisting of those squares Δ_h of S_N each of which contains at least one

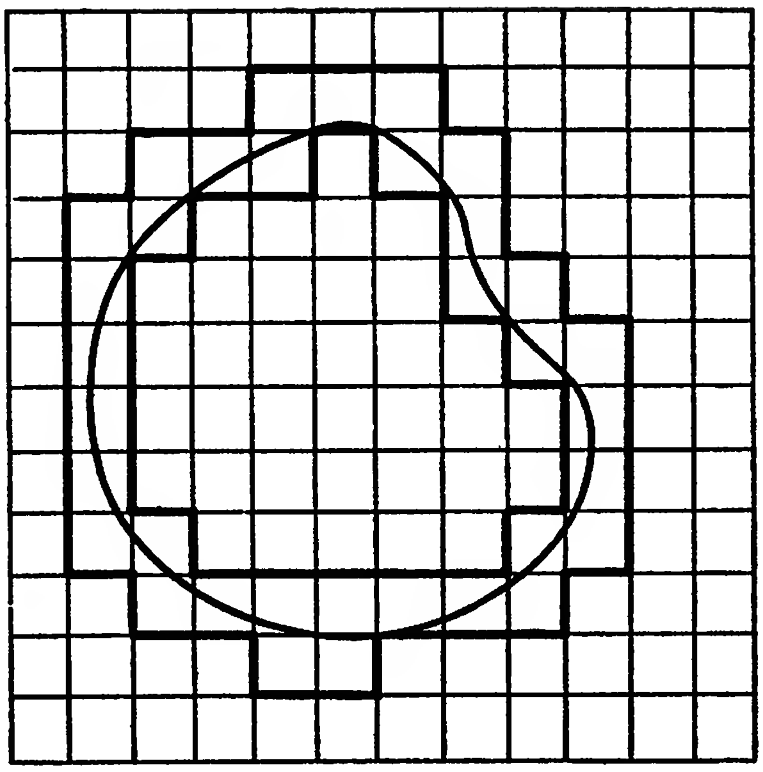

Fig. 12.1

point of the set G (see Fig. 12.1). In particular, it may happen that ω_N is an empty set; as was agreed, such a set is regarded as a figure (having zero measure).

It is evident that

$$\omega_1 \subset \omega_2 \subset \dots$$

$$\tilde{\omega}_1 \supset \tilde{\omega}_2 \supset \dots$$

and

$$\omega_N(G) \subset G \subset \tilde{\omega}_N(G)$$

where N and N' are arbitrary natural numbers. It follows that there exist the finite limits

$$m_iG = \lim_{N\to\infty} |\omega_N| \quad \text{and} \quad m_eG = \lim_{N\to\infty} |\tilde{\omega}_N| \qquad (m_iG \leqslant m_eG)$$

The number m_iG is called the *(two-dimensional) interior Jordan content (measure) of the set* G and the number m_eG is called the *(two-dimensional) exterior Jordan content (measure) of the set* G. For the sake of brevity, we shall often omit the adjective "Jordan". For the same reason in this section where we shall only deal with the Jordan two-dimensional measure we shall sometimes omit "two-dimensional".

We have proved that every bounded set $G \subset R$ possesses the interior and exterior Jordan contents m_iG and m_eG satisfying the inequality $m_iG \leqslant m_eG$.

If a set $G \subset R$ is such that $m_iG = m_eG = mG$ it is said to be *Jordan measurable*, and the number mG is called the *(two-dimensional) Jordan content (measure) of* G. As will be seen in § 12.3, the Jordan content and the Jordan measurability are defined in exactly the same manner for the case of an arbitrary dimension n. In the case $n = 2$ (and only in this case) with which we deal now a Jordan measurable set is also said to be *squarable*, and its Jordan content is called its *area*.

Now we proceed to the first of the additional properties of elementary figures σ mentioned above:

(e) *A figure σ (consisting of rectangles occupying arbitrary positions with respect to the coordinate system R) is a Jordan measurable set.*

The proof of this property can be reduced to the computation of the area of σ in the Cartesian coordinates R by using the well-known technique of the integral calculus, which results in the sum of the areas of the rectangles the figure σ is formed of.

In Fig. 12.2 we see a figure σ and two more figures σ' and σ'' such that $\sigma' \subset \sigma \subsetneq \sigma''$. It is clear that for the given figure σ and for an arbitrary $\varepsilon > 0$ there are two figures σ' and σ'' such that the following conditions hold:

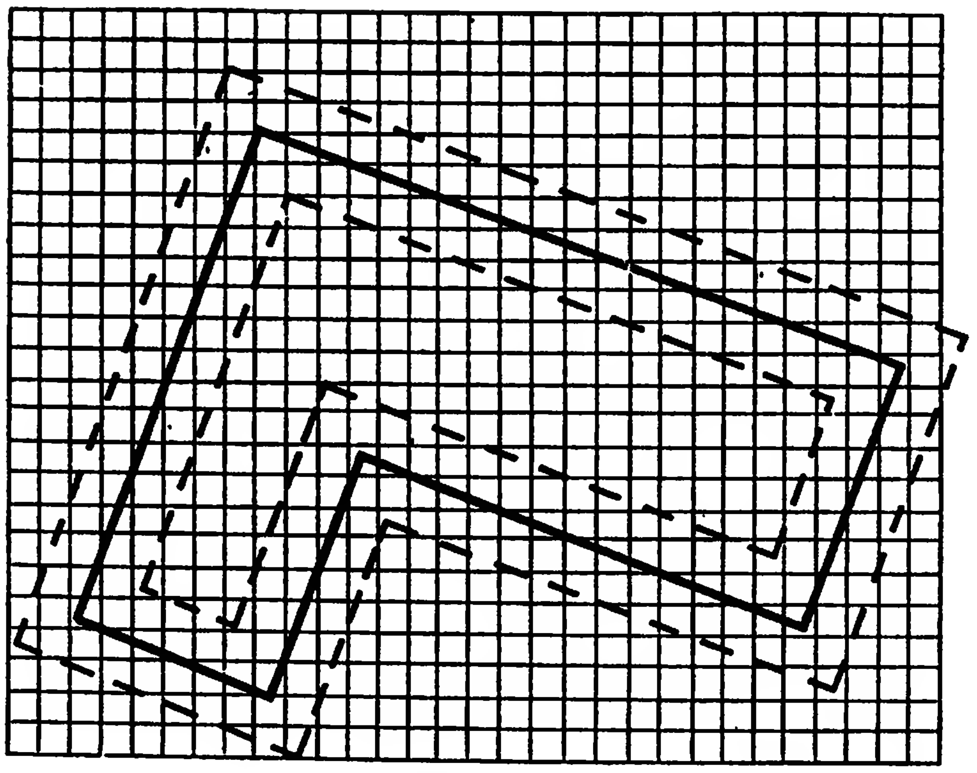

Fig. 12.2

(i) $\sigma' \subset \sigma \subset \sigma''$,

(ii) $|\sigma''|-|\sigma|' < \varepsilon$,

(iii) the points of the boundaries of σ' and σ'' are at distances exceeding a constant positive number α from any point of the boundary of σ.

Now we can take in the coordinate system R a network S_N such that $\sqrt{2}h = \sqrt{2}\cdot 2^{-N} < \alpha$; then the set $\sigma''-\sigma'$ entirely contains every square of the network which covers at least one boundary point of σ. Therefore the sum of the areas of all the squares of S_N covering the boundary of σ does not exceed $|\overline{\sigma''-\sigma'}| = |\sigma''|-|\sigma'| < \varepsilon$. It follows that

$$|\tilde{\omega}_N(\sigma)|-|\omega_N(\sigma)| < \varepsilon \tag{2}$$

and, since ε is arbitrarily small, we obtain

$$m_i\sigma = m_e\sigma = |\sigma|$$

Let us put down the following equalities expressing equivalent definitions of the exterior and interior measures of a bounded set G:

$$m_iG = \lim_{N\to\infty} |\omega_N(G)| = \sup_N |\omega_N(G)| = \sup_{\sigma_R\subset G} |\sigma_R| = \sup_{\sigma\subset G} |\sigma| \tag{3}$$

and

$$m_eG = \lim_{N\to\infty} |\tilde{\omega}_N(G)| = \inf_N |\tilde{\omega}_N(G)| = \inf_{\sigma_R\supset G} |\sigma_R| = \inf_{\sigma\supset G} |\sigma| \tag{4}$$

The first equality (3) coincides with the above definition of m_iG and provides an effective method for computing m_iG. However, it is connected with the chosen coordinate system R because the network S_N is connected with R.

The second equality (3) is obvious since $|\omega_N(G)|$ increases together with N.

Since $\omega_N(G)$ is a figure of the type of σ_R while σ_R is a figure of the type of σ we obviously have

$$\sup_N |\omega_N(G)| \leqslant \sup_{\sigma_R\subset G} |\sigma_R| \leqslant \sup_{\sigma\subset G} \leqslant |\sigma| \tag{5}$$

On the other hand, if $\sigma \subset G$ is an arbitrary figure and $\varepsilon > 0$ is an arbitrary positive number then, by the measurability of σ, there is N so large that

$$|\sigma| < |\omega_N(\sigma)|+\varepsilon \leqslant |\omega_N(G)|+\varepsilon \leqslant m_iG+\varepsilon$$

whence $\sup_{\sigma\subset G} |\sigma| \leqslant m_iG+\varepsilon$, and, by the arbitrariness of ε, we obtain

$$\sup_{\sigma\subset G} |\sigma| \leqslant m_iG \tag{6}$$

Relations (5) and (6) imply the third and fourth equalities (3) (the first two equalities (3) have already been proved).

The last term in (3) indicates that *the interior content m_iG is invariant with respect to the choice of the coordinate system*, that is it is independent of the system R in which it is considered.

Equalities (4) are proved analogously.

From equalities (3) and (4) readily follows

Lemma 1. *For a set G to be measurable it is necessary and sufficient that, given any $\varepsilon > 0$, there exist two figures $\underset{\sim}{\sigma}$ and $\tilde{\sigma}$ such that $\underset{\sim}{\sigma} \subset G \subset \tilde{\sigma}$ and $|\tilde{\sigma}|-|\underset{\sim}{\sigma}| < \varepsilon$.*

Here it is legitimate to assume that $\tilde{\sigma} = \tilde{\sigma}_R$ and $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_R$.

Indeed, if G is measurable and R is the given coordinate system, there are $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}_R \subset G$ and $\tilde{\sigma} = \tilde{\sigma}_R \supset G$ such that

$$mG-\frac{\varepsilon}{2} < |\underset{\sim}{\sigma}| \leqslant |\tilde{\sigma}| < mG+\frac{\varepsilon}{2} \quad \text{and} \quad |\tilde{\sigma}|-|\underset{\sim}{\sigma}| < \varepsilon$$

Conversely, the inclusion relations $\underset{\sim}{\sigma} \subset G \subset \tilde{\sigma}$ imply $|\underset{\sim}{\sigma}| \leqslant m_iG \leqslant m_eG \leqslant |\tilde{\sigma}|$, and if $|\tilde{\sigma}|-|\underset{\sim}{\sigma}| < \varepsilon$ then $m_eG-m_iG < \varepsilon$, and, by the arbitrariness of $\varepsilon > 0$, we obtain

$$m_eG = m_iG$$

It follows from Lemma 1 that *a measurable set is bounded.*

Lemma 2. *For a set G to be Jordan measurable it is necessary and sufficient that its boundary Γ be of Jordan (two-dimensional) measure zero, that is for any $\varepsilon > 0$ there should exist a figure σ_0 covering Γ such that $|\sigma_0| < \varepsilon$.*

Proof. Let the set G be measurable. Then, given any $\varepsilon > 0$, there are two figures $\sigma' = \sigma'_R$ and $\sigma'' = \sigma''_R$ such that $\sigma' \subset G \subset \sigma''$ and $|\sigma''|-|\sigma'| < \varepsilon$ (see Fig. 12.1). It can always be assumed that the points of the boundary Γ of the set G do not lie on the boundary of σ'' and on the boundary of σ'. Indeed, if this property did not hold for the given σ' and σ'' we could stretch the figure σ'' and contract the figure σ' in the directions of the coordinate axes so that the inequality $|\sigma''|-|\sigma'| < \varepsilon$ would remain valid. Therefore it is evident that

$$\Gamma \subset \sigma''-\sigma' \subset \overline{\sigma''-\sigma'} = \sigma_0 \text{ and } |\sigma_0| = |\sigma''|-|\sigma'| < \varepsilon$$

which means that the figure σ_0 whose area is less than ε covers Γ.

Conversely, let for any $\varepsilon > 0$ there exist a figure $\sigma_0 = \sigma_R^0$ covering Γ such that $|\sigma_0| < \varepsilon$ (see Fig. 12.1). We can assume that Γ has no points in common with the boundary of σ_0 because if this is not so for the chosen figure σ_0 it can be additionally stretched with the retention of the inequality $|\sigma_0| < \varepsilon$.

Let us put $\sigma'' = G+\sigma_0$ and $\sigma' = \overline{G-\sigma_0}$. It is readily seen that σ' and σ'' are elementary figures (see Fig. 12.1) and that $\sigma' = \sigma'_R$, $\sigma'' = \sigma''_R$, $\sigma' \subset G \subset \sigma''$, $\overline{\sigma''-\sigma'} = \sigma_0$ and $|\sigma''|-|\sigma'| = |\sigma_0| < \varepsilon$. This shows that G is a measurable set.

Lemma 3. *If two sets G_1 and G_2 are of Jordan measure zero their sum is again a measurable set with Jordan measure zero.*

Indeed, by the hypothesis, for any $\varepsilon > 0$ there exist two figures σ'_R and σ''_R such that $\sigma'_R \supset G_1$, $\sigma''_R \supset G_2$, $|\sigma'_R| < \frac{\varepsilon}{2}$ and $|\sigma''_R| < \frac{\varepsilon}{2}$. Therefore for the figure $\sigma_R = \sigma'_R + \sigma''_R$ we have

$$\sigma_R \supset G_1 + G_2 \quad \text{and} \quad |\sigma_R| \leqslant |\sigma'_R| + |\sigma''_R| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Lemma 4. *If G is a set of Jordan measure zero then so is any set $G_1 \subset G$.*

The assertion of the lemma is obvious.

Theorem 1. *If two sets G_1 and G_2 are Jordan measurable then their sum, difference and intersection are also Jordan measurable.*

Proof. Let $\Gamma(E)$ denote the boundary of a given set E. There hold the following set-theoretic relations which can easily be proved:

$$\Gamma(G_1+G_2) \subset \Gamma(G_1)+\Gamma(G_2)$$

$$\Gamma(G_1 \cdot G_2) \subset \Gamma(G_1)+\Gamma(G_2)$$

and

$$\Gamma(G_1-G_2) \subset \Gamma(G_1)+\Gamma(G_2)$$

Since G_1 and G_2 are measurable, Lemma 2 shows that $m\Gamma(G_1) = 0$ and $m\Gamma(G_2) = 0$. Therefore, by Lemma 3, the right-hand members of the above relations are of measure zero, and, by Lemma 4, the left-hand members are also of measure zero. Whence, applying again Lemma 2, we conclude that the sets mentioned in the statement of the theorem are in fact measurable.

Lemma 5. *If a Jordan measurable set G is divided into two parts G_1 and G_2 by a segment of a curve L (which, in particular, can be a line segment) of Jordan measure zero then each of the parts is also Jordan measurable.*

Proof. It is evident that

$$\Gamma(G_1) \subset \Gamma(G)+L \quad \text{and} \quad \Gamma(G_2) \subset \Gamma(G)+L$$

whence, by the foregoing lemmas, follows the assertion we intend to prove.

Thus, *if G is a Jordan measurable set, any network S_N (related to any coordinate system R') partitions G into parts each of which is Jordan measurable. The diameter of each such part does not exceed $\sqrt{2}\cdot 2^{-N}$. Hence, the diameters of the parts uniformly tend to zero as $N \to \infty$.*

Now we state the last of the properties of figures σ.

(f) *If a figure σ undergoes a parallel translation or a rotation in the plane R the resultant set is a figure σ' and $|\sigma| = |\sigma'|$.*

From this property and from the fact that an inclusion relation $A \subset B$ is retained under translation and rotation now follows

Lemma 6. *If G_* is a set resulting from a translation or a rotation in R of a measurable set G then G_* is a measurable set and $mG = mG_*$.*

Example. Let us consider an example of a nonsquarable (i.e. nonmeasurable in the two-dimensional sense) set. Let $G \subset R$ be a bounded open nonempty set and E be the set of all its rational points, that is points having rational coordinates (x, y). It is evident that $\omega_N(E)$ is an empty set for any natural N and $m_iE = 0$. On the other hand, if x^0 is a point belonging to G then there is a nondegenerate rectangle Δ belonging to G and containing x^0. It is apparent that $\tilde{\omega}_N(E) \supset \Delta$, $|\tilde{\omega}_N(E)| \geqslant |\Delta| > 0$ and $m_eE \geqslant |\Delta| > 0$.

Hence, $m_iE < m_eE$, which means that E is a nonsquarable set.

Let us prove the additive property of the Jordan measure.

Theorem 2. *If two sets G_1 and G_2 are Jordan measurable and if their common points, provided there are such, only belong to their boundaries, their union (which is measurable by Theorem 1) has a measure equal to the sum of the measures of the sets G_1 and G_2:*

$$m(G_1+G_2) = mG_1+mG_2 \tag{7}$$

Proof. Let us choose an arbitrary $\varepsilon > 0$ and take some figures $\sigma_1' = \sigma'_{1,R}$, $\sigma_1'' = \sigma''_{1,R}$, $\sigma_2' = \sigma'_{2,R}$ and $\sigma_2'' = \sigma''_{2,R}$ such that

$$\sigma_1' \subset G_1 \subset \sigma_1'', \qquad \sigma_2' \subset G_2 \subset \sigma_2''$$

$mG_1-\varepsilon < |\sigma_1'| < |\sigma_1''| < mG_1+\varepsilon$ and $mG_2-\varepsilon < |\sigma_2'| < |\sigma_2''| < mG_2+\varepsilon$. Putting $\sigma' = \sigma_1'+\sigma_2'$ and $\sigma'' = \sigma_1''+\sigma_2''$ we see that σ' and σ'' are elementary figures, $\sigma' \subset G_1+G_2 \subset \sigma''$ and

$$|\sigma''| \leqslant |\sigma_1''|+|\sigma_2''| \quad \text{and} \quad |\sigma'| = |\sigma_1'|+|\sigma_2'| \tag{8}$$

Equality (8) holds because, by the conditions imposed on G_1 and G_2, the figures σ_1' and σ_2' either do not intersect or intersect only along some parts of their boundaries.

Now it becomes clear that

$$(mG_1-\varepsilon)+(mG_2-\varepsilon) < |\sigma_1'|+|\sigma_2'| = |\sigma'| \leqslant m_i(G_1+G_2) \leqslant$$
$$\leqslant m_e(G_1+G_2) \leqslant |\sigma''| \leqslant |\sigma_1''|+|\sigma_2''| < (mG_1+\varepsilon) + (mG_2+\varepsilon)$$

whence, by the arbitrariness of the positive number ε, follows (7).

Theorem 3. *If two set G_1 and G_2 are Jordan measurable and $G_1 \subset G_2$ then*

$$m(G_2-G_1) = mG_2-mG_1 \tag{9}$$

Proof. The measurability of G_2-G_1 was proved in Theorem 1, and therefore the measurable set G_2 splits into two nonintersecting sets G_1 and G_2-G_1: $G_2 = G_1+(G_2-G_1)$. Equality (9) now follows from the foregoing theorem.

§ 12.3. Some Important Examples of Squarable Sets

Let $f(x)$ be a nonnegative function defined on a closed interval $[a, b]$. Suppose that $f(x)$ is integrable on $[a, b]$ (in particular, this is the case when $f(x)$ is continuous on $[a, b]$). We shall denote by Γ the graph of the function

f, that is the set of all points $(x, f(x))$ with $x \in [a, b]$, and by Ω the set of points (x, y) in the plane which satisfy the inequalities

$$a \leq x \leq b \quad \text{and} \quad 0 \leq y \leq f(x)$$

Theorem 1. *The set Ω is (Jordan) measurable and its (two-dimensional) measure is given by the formula*

$$m\Omega = \int_a^b f(x)\,dx = I$$

Indeed, by the integrability of f on $[a, b]$, for any $\varepsilon > 0$ there exists a partition R of the closed interval $[a, b]$ such that

$$I - \frac{\varepsilon}{2} < \underline{S}_R \leq \bar{S}_R < I + \frac{\varepsilon}{2}$$

where $\underline{S}_R$ and $\bar{S}_R$ are, respectively, the lower and upper integral sums of the function f corresponding to the partition R; the sum $\underline{S}_R$ is equal to the area of a figure contained in Ω while $\bar{S}_R$ is the area of a figure enveloping Ω. This proves the theorem.

Theorem 2. *A continuous curve Γ in the xy-plane which is projectable in a one-to-one manner onto a closed interval $[a, b]$ lying in a straight line L is a point set of two-dimensional measure zero.*

To prove the theorem we can suppose that Γ is entirely on one side of the line L because, if otherwise, we can take as L another straight line L' parallel to the former and satisfying the above requirement. Let us consider the rectangular coordinates x, y with x-axis coinciding with L. Then Γ can be regarded as the graph of a continuous function $f(x)$ defined on the interval $[a, b]$.

Now we see that the set Ω constructed for the function f as was indicated in Theorem 1 is measurable by virtue of this theorem, and Γ has a Jordan (two-dimensional) measure zero as a part of the boundary of Ω.

Theorem 3. *A bounded set Ω in the plane whose boundary consists of a finite number of points and of a finite number of continuous curvilinear segments each of which is projectable in a one-to-one way onto one of the rectangular coordinate axes is measurable (in the two-dimensional sense).*

Indeed, the boundary of such a set Ω is the union of a finite number of sets of two-dimensional measure zero.

It should be noted that a segment Γ of a smooth curve $x = \varphi(t)$, $y = \psi(t)$ ($a \leq t \leq b$, φ' and ψ' are continuous and $\varphi'^2 + \psi'^2 > 0$) can always be broken into a finite number of parts each of which can be projected in a one-to-one manner on one of the coordinate axes. Indeed, according to § 6.5, each point $t \in [a, b]$ can be covered by an open interval (t', t'') (in case $t = a$ or $t = b$ it is meant that the point can be covered by a half-open interval) such that the part of the smooth curve Γ corresponding to it is projectable on one of the coordinate axes. By the Heine-Borel lemma, the system of these intervals contains a finite subsystem covering the interval $[a, b]$, whence follows the assertion we wanted to prove. The fact that a portion of a smooth curve is of two-dimensional measure zero also follows from a general property established in § 12.5, Theorem 3.

In conclusion we mention that an arbitrary continuous curve in the plane does not necessarily have zero two-dimensional measure. In this connection we remind the reader of Peano's curve (see § 6.5) whose points fill unit square.

§ 12.4. One More Test for Measurability of a Set. Area in Polar Coordinates

The interior and exterior measures of a bounded set Ω can also be defined by the relations

$$m_i\Omega = \sup_{\Omega'\subset\Omega} m\Omega' \quad \text{and} \quad m_e\Omega = \inf_{\Omega'\supset\Omega} m\Omega' \tag{1}$$

where in the first relation Ω' is an arbitrary measurable set belonging to Ω and in the second relation a measurable set containing Ω. For, on the one hand, we have

$$m_i\Omega = \sup_{\sigma\subset\Omega} |\sigma| \leqslant \sup_{\Omega'\subset\Omega} m\Omega' = I$$

because the elementary figures σ are measurable and, on the other hand, if $\varepsilon > 0$ is an arbitrary number and Ω' is a measurable set such that $\Omega' \subset \Omega$ and $I-\varepsilon < m\Omega'$, then there is $\sigma \subset \Omega'$ such that $m\Omega' < |\sigma|+\varepsilon$. Consequently, $I-2\varepsilon < |\sigma| \leqslant m_i\Omega$, and, by the arbitrariness of ε, we have $I \leqslant m_i\Omega$. This proves the first relation (1). The second relation is proved in like manner.

From (1) it obviously follows that *for a set Ω to be measurable it is necessary and sufficient that, given any $\varepsilon > 0$, there should exist two measurable sets Ω' and Ω'' such that $\Omega' \subset \Omega \subset \Omega''$ and $m\Omega''-m\Omega' < \varepsilon$.*

The area (two-dimensional Jordan measure) of the figure Ω bounded by two rays $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$ and by a curve Γ represented by a continuous function $\varrho = f(\theta)$ in polar coordinates ϱ, θ is given by the formula

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} f(\theta)^2\, d\theta \tag{2}$$

(see § 10.1 and the question posed there).

Let us show that $m\Omega = S$. Indeed, an arbitrary sector of a circle is a measurable set since its boundary is a continuous piecewise smooth curve. Further, the existence of integral (2) implies that for any $\varepsilon > 0$ there is a partition of the interval $[\theta_1, \theta_2]$ such that the upper integral sum corresponding to it differs from the corresponding lower integral sum by less than ε. Now, since the upper sum equals the measure of a finite union of circular sectors containing Ω while the lower sum is the measure of a finite union of circular sectors contained in Ω, we conclude, by virtue of (1), that $m\Omega = S$.

§ 12.5. Jordan Measurable Three-dimensional and n-dimensional Sets

The theory presented in the foregoing sections can readily be extended to the case of an arbitrary dimension n.

We shall begin with the three-dimensional measure. In this case we consider as the simplest figures and denote by the symbol Δ the closed rectangles (rectangular parallelepipeds). By the measure of Δ we mean the volume of Δ, that is the number $|\Delta|$ equal to the product of the lengths of the three edges of Δ (of its dimensions). We agree to deal only with nondegenerate rectangles whose all dimensions are positive.

An elementary figure σ is now understood as a point set in space representable in the form of a finite union of rectangular parallelepipeds Δ which are only allowed to intersect along some parts of their boundaries.

The volume $|\sigma|$ of a figure σ is defined as the sum of the volumes $|\Delta|$ of the rectangles Δ it consists of, and an empty set is regarded as a figure with zero volume.

If R' is a rectangular coordinate system, the rectangles with edges parallel to the coordinate axes of the system R' are denoted $\Delta = \Delta_{R'}$ and the figures formed of such rectangles are denoted $\sigma = \sigma_{R'}$. Further, in the three-dimensional space $R = R_3$ we consider for every natural N a network S_N formed by the three families of planes

$$x = kh, y = lh, z = mh \qquad (h = 2^{-N}, k, l, m = 0, \pm 1, \pm 2, \ldots)$$

The network S_N partitions the space R into the cubes Δ_h with edges of length h parallel to the coordinate axes. For an arbitrary bounded set $G \subset R$ we define the set $\omega_N(G)$ as the sum of those Δ_h's which are entirely contained within G and the set $\bar{\omega}_N(G)$ as the sum of all the cubes Δ_h each of which contains at least one point belonging to G.

By analogy with § 12.2, the *exterior measure (Jordan content)* $m_e G$ and the *interior measure (Jordan content)* $m_i G$ of the set G are introduced as the limits of the corresponding monotone sequences:

$$m_e G = \lim_{N\to\infty} |\bar{\omega}_N(G)| \quad \text{and} \quad m_i G = \lim_{N\to\infty} |\omega_N(G)|$$

The set G is said to be *Jordan measurable in the three-dimensional sense* if $m_i G = m_e G = mG$, and the number mG is called the *Jordan three-dimensional measure (content) of* G.

The properties of the three-dimensional measure are completely analogous to those of the two-dimensional measure presented in § 12.2. To show this we must only check the validity of properties (a)-(f) of the figures σ which are now three-dimensional. This can be done elementary. The theorems in § 12.2 were proved on the basis of properties (a)-(f) alone and therefore they apply to the three-dimensional case as well. The statement of Lemma 5, § 12.2, should be slightly modified to read:

Lemma. *If a measurable set $G \subset R$ is divided into parts with the aid of a finite number of surfaces having the Jordan (three-dimensional) measure zero, these parts are measurable (in the three-dimensional sense).*

By analogy, this theory can be extended to the general case of an arbitrary dimension n. In this case $R = R_n$ is the n-dimensional space of points (n-tuples) $x = (x_1, \ldots, x_n)$. Let the letter R simultaneously denote a concrete rectangular coordinate system $R(x_1, \ldots, x_n)$ in the space $R = R_n$; any other rectangular coordinate system R' $(\xi_1, \ldots, \xi_n)$ is connected with the system $R(x_1, \ldots, x_n)$ by means of transformation formulas of the form

$$x_k = \sum_{l=1}^{n} a_{kl}\xi_l \qquad (k = 1, \ldots, n)$$

where the coefficients a_{kl} satisfy the relations

$$\sum_{l=1}^{n} a_{kl}a_{k_1l} = \begin{cases} 0 & \text{for} \quad k \neq k_1 \\ 1 & \text{for} \quad k = k_1 \end{cases}$$

Such a transformation from the old system R to a new system R' is written in the contracted notation as $R \rightleftharpoons R'$.

By a rectangular parallelepiped $\Delta \subset R$ is meant a set of points belonging to the space R for which there are rectangular coordinates R' $(\xi_1, \ldots, \xi_n)$ and n pairs of numbers $a_j < b_j$ $(j = 1, \ldots, n)$ such that $\Delta = \{a_j \leqslant \xi_j \leqslant b_j; j = 1, \ldots, n\}$.

It can easily be shown that for a given Δ such a coordinate system R' is determined uniquely. The *n-dimensional volume* of Δ is defined as the number

$$|\Delta| = \prod_{j=1}^{n} (b_j - a_j)$$

In this case it is natural to say that Δ has edges parallel to the coordinate axes of the system R' and to write $\Delta = \Delta_{R'}$. An elementary figure σ in R is defined as a finite union of rectangular parallelepipeds Δ which are only allowed to intersect along some parts of their boundaries, and the measure of σ is understood as the number $|\sigma|$ equal to the sum of the volumes of the rectangles Δ of which σ is formed.

An empty set in R_n is regarded as an elementary figure with the n-dimensional volume zero.

Further, for any natural N we consider the network S_N as the system of n families of planes

$$x_j = k_j h \qquad (j = 1, \ldots, n;\ h = 2^{-N};\ k_j = 0, \pm 1, \pm 2, \ldots)$$

which break up R_n into the n-dimensional cubes

$$\Delta_h = \{k_j h \leqslant x_j \leqslant (k_j + 1) h\}$$

If G is a bounded point set in R, the set $\omega_N(G)$ is defined as the set-theoretic sum of the cubes Δ_h entirely contained in G and the set $\bar{\omega}_N(G)$ as the sum of the cubes Δ_h each of which contains at least one point x belonging to G.

It is evident that there exist the limits

$$\lim_{N\to\infty} |\underset{\sim}{\omega}_N(G)| = m_iG \quad \text{and} \quad \lim_{N\to\infty} |\tilde{\omega}_N(G)| = m_eG$$

of the monotone sequences $|\underset{\sim}{\omega}_N(G)|$ and $|\tilde{\omega}_N(G)|$; m_iG and m_eG are called the *interior* and *exterior Jordan n-dimensional measures (contents) of G.*

Finally, a set G is said to be *Jordan measurable in the sense of the n-dimensional measure* if $m_iG = m_eG = mG$, the number mG being the *n-dimensional Jordan measure* (*content*) *of G.*

The theory of the Jordan n-dimensional measure is completely analogous to the theory of the two-dimensional measure given in § 12.2. To show this we must only check the validity of properties (a)-(f) for the n-dimensional figures σ, which requires some work but can be done by analogy with the three-dimensional case.

The following theorem generalizes Theorem 2, § 12.3.

Theorem 1. *A surface S in the n-dimensional space represented by an equation*

$$x_n = f(x_1, \ldots, x_{n-1}) = f(Q) \qquad (G = (x_1, \ldots, x_{n-1}) \in \Lambda)$$

where f is a continuous function defined on a bounded closed set $\Lambda \subset R_{n-1}$ *is a measurable set of Jordan n-dimensional measure zero.*

Proof. The function f being uniformly continuous in Λ, for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$|f(Q)-f(Q')| < \varepsilon, \ |Q-Q'| < \delta; \qquad Q, Q' \in \Lambda$$

Let us divide R_n with the aid of the network

$$\begin{aligned} x_i &= \alpha_i h, \qquad \alpha_i = 0, \pm 1, \pm 2, \ldots; \qquad i = 1, \ldots, n-1 \\ x_n &= k\varepsilon, \qquad k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

into congruent rectangles (rectangular parallelepipeds) Δ. The altitude of each Δ (in the direction of the x_n-axis) is equal to ε, its base Δ' (i.e. the projection of Δ on R_{n-1} $(x_1, \ldots, x_{n-1})$) being a cube with edge h. Let us choose $h > 0$ so small that the diameter of Δ' is less than δ. The network we have taken partitions the space R_{n-1} into the cubes Δ'. Since the set Λ is bounded it is entirely contained within an $(n-1)$-dimensional cube, and hence the sum of the volumes of the cubes Δ' each of which contains at least one point belonging to Λ does not exceed a constant M. Let us consider one of these cubes Δ'. Among the rectangles Δ with Δ' as projection on R_{n-1} there can obviously be not more than three containing points of the surface S. Their total volume does not exceed $3\varepsilon\,|\Delta'|$ where Δ' is the $(n-1)$-dimensional measure of Δ'. Therefore the total volume of the cubes Δ covering S does not exceed $3\varepsilon M$, that is it can be made arbitrarily small, which completes the proof.

Example. Consider a rectangular parallelepiped Δ in the n-dimensional space with coordinate system R, the sides of Δ being parallel to the axes of another rectangular system R'. Let S be one of its faces; it is projectable in a one-to-one manner onto one of the $(n-1)$-dimensional coordinate subspaces, say on the subspace $(x_1, \ldots, x_{n-1})$, and is represented by a (continuous) linear function of the form $x_n = \sum_{j=1}^{n-1} d_j x_j$ defined on a bounded closed set. By Theorem 1, we have $mS = 0$. Since the rectangle Δ has a finite number of faces we conclude that it is a measurable set.

Theorem 2. *A bounded open nonempty set G in the space $R_n = R$ is representable as a countable union*

$$G = \sum_1^{\infty} \Delta_k \tag{1}$$

of closed cubes Δ_k (with edges parallel to the coordinate axes) such that any two of them either do not intersect or intersect only along some parts of their boundaries. Although G may not be Jordan measurable (see § 19.7) *representation* (1) *possesses the property that the Jordan interior measure of G is expressed as*

$$m_i G = \sum_1^{\infty} |\Delta_k| \tag{2}$$

Proof. The sequence of networks $S_N (N = 1, 2, \ldots)$ specifies the (closed) figures

$$\omega_1(G) \subset \omega_2(G) \subset \omega_3(G) \subset \ldots \subset G$$

The set G being open, its every point x^0 can be covered by an open cube $\Delta \subset G$ with centre at x^0. Therefore the cube of the network S_N containing x^0 belongs to Δ for a sufficiently large N; consequently it belongs to G and thus enters into the figure $\omega_N(G)$. This shows that there holds the equality

$$G = \omega_1(G) + (\omega_2(G) - \omega_1(G)) + (\omega_3(G) - \omega_2(G)) + \ldots$$

If the figure $\omega_1(G)$ is nonempty let us number the cubes entering into it with the aid of consecutive integral indices; after this let us number the cubes belonging to the closure of $\omega_2(G) - \omega_1(G)$ with the aid of the remaining consecutive integral indices and then, in the same manner, the cubes of the closure of $\omega_3(G) - \omega_2(G)$, and so on. This procedure results in a sequence $\Delta_1, \Delta_2, \ldots$ for which (1) holds. This sequence is infinite because the closed set $\sum_1^m \Delta_k$ $(m = 1, 2, \ldots)$ is different from the bounded open set G in which it is contained for any m. Finally, we have

$$\sum_1^{\infty} |\Delta_k| = \lim_{N \to \infty} \omega_N(G) = m_i G$$

Theorem 3. *Let a surface S be representable as*

$$x_i = \varphi_i(u) = \varphi_i(u_1, \ldots, u_{n-1}) \qquad (i = 1, \ldots, n; \, u \in \Omega)$$

where φ_i are functions continuous on Ω together with their partial derivatives of the first order and Ω is the closure of a bounded open and convex $(n-1)$-dimensional set; then S is a set of (n-dimensional) measure zero.

Proof. Let

$$K \geqslant \left|\frac{\partial\varphi_i}{\partial u_j}\right| \text{ on } \Omega \qquad (i = 1, \ldots, n; j = 1, \ldots, n-1)$$

By the mean value theorem (see § 7.13 (12)), the distance between two points $\boldsymbol{x}', \boldsymbol{x}'' \in S$ corresponding to some values $\boldsymbol{u}'$ and $\boldsymbol{u}''$ of the parameters satisfies the inequality

$$|\boldsymbol{x}'-\boldsymbol{x}''|^2 = \sum_1^n (x_i'-x_i'')^2 = \sum_{i=1}^n \left[\sum_{j=1}^{n-1} \left(\frac{\partial\varphi_i}{\partial u_j}\right)_1 (u_j'-u_j'')\right]^2 \leqslant CK^2|\boldsymbol{u}'-\boldsymbol{u}''|^2 \quad (3)$$

-where C depends solely on n. (The symbol $(\ )_1$ indicates that into the expres sion $(\ )$ is substituted an intermediate point lying between $\boldsymbol{u}'$ and $\boldsymbol{u}''$.) Since Ω is a bounded set it can be embedded in an $(n-1)$-dimensional cube Δ. Let us break up Δ into congruent cubes σ with edge h. By virtue of inequality (3), the part of S corresponding to the points $\boldsymbol{u}$ of some cube σ can be embedded in an n-dimensional cube with volume of the order of h^n. The total number of such cubes σ is of the order of $h^{-(n-1)}$. Consequently, S can be embeded in a figure having an n-dimensional volume of the order of h, that is in a figure of an arbitrarily small volume, which is what we had to prove.

§ 12.6. The Notion of Multiple Integral

We shall give the definition of a multiple integral for the general case of an arbitrary dimension n. Its special cases for the two- and three-dimensional integrals were already outlined schematically in § 12.1.

Let $R = R_n$ be the n-dimensional Euclidean space of points $\boldsymbol{x} = (x_1, \ldots, x_n)$ and let $\Omega \subset R_n$ be a Jordan measurable (and therefore bounded) set on which a function $f(P)$ $(P \in \Omega)$ is defined.

Let us partition Ω into subsets, that is represent it as a finite union of the form

$$\Omega = \Omega_1 + \ldots + \Omega_N \tag{1}$$

under the assumption that each set Ω_j $(j = 1, \ldots, N)$ is Jordan measurable in the n-dimensional sense and that any two of them either do not intersect or intersect only along some parts of their boundaries. The various partitions of Ω will be denoted by the symbols $\varrho, \varrho^1, \ldots$

Now we choose an arbitrary point $P_j \in \Omega_j$ in each subset Ω_j $(j = 1, \ldots, N)$ entering into the partition ϱ and form the *(Riemann) integral sum*

$$S_\varrho = S_\varrho(f) = \sum_1^N f(P_j) m\Omega_j \tag{2}$$

where $m\Omega_j$ is the Jordan measure of Ω_j $(j = 1, \ldots, N)$.

Note that S_ϱ depends on the function f, on the way in which Ω is partitioned into measurable parts Ω_j and on the choice of the points P_j ($j = 1, \ldots, N$) in each of the subsets Ω_j of the partition.

Let us denote by δ the maximum diameter of the subsets Ω_j (it is referred to as the *fineness* of partition (1)):

$$\delta = \delta(\varrho) = \max_{1 \leq j \leq N} d(\Omega_j)$$

The *Riemann definite (n-fold multiple) integral of the function f over the set* Ω is defined as the limit

$$\lim_{\delta \to 0} \sum_{j=1}^{N} f(P_j) m\Omega_j = I \tag{3}$$

Thus, *the definite integral of the function f on the set Ω is the limit to which tends the integral sum of f corresponding to a variable partition of Ω when the maximum diameter of the subsets entering into the partition tends to zero (it is meant that this limit does not depend on the choice of the points $P_j \in \Omega_j$ and on the way in which the partitions are formed).*

As usual, this definition can be expressed analytically in the two (equivalent) ways: in terms of ε, δ formalism and in terms of sequences.

The ε, δ definition reads:

The Riemann integral of the function f over the set Ω is the number I satisfying the following requirement: given any $\varepsilon > 0$, there is $\delta > 0$ dependent solely on ε such that for any partition of Ω with maximum diameter less than δ and for any choice of the points $P_j \in \Omega_j$ ($j = 1, 2, 3, \ldots$) there holds the inequality

$$\left| I - \sum_{j=1}^{N} f(P_j) m\Omega_j \right| < \varepsilon \tag{4}$$

The definition in terms of sequences states:

The Riemann integral of the function f over the set Ω is the limit to which tends any sequence of integral sums S_{ϱ_k} of the function f corresponding to an arbitrary sequence of partitions ϱ_k ($k = 1, 2, \ldots$) with maximum diameter δ_k tending to zero:

$$I = \int_\Omega f\,dx = \int_\Omega f\,d\Omega = \lim S_{\varrho_k} \qquad (\delta_k \to 0) \tag{5}$$

As will be shown in § 12.7, Theorem 2, *if a function f defined on a measurable set Ω is bounded and if there is a definite sequence of partitions S_{ϱ_k} for which limit* (5) *equal to I exists and is independent of the choice of the points $P_j \in \Omega_j$, then the definite integral of f on Ω exists and is equal to the number I; this means that the existence of such a definite sequence of partitions automatically implies the validity of equality* (5) *for any sequence of partitions $\varrho_1, \varrho_2 \ldots$ with $\delta_k \to 0$.*

We remind the reader that, as was shown in § 12.2 (after Lemma 5), given any $\varepsilon > 0$, a measurable set can always be divided into parts whose diameters are less than ε.

If the Riemann integral of f over Ω exists it is denoted as

$$I = \int_{\Omega} f(P)\, d\Omega = \int_{\Omega} f(P)\, dP \tag{6}$$

In this case the function f is said to be *Riemann integrable on* Ω.

The condition of the Riemann integrability of f on Ω can also be stated in terms of Cauchy's criterion: given any $\varepsilon > 0$, there is $\delta > 0$ dependent solely on ε such that for any two partitions ϱ' and ϱ'' of the set Ω into measurable subsets with diameters less than δ there holds the inequality

$$|S_{\varrho'} - S_{\varrho''}| < \varepsilon$$

The n-fold multiple integral of f on Ω is also denoted as

$$I = \underset{\Omega}{\int \ldots \int} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n$$

This type of notation is convenient because, as will be seen later, the computation of a multiple integral can be reduced to the successive computation of the corresponding one-fold integrals with respect to each of the variables $x_1, x_2, \ldots, x_n$.

If $f(P) = A = \text{const}$ is a constant function on a measurable set Ω its integral sum is equal to the number

$$S_{\varrho} = \sum_{j=1}^{N} Am\Omega_j = Am\Omega$$

independent of the way in which Ω is partitioned. Therefore

$$\int_{\Omega} A\, d\Omega = A \int_{\Omega} d\Omega = Am\Omega \tag{7}$$

It should also be noted that if Ω is of Jordan measure zero ($m\Omega = 0$) then

$$\int_{\Omega} f\, d\Omega = \lim_{\max d(\Omega_j) \to 0} \sum_{j=1}^{N} f(P_j) m\Omega_j = \lim_{\max d(\Omega_j) \to 0} 0 = 0$$

for any function f defined on Ω and assuming finite values even if the function is unbounded. Hence, the integrability of f on Ω does not necessarily imply the boundedness of f on Ω. *In the study of the integrability of a function f defined on an arbitrary measurable set Ω we shall always suppose* (see the footnote on p. 38) *that the function is bounded on Ω*. By the way, *if Ω is a measurable open set the integrability of a function f on Ω implies its boundedness on Ω* (see § 12.10).

In what follows, when speaking of a function f Riemann integrable on a set $\Omega \subset R_n$, we shall mean, without further stipulations, that Ω is a Jordan measurable set (in the n-dimensional sense). This is quite natural since the definition of the Riemann integral on Ω is closely connected with the Jordan measurability of Ω.

§ 12.7. Upper an Lower Integral Sums. Key Theorem

Let R be the n-dimensional space. In the first reading of the book the reader may, for simplicity, interpret the subject matter of this section as referring to the case $n = 2$ or $n = 3$; however, all the considerations and statements are applicable in a quite analogous form to any natural n including $n = 1$.

Let a set Ω be Jordan measurable (and therefore bounded) in the n-dimensional sense, and let $f(P)$ be a bounded function defined on Ω:

$$|f(P)| \leqslant K < \infty \qquad (P \in \Omega)$$

The set Ω can be partitioned into subsets (Jordan measurable and such that any two of them either do not intersect or intersect only along some parts of their boundaries) in infinitely many ways. Let ϱ and ϱ' be two such partitions:

$$\Omega = \Omega_1 + \ldots + \Omega_N \quad \text{and} \quad \Omega = \Omega_1' + \ldots + \Omega_N'$$

We shall call ϱ' a *refinement* of ϱ and shall write $\varrho \subset \varrho'$ if any subset Ω_k' $(k = 1, \ldots, N')$ entering into the partition ϱ' is a part of one of the subsets Ω_j entering into the partition ϱ. In other words, the partition ϱ' can be obtained from ϱ by dividing some of the subsets entering into ϱ into a finite number of parts:

$$\Omega_j = \sum_{k=1}^{l_j} \Omega_{jk} \qquad (k = 1, \ldots, l_j;\ j = 1, \ldots, N)$$

Thus, the partition ϱ' can be represented as the double sum

$$\Omega = \sum_{j=1}^{N} \sum_{k=1}^{l_j} \Omega_{jk} \tag{1}$$

Let us take an arbitrary partition ϱ. The corresponding integral sum of the function f is

$$S_\varrho = \sum_{j=1}^{N} f(P_j) m\Omega_j = \sum_{\varrho} f(P_j) m\Omega_j$$

$(P_j \in \Omega_j)$. We shall use both forms of notation for the sum S_ϱ indicated here.

Let us put

$$M_j = \sup_{P \in \Omega_j} f(P), \quad m_j = \inf_{P \in \Omega_j} f(P),$$

$$\bar{S}_\varrho = \sum_{\varrho} M_j m\Omega_j \quad \text{and} \quad \underline{S}_\varrho = \sum_{\varrho} m_j m\Omega_j$$

The sums $\bar{S}_\varrho$ and $\underline{S}_\varrho$ are called the *upper* and *lower integral sums of the functions f (corresponding to the partition* ϱ*)* respectively.

For an arbitrary point $P_j \in \Omega_j$ we have the inequalities $m_j \leqslant f(P_j) \leqslant M_j$ $(j = 1, \ldots, N)$. Therefore, taking into account that $m\Omega_j \geqslant 0$ we can write

$$m_j m\Omega_j \leqslant f(P_j) m\Omega_j \leqslant M_j m\Omega_j$$

whence

$$\underline{S}_\varrho \leqslant S_\varrho \leqslant \bar{S}_\varrho \tag{2}$$

Thus, *irrespective of the choice of the points* P_j, *any integral sum of the function* f *corresponding to the partition* ϱ *is not greater than its upper sum and not less than its lower sum corresponding to the same partition* ϱ.

Another important property of upper and lower sums is that if $\varrho \subset \varrho'$ then the corresponding sums satisfy the inequalities

$$\underline{S}_\varrho \leqslant \underline{S}_{\varrho'} \leqslant \bar{S}_{\varrho'} \leqslant \bar{S}_\varrho \tag{3}$$

The second of them has already been proved.

Let us prove, for example, the third inequality. To this end we write $\bar{S}_{\varrho'}$ in the form

$$\bar{S}_{\varrho'} = \sum_{j=1}^{N} \sum_{k=1}^{l_j} M_{jk} m\Omega_{jk}$$

where

$$M_{jk} = \sup_{P \in \Omega_{jk}} f(P)$$

To compare $\bar{S}_\varrho$ with $\bar{S}_{\varrho'}$ we write the former in the similar form:

$$\bar{S}_\varrho = \sum_{j=1}^{N} M_j m\Omega_j = \sum_{j=1}^{N} M_j \sum_{k=1}^{l_j} m\Omega_{jk} = \sum_{j=1}^{N} \sum_{k=1}^{l_j} M_j m\Omega_{jk}$$

Now it becomes clear that $\bar{S}_{\varrho'} \leqslant \bar{S}_\varrho$ because the inclusion relations $\Omega_{jk} \subset \Omega_j$ imply $M_{jk} \leqslant M_j$.

Let ϱ_1 and ϱ_2 be some partitions of Ω and $\varrho = \varrho_1 + \varrho_2$ be the new partition obtained by the superposition of ϱ_1 and ϱ_2. Then ϱ serves as the refinement both of ϱ_1 and of ϱ_2, and we have

$$\underline{S}_{\varrho_1} \leqslant \underline{S}_\varrho \leqslant \bar{S}_\varrho \leqslant \bar{S}_{\varrho_2}$$

Hence,

$$\underline{S}_{\varrho_1} \leqslant \bar{S}_{\varrho_2} \tag{4}$$

for any partitions ϱ_1 and ϱ_2.

Let us fix ϱ_2 and consider ϱ_1 (which we denote by ϱ) as a variable partition; then

$$\underline{I}(f) = \sup_\varrho \underline{S}_\varrho \leqslant \bar{S}_{\varrho_2}$$

Finally, considering ϱ_2 (denoted by ϱ) as a variable partition we obtain

$$\underline{I}(f) \leqslant \bar{I}(f) = \inf \bar{S}_\varrho$$

The numbers $\bar{I}(f) = \bar{I}$ and $\underline{I}(f) = \underline{I}$ are called the *upper* and *lower* (*Riemann*) *integrals of the function* f *on* Ω respectively. Our considerations show that *the upper and lower integrals of* f *on* Ω *exist for any bounded function* f *defined on* Ω.

Let us proceed to the following important theorem.

Theorem 1 (Key Theorem). *Let $\Omega \subset R$ be a measurable set (i.e. a Jordan measurable set in the sense of dimension n) on which a bounded function f is defined* ($|f(x)| \leq K$).

Then the following properties are equivalent:

(i) $\bar{I} = \underline{I}$;

(ii) *for any $\varepsilon > 0$ there is a partition ϱ such that*

$$\bar{S}_\varrho - \underline{S}_\varrho < \varepsilon \tag{5}$$

(iii) *for any $\varepsilon > 0$ there is $\delta > 0$ such that for all partitions ϱ with diameters $d(\Omega_j) < \delta$ there holds inequality* (5);

(iv) *the integral*

$$\int\limits_\Omega f\, d\Omega = I \tag{6}$$

exists (in this case $I = \underline{I} = \bar{I}$).

Here it is of course meant that $\underline{I}$ and $\bar{I}$ are the lower and upper integrals of f on Ω respectively and that $\underline{S}_\varrho$ and $\bar{S}_\varrho$ are the lower and upper integral sums of f corresponding to the partition ϱ.

The theorem can be restated in the following equivalent way: *the integral of f on Ω exists if and only if at least one of properties* (i)-(iii) *holds (and then the other properties hold as well); if this is the case the value I of the integral is equal to $\underline{I} = \bar{I}$.*

Proof. The implication (i) → (ii). As follows from (i), there are partitions ϱ_1 and ϱ_2 such that

$$\underline{I} - \frac{\varepsilon}{2} < \underline{S}_{\varrho_1} \quad \text{and} \quad \bar{S}_{\varrho_2} < \bar{I} + \frac{\varepsilon}{2}$$

Then for $\varrho = \varrho_1 + \varrho_2$ we have

$$\underline{I} - \frac{\varepsilon}{2} < \underline{S}_{\varrho_1} \leq \underline{S}_\varrho \leq \bar{S}_\varrho \leq \bar{S}_{\varrho_2} \leq \bar{I} + \frac{\varepsilon}{2}$$

whence follows (5).

Implication (ii) → (i). Let ϱ be a partition satisfying (5). Then, by the inequalities $\underline{S}_\varrho \leq \underline{I} \leq \bar{I} \leq \bar{S}_\varrho$, we have $\bar{I} - \underline{I} < \varepsilon$. Since $\varepsilon > 0$ can be made arbitrarily small and $\underline{I}$ and $\bar{I}$ are definite numbers independent of ε there must be $\underline{I} = \bar{I}$.

Implication (iv) → (iii). Let integral (6) exist. The definition of the integral indicates that, given any $\varepsilon > 0$, there is $\delta > 0$ such that for any partition ϱ with $d(\Omega_j) < \delta$ there hold the inequalities

$$I - \frac{\varepsilon}{2} < \sum_{j=1}^{N} f(P_j)\, |\Omega_j| < I + \frac{\varepsilon}{2}$$

for any points $P_j \in \Omega_j$. On taking the supremum and the infimum with respect to $P_j \in \Omega_j$ of the sum entering into these inequalities we obtain

$$I - \frac{\varepsilon}{2} \leq \underline{S}_\varrho \leq \bar{S}_\varrho \leq I + \frac{\varepsilon}{2}$$

Consequently (iii) holds.

Implication (iii) → (ii) is trivial.

Implication (ii) → (iii). This is the most intricate part of the theorem asserting that if, given any $\varepsilon > 0$, there is a partition ϱ_* (dependent on ε) which we write as

$$\Omega = \sum_{j=1}^{N} \Omega_j^*$$

for which $\bar{S}_{\varrho_*} - \underline{S}_{\varrho_*} < \varepsilon$, then there also is $\delta > 0$ such that inequality (5) holds for all the partitions ϱ with $d(\Omega_j) < \delta$.

Let us denote by Γ_* the union of all the boundary points of Ω_j^* for all $j = 1, \ldots, N$. This union has measure zero since the sets Ω_j^* are measurable, and therefore there is a figure σ' covering Γ_* such that $|\sigma'| < \frac{\varepsilon}{2K}$. Let us take another figure σ such that σ' lies strictly inside it and $|\sigma| < \frac{\varepsilon}{2K}$.

Let $\delta > 0$ be a sufficiently small positive number such that the distance between any two points belonging to the boundaries of σ and σ' respectively exceeds δ. Then, moreover, the distance from any point of Γ_* to the boundary of σ is greater than δ.

Let us choose an arbitrary partition ϱ on which the only condition is imposed that its every subset (we denote these subsets ω) has a diameter $d(\omega)$ not exceeding δ (it is convenient to write ω without indices; the corresponding numbers m and M will also be written without indices). We have

$$\bar{S}_\varrho - \underline{S}_\varrho = \sum{}'(M-m)\, m\omega + \sum{}''(M-m)\, m\omega$$

where the sum $\sum'$ extends over all the subsets ω entering into the partition ϱ each of which contains a point belonging to Γ_*. Since $d(\omega) < \delta$, we have $\omega \subset \sigma$ for all such subsets, and their total measure does not exceed $m\sigma < \frac{\varepsilon}{2K}$. Consequently,

$$\sum{}'(M-m)\, m\omega \leqslant 2K \sum{}' m\omega \leqslant 2K \frac{\varepsilon}{2K} = \varepsilon$$

Let us write the sum $\sum''$ as the double sum $\sum'' = \sum_i \sum^i$ where $\sum^i$ is the sum of the terms entering into $\sum''$ which correspond to the subsets ω entering into the partition ϱ and lying entirely within the subset Ω_i^* of the old partition ϱ_*. We have

$$\sum{}''(M-m)\, m\omega = \sum_i \sum{}^i (M-m)\, m\omega \leqslant$$

$$\leqslant \sum_i (M_i^* - m_i^*) \sum{}^i m\omega \leqslant \sum_i (M_i^* - m_i^*)\, m\Omega_i^* < \varepsilon$$

and therefore $\bar{S}_\varrho - \underline{S}_\varrho < 2\varepsilon$ for all the partitions ϱ with $d(\omega) < \delta$, that is (iii) takes place.

Implication (iii) → (iv). Let (iii) hold. Then, as has already been proved, (i) also holds. Let us take an arbitrary $\varepsilon > 0$ and choose $\delta > 0$ as was indicated in the proof of (iii). Then for the partition ϱ mentioned in (iii) we have

$$\underline{S}_\varrho \leqslant \sum f(P_j)\,\Omega_j \leqslant \bar{S}_\varrho, \quad \underline{S}_\varrho \leqslant \underline{I} = \bar{I} \leqslant \bar{S}_\varrho \tag{7}$$

On putting $I = \underline{I} = \bar{I}$ we obtain

$$|I - \sum f(P_j)\,|\Omega_j|| < \varepsilon \tag{8}$$

that is I is the integral of f over Ω. Thus, we have proved (iv).

The proof of the theorem is completed.

Theorem 2. *Let*

$$\Omega = \sum_{j=1}^{N_k} \Omega_j^k \qquad (k = 1, 2, \ldots) \tag{9}$$

be a sequence of partitions ϱ_k $(k = 1, 2, \ldots)$ *of a measurable set* Ω, *the maximum diameter* δ_k *of the subsets* Ω_j^k *tending to zero.*

If for a bounded function $f(x)$ *defined on* Ω *there holds at least one of the conditions*

$$\text{(i)} \ \lim_{k\to\infty} (\bar{S}_{\varrho_k}(f) - \underline{S}_{\varrho_k}(f)) = 0 \tag{10}$$

$$\text{(ii)} \ \lim_{k\to\infty} \sum_j f(\xi_j^k)\, m\Omega_j^k = I \ (\xi_j^k \in \Omega_j^k) \tag{11}$$

and

$$\text{(iii)} \ \lim_{k\to\infty} \underline{S}_{\varrho_k}(f) = \lim_{k\to\infty} \bar{S}_{\varrho_k}(f) = I \tag{12}$$

then the integral of f *on* Ω *exists.*

Conversely, the existence of the integral of f *on* Ω *implies the fulfilment of all the conditions* (i), (ii) *and* (iii).

The existence of only one of limits (iii) *does not necessarily imply the existence of the integral of* f *on* Ω.

The proof of this theorem is completely analogous to that of Theorem 2 in § 9.4 stated for the one-dimensional case.

Theorem 3. *Let there be given a sequence of partitions of type* (9) *of a measurable set* Ω *with* $\delta_{\varrho_k} = \max_j d(\Omega_j^k) \to 0$. *Then the sum*

$$\sum_j{}' m\Omega_j^k = m\Omega^k$$

of the measures of those subsets of the partition which adjoin the boundary Γ *of the set* Ω *(that is the sum of the measures of those subsets* Ω_j^k *for each of which the closure* $\bar{\Omega}_j^k$ *contains points belonging to* Γ*) tends to zero as* $k \to \infty$.

Proof. Let us take an arbitrary $\varepsilon > 0$ and choose a figure σ covering Γ such that $|\sigma| < \varepsilon$. Now let us stretch σ along the directions of all the coordinate axes so that the new figure $\sigma' \supset \sigma$ has a measure $|\sigma'| < \varepsilon$. Denoting by η the distance between the boundaries of σ' and Ω we can assert that there is k_0 such that $\delta_{\varrho_k} < \eta$ for $k > k_0$; for such k's the subsets Ω_j^k adjoining Γ belong to σ', that is

$$m\Omega^k = \sum_j{}' m\Omega_j^k < |\sigma'| < \varepsilon$$

Theorem 4. *The definite integral of a bounded function f on a measurable set Ω can be defined as the limit*

$$\lim_{\delta_{\varrho_k}\to 0} \sum_j{}' f(P_j^k)\, m\Omega_j^k = \int_\Omega f\, dx$$

for a concrete sequence of partitions (9) *with* $\delta_{\varrho_k} \to 0$ *where the integral sums only extend over the subsets* Ω_j^k *not adjoining* Γ.

For, by the foregoing theorem, the parts of the integral sum corresponding to these Ω_j^k's which adjoin Γ can be estimated by the inequality

$$\Big|\sum_j{}' f(P_j^k)\, m\Omega_j^k\Big| \leqslant K \sum_j{}' m\Omega_j^k \to 0, \qquad k \to 0 \qquad (K > |f(x)|)$$

§ 12.8. Integrability of a Continuous Function on a Measurable Closed Set. Some Other Integrability Conditions

Theorem 1. *A function $f(P)$ continuous on a Jordan measurable closed set Ω is Riemann integrable on Ω.*

Proof. Since the set Ω is measurable it is bounded. Besides, it is closed, and therefore the function f is uniformly continuous on Ω. This means that for any $\varepsilon > 0$ there is $\delta > 0$ such that if $P', P'' \in \Omega$ and $|P''-P'| < \delta$ then $|f(P'')-f(P')| < \varepsilon$.

Let ϱ be an arbitrary partition $\Omega = \sum_1^N \Omega_j$ of the set Ω into measurable parts Ω_j with diameters $d(\Omega_j) < \delta$, and let, as usual, $M_j = \sup_{x\in\Omega_j} f(x)$ and $m_j = \inf_{x\in\Omega_j} f(x)$. We have

$$M_j - m_j = \sup_{P',P''\in\Omega_j} [f(P'')-f(P')] \leqslant \varepsilon$$

because, by the hypothesis, the distance between any points $P', P'' \in \Omega_j$ does not exceed δ. Consequently,

$$\bar{S}_\varrho - \underline{S}_\varrho = \sum (M_j - m_j)\, m\Omega_j < \varepsilon \sum m\Omega_j = \varepsilon m\Omega = \eta$$

Since $\eta > 0$ can be made arbitrarily small, the key theorem shows that the integral of f on Ω does in fact exist.

Theorem 2. *A function f defined and bounded on a measurable closed set Ω and continuous on Ω everywhere except possibly at points forming a subset $\Lambda \subset \Omega$ of measure zero is integrable on Ω.*

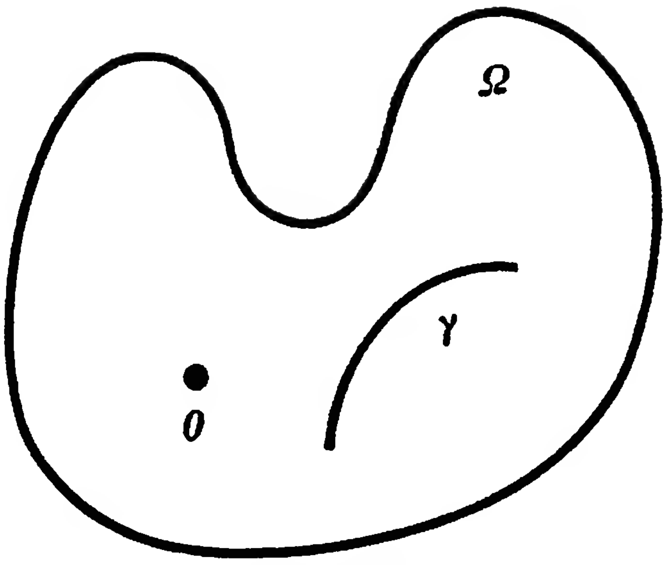

Fig. 12.3

For instance, if a function f is bounded on a closed set Ω in the plane whose boundary is a smooth curve (see Fig. 12.3) and if f is continuous on Ω except at a separate point O and at the points belonging to a smooth arc γ, the function f is integrable on Ω.

Proof. Let $\varepsilon > 0$ be an arbitrary positive number and let $\sigma \supset \Lambda$ be an open (without boundary) figure such that $|\sigma| < \varepsilon$. Then $\Omega - \sigma$ is a measurable closed set on which f is continuous and, consequently, integrable. Let us take a partition ϱ' of the set $\Omega - \sigma$ of the form $\Omega - \sigma = \Omega_1 + \ldots + \Omega_N$ such that $\bar{S}_{\varrho'} - \underline{S}_{\varrho'} < \varepsilon$ and then construct the partition ϱ of the form $\Omega = \Omega_1 + \ldots + \Omega_N + \sigma\Omega$ of the set Ω. On putting $M = \sup\limits_{x \in \sigma\Omega} f(x)$ and $m = \inf\limits_{x \in \sigma\Omega} f(x)$ we write

$$\bar{S}_\varrho - \underline{S}_\varrho = (\bar{S}_{\varrho'} - \underline{S}_{\varrho'}) + (M - m)m(\sigma\Omega) < \varepsilon + 2K\varepsilon = \eta$$
$$(K > |f(x)|, \ x \in \Omega)$$

where η can be made arbitrarily small. Therefore, according to property (ii) mentioned in the key theorem, the function f is integrable on Ω.

It turns out that Theorem 2 can be generalized to the case when the set Λ is of Lebesgue measure zero (instead of Jordan measure zero; see § 12.9).

This generalization provides a necessary and sufficient condition for the Jordan integrability of a bounded function on a measurable closed set because there holds the following theorem:

Theorem (H. L. Lebesgue). *A function $f(x)$ defined and bounded on a Jordan measurable closed set Ω is Riemann integrable on Ω if and only if the function f is continuous on Ω except possibly on a subset of Ω having Lebesgue measure zero.*

Example. Let us take the function $\psi(x, y) = \sin \frac{1}{xy}$ and consider it as defined in the half-open rectangle $\Delta' = \{0 < x, y \leqslant \pi/2\}$. To apply Theorem 2 to this function we can argue in the following way. Let us extend ψ to the closed interval $0 \leqslant x \leqslant \pi/2$ of the x-axis and to the closed interval $0 \leqslant y \leqslant \pi/2$ of the y-axis in an arbitrary manner so that the new values of the function are uniformly bounded. The extended function ψ is then defined on the closed rectangle $\Delta = \bar{\Delta}'$ and is bounded on Δ and continuous on Δ everywhere except on the set of Jordan two-dimensional measure zero (consisting of these two closed intervals). Consequently, by Theorem 2, the integral

$$\iint\limits_{\Delta} \psi(x, y)\, dx\, dy = \iint\limits_{\Delta'} \psi(x, y)\, dx\, dy$$

exists (see § 12.11, Theorem 1 and the corollary of this theorem).

§ 12.9. Set of Lebesgue Measure Zero*

An arbitrary open rectangular parallelepiped (rectangle)

$$\Delta = \{a_j < x_j < b_j;\ j = 1, \ldots, n\}$$

in the n-dimensional space $R = R_n$ will also be referred to as an *interval* in R. The volume (n-dimensional measure) of Δ is equal to

$$|\Delta| = \prod_{j=1}^{n} (b_j - a_j)$$

because a closed rectangle differs from the corresponding open rectangle by a set of (n-dimensional) measure zero.

We say that a *set E is of Lebesgue measure zero* if for any $\varepsilon > 0$ there is a finite or countable number of intervals $\Delta^1, \Delta^2, \ldots$ covering E $(E \subset \sum \Delta^k)$ the sum of whose volumes is less than ε: $\sum |\Delta^k| < \varepsilon$.

Lemma 1. *The sum of a finite or countable number of sets $E^1, E^2, \ldots$ each of which has Lebesgue measure zero is in its turn a set of Lebesgue measure zero.*

For, if $\varepsilon > 0$ is an arbitrary positive number we can construct the following coverings of the given sets: each E^k is covered by a countable (or finite) system of intervals Δ_j^k $\left(E^k \subset \sum_j \Delta_j^k\right)$ such that $\sum_j |\Delta_j^k| < \varepsilon/2^k$ $(k = 1, 2, \ldots)$. Since the intervals Δ_j^k $(j, k = 1, 2, \ldots)$ can be renumbered as $\Delta_1, \Delta_2, \ldots$ and since their union covers $E = \sum E^k$ and the sum of their volumes is less than $\varepsilon = \sum_k (\varepsilon/2^k)$ we see that, by the arbitrariness of ε, the set E is of Lebesgue measure zero.

If a set E has a Jordan measure zero this means that for any $\varepsilon > 0$ it can be covered by a finite number of intervals with total volume less than ε, and consequently such a set E is also of Lebesgue measure zero. This shows that it is permissible to use one and the same notation (i.e. to write $mE = 0$) for the Jordan and the Lebesgue measure when the Jordan measure exists and is equal to zero. However, here we have only discussed a very special case of the Lebesgue measure, namely, the case when it is equal to zero.

It should be stressed that a set can be of Lebesgue measure zero and at the same time be Jordan nonmeasurable. For example, the set of rational points contained in the closed interval [0, 1] has Lebesgue measure zero (since it is countable). But it has no (one-dimensional) Jordan measure because its exterior Jordan content is equal to 1 while the interior Jordan content is equal to zero.

We also note that if a set F is bounded and closed and has Lebesgue measure zero then, given any $\varepsilon > 0$, there is a countable system of intervals covering F whose total volume is less than ε. By the Heine-Borel lemma, since the set F is bounded and closed, this system contains a finite subsystem covering F, and the sum of the volumes of the intervals in this subsystem is less than ε. Therefore F is Jordan measurable and has the Jordan measure equal to zero. We have thus proved the following lemma.

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*The subject matter of this section is a part of the material presented in full in § 19.1.

Lemma 2. *A bounded closed set of Lebesgue measure zero is Jordan measurable and its Jordan measure is also equal to zero.*

§ 12.10. Proof of Lebesgue's Theorem. Connection Between Integrability and Boundedness of a Function

In § 7.10 we stated the definition of the oscillation $\omega(x)$ $\big(\omega(x) \geqslant 0\big)$ of a function f, defined on a set Ω, at a point $x \in \Omega$ and showed (see § 7.10, Theorem 5) that f is continuous at x if and only if its oscillation at that point is equal to zero $\big(\omega(x) = 0\big)$. Hence, the oscillation of a function at its point of discontinuity is sure to be positive. Let E_λ denote the set of all points $x \in \Omega$ at which the oscillation of f is not less than $\lambda > 0$ $\big(\omega(x) \geqslant \lambda\big)$. It is important to note that if Ω is a closed set then so is the set E_λ for any λ (see § 7.10, Theorem 6).

Proof of sufficiency of the condition of Lebesgue's theorem. Let the function f be bounded on a measurable closed set Ω, that is

$$|f(x)| \leqslant K \qquad (x \in \Omega) \tag{1}$$

and let the set E of its points of discontinuity be of Lebesgue measure zero ($E \subset \Omega$).

We shall suppose that $m\Omega > 0$ since if otherwise the assertion we have to prove becomes quite obvious. Let us take an arbitrary $\varepsilon > 0$, and let $\lambda > 0$ satisfy the inequality $4\lambda m\Omega < \varepsilon$.

Since we have $mE = 0$ for the Lebesgue measure of E, the Lebesgue measure of E_λ is also equal to zero for any $\lambda > 0$ ($mE_\lambda = 0$). But the set E_λ is bounded and closed, and therefore the Jordan measure of E_λ is also equal to zero. Consequently, given any $\varepsilon > 0$, there is a set G covering E_λ which is a finite system of intervals with measure $|G| < \frac{\varepsilon}{4K}$. The set G is open and measurable and, since the set Ω is bounded and closed it is representable as a union of two nonintersecting Jordan measurable sets: $\Omega = \Omega_1 + \Omega''$ where $\Omega_1 = G\Omega$, $\Omega'' = \Omega - G$, $m\Omega_1 \leqslant |G| < \frac{\varepsilon}{4K}$ and Ω'' is a bounded closed set. The oscillation of the function in question is less than λ at each point of the set Ω''. Therefore, by Theorem 3, § 7.10, there is $\delta > 0$ such that $|f(P) - f(P')| < 2\lambda$ for any two points $P, P' \in \Omega''$ satisfying the condition $|P - P'| < \delta$. Let us partition Ω'' into parts of diameters less than the number δ: $\Omega'' = \sum_{2}^{N} \Omega_j$. These parts together with the set Ω_1 defined above form a partition ϱ of the whole set Ω. For this partition we obviously have the inequalities

$$\bar{S}_\varrho - \underline{S}_\varrho = (M_1 - m_1) m\Omega_1 - \sum_{j=2}^{N} (M_j - m_j) m\Omega_j \leqslant$$

$$\leqslant 2K \frac{\varepsilon}{4K} + 2\lambda \sum_{j=2}^{N} m\Omega_j < \frac{\varepsilon}{2} + 2\lambda m\Omega < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Consequently, by the arbitrariness of ε and by the key theorem, the function f is integrable on Ω.

Proof of necessity of the condition of Lebesgue's theorem. Let the function f be bounded on a measurable closed set Ω and integrable on it. Then, according to the key theorem, for any $\varepsilon > 0$ and $\lambda > 0$ there exists a partition ϱ of the set Ω such that (see the explanations below)

$$\varepsilon\lambda > \bar{S}_\varrho - \underline{S}_\varrho = \sum (M_j - m_j) m\Omega_j > \sum{}' (M_j - m_j) m\Omega_j \geqslant \lambda \sum{}' m\Omega_j \quad (2)$$

The sum $\sum'$ extends only over those summands which correspond to the sets Ω_j each of which contains at least one interior point P of E_λ. Such a point can be covered by a ball V_δ with centre at that point lying within Ω_j, and therefore we have the inequalities

$$M_j - m_j \geqslant M_\delta - m_\delta \geqslant \omega(P) \geqslant \lambda \text{ and } M_\delta = \sup_{x \in V_\delta} f(x), \quad m_\delta = \inf_{x \in V_\delta} f(x)$$

On cancelling (2) by λ we obtain the inequalities

$$\varepsilon > \sum{}' m\Omega_j = m\left(\sum{}' \Omega_j\right)$$

where $\sum' \Omega_j$ contains the points of E_λ each of which is an interior point of one of the subsets Ω_j entering into the partition ϱ. The remaining points of E_λ not belonging to $\sum' \Omega_j$ can only lie on the boundaries of the sets Ω_j ($j = 1, \ldots, N$) whose total measure is zero. Thus, for any $\lambda > 0$ and $\varepsilon > 0$ we have $mE_\lambda < \varepsilon$, that is $mE_\lambda = 0$. Now, since the set of all points of discontinuity of f can be represented in the form

$$E = \sum_{k=1}^{\infty} E_{1/k}$$

and since $mE_{1/k} = 0$ ($k = 1, 2, \ldots$) we conclude that $mE = 0$.

The necessity of the condition of the theorem has been proved.

Let us state the following useful definition. We shall say that a set $\Omega \subset R$ *possesses property* (A) if it is measurable and if for any $\varepsilon > 0$ there is a partition $\Omega = \sum_1^N \Omega_j$ of Ω into measurable parts of positive measures ($m\Omega_j > 0$) with diameters $d(\Omega_j) < \varepsilon$.

An arbitrary measurable open set Ω possesses property (A). For, if we take a rectangular network with cubes Δ of diameter less than ε, it partitions Ω into nonempty measurable parts Ω_j. Let a set Ω_j of this kind be a part of Δ and let $x^0 \subset \Omega_j \subset \Omega$; then there is a ball V_{x^0} with centre at x^0 contained in

Ω, and the intersection of V_{x^0} and Δ belongs to Ω_j: $V_{x^0}\Delta \subset \Omega_j$. Now it is readily seen that $m(V_{x^0}\Delta) > 0$.

Of course, together with the measurable open set Ω, its closure $\bar{\Omega}$ also possesses property (A).

A rectangle, a circle and an ellipse are all examples of two-dimensional sets possessing property (A). Examples of one-dimensional sets possessing property (A) are a closed interval, a finite system of closed intervals and an open interval; as examples of three-dimensional sets with property (A) we can take a ball, a cube, an elementary figure, an ellipsoid and a torus.

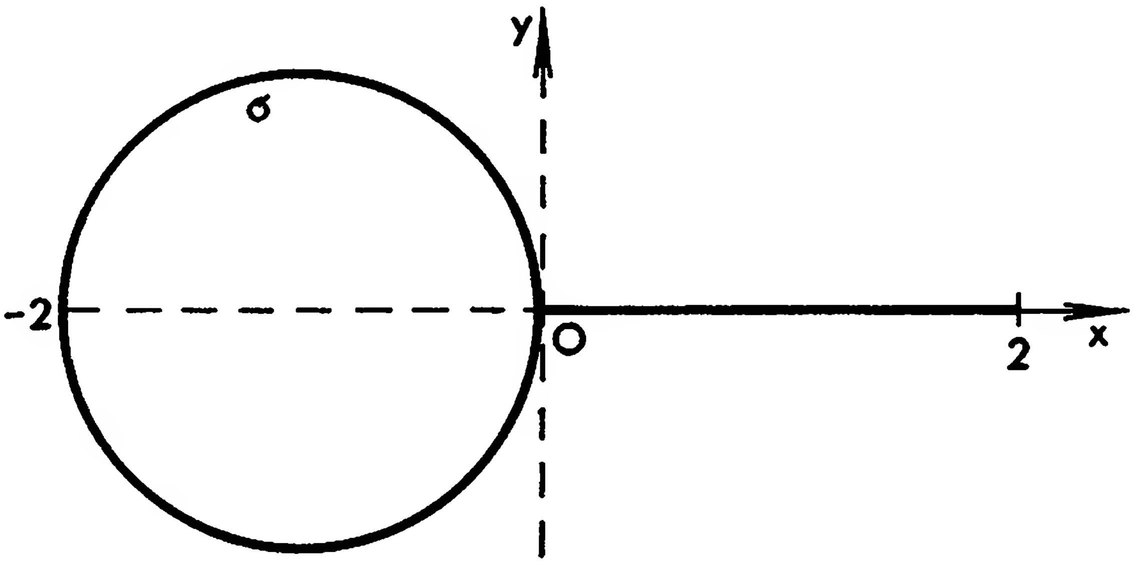

Fig. 12.4

On the other hand, the plane set Ω shown in Fig. 12.4 which consists of the circle σ and of the closed interval [0, 2] lying in the x-axis does not possess property (A). This set is measurable because its boundary consists of a finite number of smooth curvilinear segments. However, if, for instance, we take $\varepsilon < 1/2$, it is readily seen that it is impossible to break up Ω into measurable (in the two-dimensional sense) parts of positive measures with diameters less than ε.

Theorem 1. *A function f integrable on a set Ω possessing property* (A) *is bounded on Ω.*

Indeed, let us take an arbitrary $\delta > 0$ and construct a partition ϱ of the set Ω into parts Ω_j ($j = 1, \ldots, N$) of positive measures so that

$$\delta_\varrho = \max_j d(\Omega_j) < \delta$$

Suppose that the function f is unbounded on Ω; then it is unbounded on at least one of the sets Ω_j, say on Ω_1. The corresponding integral sum can be represented as

$$S_\varrho = f(p_1)m\Omega_1 + \sum_{j=2}^{N} f(p_j)m\Omega_j \tag{3}$$

For the given ϱ and fixed p_j ($j = 2, \ldots, N$) the sum $\sum_{j=2}^{N} f(p_j)m\Omega_j$ is constant while the product $f(p_1)m\Omega_1$ where p_1 is a variable point belonging to Ω_1 is unbounded (because $m\Omega_1 > 0$). Consequently, the integral sum S_ϱ is unbounded. This shows that S_ϱ cannot tend to a finite limit as $\delta_\varrho \to 0$, and consequently the function f is nonintegrable on Ω (cf. Theorem 1, § 9.2).

For the sets possessing property (A) Lebesgue's theorem can obviously be stated as follows.

Theorem 2. *Let Ω be a closed set possessing property (A). Then a function f defined on Ω is integrable on Ω if and only if it is bounded on Ω and continuous on Ω except possibly at the points forming a subset of Ω of measure zero.*

§ 12.11. Properties of Multiple Integrals

Theorem 1. *If a function f is bounded and integrable on a set $\Omega = \Omega' + \Omega''$ where Ω' and Ω'' are measurable sets which either do not intersect or intersect only along some parts of their boundaries then f is also integrable on each of the sets Ω' and Ω'' and vice versa. In this case*[*]

$$\int_{\Omega} f\,dx = \int_{\Omega'} f\,dx + \int_{\Omega''} f\,dx \tag{1}$$

To prove the theorem we take an arbitrary sequence of partitions ϱ_k ($k = 1, 2, \ldots$) of the set Ω such that the boundaries of the sets Ω' and Ω'' consist only of the boundary points of some of the subsets into which Ω is partitioned. These partitions induce on Ω' and Ω'' the corresponding partitions ϱ'_k and ϱ''_k. The further course of the proof is exactly the same as that of the proof of Theorem 1 in § 9.7 for the one-dimensional integral, the role of the closed intervals $[a, c]$ and $[c, b]$ being played by the sets Ω' and Ω''.

Corollary. *If a bounded and integrable function f defined on Ω is redefined on any subset $E \subset \Omega$ of Jordan measure zero so that the modified function f_1 remains bounded on Ω, then f_1 is integrable on Ω and*

$$\int_{\Omega} f\,d\Omega = \int_{\Omega} f_1\,d\Omega$$

Indeed, the set $\Omega - E$ is measurable together with Ω, and therefore f is integrable on $\Omega - E$ and, besides,

$$\int_{E} f\,d\Omega = \int_{E} f_1\,d\Omega = 0$$

---

[*] For an unbounded function f equality (1) may not hold. For instance, if $f = 0$ on σ (see Fig. 12.4) and $f = 1/x$ on the half-open interval $\gamma = (0, 2]$ of the x-axis then $\iint_{\sigma} f\,dx\,dy = \iint_{\gamma} f\,dx\,dy = 0$. At the same time the integral $\iint_{\sigma+\gamma} f\,dx\,dy$ does not exist.

Consequently, f_1 is integrable on Ω and

$$\int_{\Omega} f_1\,d\Omega = \int_{\Omega-E} f_1\,d\Omega + \int_{E} f_1 d\Omega = \int_{\Omega-E} f d\Omega + \int_{E} f d\Omega = \int_{\Omega} f d\Omega$$

By virtue of this assertion, in the case when a function f is bounded on a measurable nonclosed set Ω and integrable on Ω it makes sense to write

$$\int_{\Omega} f\,dx = \int_{\bar{\Omega}} f\,dx$$

although the function f may not be defined on $\bar{\Omega}-\Omega$. For, if f is not originally defined on $\bar{\Omega}-\Omega$ and is then extended to $\bar{\Omega}-\Omega$ in an arbitrary way so that its values assumed on $\bar{\Omega}-\Omega$ are uniformly bounded, the integrals of f over Ω and over $\bar{\Omega}$ coincide.

Theorem 2. *If $f(x)$ and $\varphi(x)$ are bounded integrable functions on Ω and c is a constant then the functions*

(i) $f(x)\pm\varphi(x)$, (ii) $cf(x)$, (iii) $|f(x)|$, (iv) $f(x)\varphi(x)$ and (v) $1/f(x)$

(in the last case it is supposed that $|f(x)| > d > 0$) are integrable on Ω. Besides,

$$\int_{\Omega} (f\pm\varphi)\,dx = \int_{\Omega} f\,dx \pm \int_{\Omega} \varphi\,dx \tag{2}$$

and

$$\int_{\Omega} cf\,dx = c\int_{\Omega} f\,dx \tag{3}$$

It should be noted that the integrability of these functions readily follows from Lebesgue's theorem because the Lebesgue measure of the sum of two sets of measure zero is again a set of measure zero.

Equalities (2) and (3) are proved by complete analogy with Theorem 2, § 9.7. The existence of the integrals of functions (i)-(v) can also be proved without resorting to Lebesgue's theorem as it was done in the above-mentioned one-dimensional theorem.

Theorem 3. *If functions f_1, f_2 and φ are bounded and integrable on Ω and if*

$$f_1(P) \leqslant f_2(P) \quad \text{and} \quad \varphi(P) \geqslant 0 \qquad (P \in \Omega) \tag{4}$$

then

$$\int_{\Omega} f_1\varphi\,dP \leqslant \int_{\Omega} f_2\varphi\,dP \tag{5}$$

In particular, if

$$A \leqslant f(P) \leqslant B, \quad \varphi(P) \geqslant 0 \tag{6}$$

where A and B are constants then

$$A\int_{\Omega}\varphi\,dP \leqslant \int_{\Omega} f\varphi\,dP \leqslant B\int_{\Omega}\varphi\,dP \tag{7}$$

and for some C we have

$$\int_{\Omega} f\varphi\,dP = C\int_{\Omega}\varphi\,dP \qquad (A \leqslant C \leqslant B) \tag{8}$$

Proof. It follows from (4) that

$$f_1(P)\varphi(P) \leqslant f_2(P)\varphi(P) \qquad (P \in \Omega)$$

whence, for any partition ϱ of the set Ω we obtain

$$S_{\varrho}(f_1\varphi) \leqslant S_{\varrho}(f_2\varphi)$$

On passing to the limit as $\delta \to 0$ where δ is the maximum diameter of the subsets forming the partition ϱ we arrive at (5).

Equality (8) is spoken of as the *mean value theorem for multiple integrals.*

Remark. If Ω is a closed connected set and f is a function continuous on Ω then

$$\int_{\Omega} f\varphi\,dP = f(Q)\int_{\Omega}\varphi\,dP$$

where Q is a point belonging to Ω.

Indeed, the continuity of f on the closed measurable set Ω implies that f is integrable on Ω and, besides, there are some points Q_1 and Q_2 belonging to Ω at which f attains its maximum and minimum (on Ω) respectively:

$$\min_{P\in\Omega} f(P) = f(Q_1) = A \quad \text{and} \quad \max_{P\in\Omega} f(P) = f(Q_2) = B$$

By the connectedness of Ω, there is a continuous curve $P = P(t) = \{\varphi_1(t), \ldots, \varphi_n(t)\}$ $(t_1 \leqslant t \leqslant t_2)$ lying within Ω which joins the points $Q_1 = P(t_1)$ and $Q_2 = P(t_2)$. The function

$$z = f\big(P(t)\big) = f\big(\varphi_1(t), \ldots, \varphi_n(t)\big) = \psi(t) \qquad [t_1 \leqslant t \leqslant t_2]$$

continuous on the closed interval $[t_1, t_2]$ attains for some $t_0 \in [t_1, t_2]$ the value $\psi(t_0) = f(Q) = C$ where $Q = P(t_0)$, which completes the proof.

Theorem 4. *For a bounded integrable function f on Ω there hold the inequalities*

$$\left|\int_{\Omega} f\,dx\right| \leqslant \int_{\Omega} |f|\,dx \leqslant Km\Omega \qquad \left(K = \sup_{x\in\Omega}|f(x)|\right) \tag{9}$$

Indeed, the integrability of $|f|$ was proved in Theorem 2 and, besides, we have

$$-|f(x)| \leqslant f(x) \leqslant |f(x)| \qquad (x \in \Omega)$$

whence

$$-\int\limits_{\Omega}|f|\,dx \leqslant \int\limits_{\Omega} f\,dx \leqslant \int\limits_{\Omega}|f|\,dx$$

It should be noted that if we only suppose that $|f(x)|$ is integrable on Ω inequality (9) may have no sense (see the remark at the end of § 9.7).

§ 12.12. Reduction of Multiple Integral to Iterated Integral

The theorems proved below make it possible to reduce the computation of a multiple integral to successive integrations with respect to each of the variables $x_1, x_2, x_3, \ldots$

Theorem 1. *Let a function $f(u, v)$ be defined and integrable on a rectangle $\Delta = \{a \leqslant u \leqslant b;\ c \leqslant v \leqslant d\} = [a, b]\times[c, d]$ in the uv-plane. Then*

$$\iint\limits_{\Delta} f(u, v)\,du\,dv = \int\limits_{a}^{b} du\left(\int\limits_{c}^{d} f(u, v)\,dv\right) = \int\limits_{c}^{d} dv\left(\int\limits_{a}^{b} f(u, v)\,du\right) \qquad (1)$$

where the expression

$$\int\limits_{c}^{d} f(u, v)\,dv \qquad (2)$$

is understood as the Riemann integral of $f(u, v)$ with respect to the variable v for a fixed value of u provided this integral exists; if it does not exist expression (2) is understood as an arbitrary number lying between the upper and lower integrals of $f(u, v)$ with respect to $v \in [c, d]$. The integral with respect to u over the interval $[a, b]$ in the second term in (1) *exists in the Riemann sense. The same refers to the third term in* (1).

According to what was shown in § 12.10, the rectangle Δ is a set possessing property (A), and therefore the integrability of the function f on Δ implies its boundedness on Δ.

If not only the double integral of f over Δ exists but integral (2) also exists for every $u \in [a, b]$ then the second term in (1) is the result of the Riemann integration of f first with respect to v (in the one-fold sense) and then with respect to u. Similar assertion applies to the third term in (1).

Proof. Let us consider $f(u, v)$ as a function of v defined on $[c, d]$ for every fixed $u \in [a, b]$. This function is bounded and consequently its upper and lower integrals exist and satisfy the inequality $\underline{I}(u) \leqslant \overline{I}(u)$ $(a \leqslant u \leqslant b)$. Let $\Phi(u)$ be a function satisfying the inequalities $\underline{I}(u) \leqslant \Phi(u) \leqslant \overline{I}(u)$ $(a \leqslant u \leqslant b)$. We shall show that such a function $\Phi(u)$ is integrable on $[a, b]$ and that its integral over $[a, b]$ is equal to the double integral of f over Δ. To this end it suffices (see Theorem 2, § 9.4) to take the partition r of the closed interval $[a, b]$ into N equal parts with the aid of the points of division $a =$

$= u_0 < u_1 < \ldots < u_N = b$, $h = u_i - u_{i-1} = \frac{b-a}{N}$ and to prove that the limit

$$\lim_{h\to 0} S_r(\Phi) = \lim_{N\to\infty} \sum_{i=1}^{N} \Phi(\xi_i)\, h = \int_a^b \Phi(u)\, du = \iint_\Delta f\, du\, dv \tag{3}$$

exists for any choice of the intermediate points $\xi_i \in [u_{i-1}, u_i]$ $(i = 1, \ldots, N)$ and is independent of the choice.

Let us also form the partition ϱ of the closed interval $[c, d]$ into N equal parts with points of division $c = v_0 < v_1 < \ldots < v_N = d$, $k = v_j - v_{j-1} = \frac{d-c}{N}$. This results in the partition of the rectangle Δ into congruent rectangles $\Delta_{ij} = [u_{i-1}, u_i] \times [v_{j-1}, v_j]$ to which there correspond the upper and lower integrals

$$\bar{S}_{r\times\varrho}(f) = hk \sum_i \sum_j M_{ij} \quad \text{and} \quad \underline{S}_{r\times\varrho}(f) = hk \sum_i \sum_j m_{ij}$$

of the function f on Δ.

Since

$$\Phi(\xi_i) \leqslant \bar{I}(\xi_i) \leqslant k \sum_j \sup_{v_{j-1} \leqslant v \leqslant v_j} f(\xi_i, v) \leqslant k \sum_j M_{ij}$$

and, similarly,

$$\Phi(\xi_i) \geqslant k \sum_j m_{ij}$$

we have

$$\underline{S}_{r\times\varrho}(f) \leqslant \sum_{j=1}^{N} \Phi(\xi_j)\, h \leqslant \bar{S}_{r\times\varrho}(f)$$

By the hypothesis, the function f is integrable on Δ and therefore there exist the limits

$$\lim_{N\to\infty} \underline{S}_{r\times\varrho}(f) = \lim_{N\to\infty} \bar{S}_{r\times\varrho}(f) = \int_\Delta f\, du\, dv$$

Consequently, limit (3) exists, which proves the first equality (1). The second is proved analogously.

Now let us consider an n-dimensional rectangle $\Delta = \{a_j \leqslant x_j \leqslant b_j;\ j = 1, \ldots, n\}$. We shall denote by Δ' the projection of Δ on the coordinate subspace of points $\boldsymbol{u} = (x_1, \ldots, x_m)$: $\Delta' = \{a_j \leqslant x_j \leqslant b_j;\ j = 1, \ldots, m\}$ $(1 \leqslant m < n)$; similarly, by Δ'' we shall denote the projection of Δ on the coordinate subspace of the points $\boldsymbol{v} = (x_{m+1}, \ldots, x_n)$: $\Delta'' = \{a_j \leqslant x_j \leqslant b_j;\ j = m+1, \ldots, n\}$. We shall also write

$$\boldsymbol{x} = (\boldsymbol{u}, \boldsymbol{v}) \quad \text{and} \quad \Delta = \Delta' \times \Delta''$$

There holds the following theorem generalizing Theorem 1:

Theorem 2. *Let a function* $f(x_1, \ldots, x_n) = f(u, v)$ *be integrable on the rectangle* $\Delta = \Delta' \times \Delta''$ $(u \in \Delta', v \in \Delta'')$. *Then*

$$\iint_{\Delta} f(u, v)\, du\, dv = \int_{\Delta'} du \int_{\Delta''} f(u, v)\, dv \tag{4}$$

where the expression $\int_{\Delta''} f(u, v)\, dv$ *is understood as the Riemann integral of* f *with respect to* v *for a fixed* u *provided that this integral exists and as an arbitrary number lying between the upper and lower integrals of* $f(u, v)$ *with respect to* $v \in \Delta''$ *if it does not exist. The integral with respect to* u *over* Δ' *entering into the right-hand side of* (4) *exists in the Riemann sense. In particular, if, in addition to the above hypothesis, it is known that the function* $f(u, v)$ *is integrable with respect to* $v \in \Delta''$ *for any fixed* $u \in \Delta'$, *then, without any further stipulations, the right-hand side of* (4) *is the result of the successive integration of* f *in the Riemann sense first with respect to* $v \in \Delta''$ *and then with respect to* $u \in \Delta'$.

Proof. The argument is completely analogous to that in the proof of Theorem 1. We take the partition r of the (m-dimensional) rectangle Δ' into N^m congruent rectangles $\Delta'_1, \Delta'_2, \ldots$ with m-dimensional measures

$$h = |\Delta'_i|^m = h_1 \ldots h_m = \frac{(b_1 - a_1) \ldots (b_m - a_m)}{N^m}$$

and the partition ϱ of the $(n-m)$-dimensional rectangle Δ'' into N^{n-m} congruent rectangles $\Delta''_1, \Delta''_2, \ldots$ with $(n-m)$-dimensional measures

$$g = |\Delta''_j|^{n-m} = h_{m+1} \ldots h_n = \frac{(b_{m+1} - a_{m+1}) \ldots (b_n - a_n)}{N^{n-m}}$$

This results in the partition $r \times \varrho$

$$\Delta = \sum_i \sum_j \Delta'_i \times \Delta''_j$$

of the rectangle Δ. To this partition there correspond the upper and lower sums $\bar{S}_{r\times\varrho}(f) = hg \sum_i \sum_j M_{ij}$ and $\underline{S}_{r\times\varrho}(f) = hg \sum_i \sum_j m_{ij}$.

Next we consider a function $\Phi(u)$ defined as an arbitrary function satisfying the inequalities $\underline{I}(u) \leqslant \Phi(u) \leqslant \bar{I}(u)$ where $\underline{I}(u)$ and $\bar{I}(u)$ are the lower and upper integrals of $f(u, v)$ with respect to $v \in \Delta''$.

The further course of the proof is quite similar to that in the proof of the foregoing theorem with the replacement of $\xi_i \in [u_{i-1}, u_i]$ by $\xi_i \in \Delta'_i$.

Theorem 3. *Let the function* $f(x) = f(x_1, \ldots, x_n)$ *be not only integrable on the rectangle* Δ *but also satisfy the condition that, given any natural number* $k = 1, \ldots, n-1$ *and any admissible set of values* $(x_1, \ldots, x_k)$, *the function* f *is integrable on the projection* Δ^k *of* Δ *on the coordinate subspace of points* $u^k = (x_{k+1}, \ldots, x_n)$. *Then for the rectangle* $\Delta = \{a_j \leqslant x_j \leqslant b_j;\ j = 1, \ldots, n\}$

there holds the equality

$$\int_{\Delta} \cdots \int f\,dx_1 \ldots dx_n = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\,dx_n$$

where the integrals on the right-hand side are understood in the Riemann (one-fold) sense.

Indeed, applying consecutively the foregoing theorem we obtain

$$\int_{\Delta} f\,dx = \int_{a_1}^{b_1} dx_1 \int_{\Delta'} f(x_1, u^1)\,du^1 = \int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \int_{\Delta^2} f(x_1, x_2, u^2)\,du^2 = \ldots$$

$$\ldots = \int_{a_1}^{b_1} dx_1 \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\,dx_n \qquad (5)$$

An integral of the type of $\int_{a_1}^{b_1} dx_1 \int_{a_2}^{b_2} dx_2 \ldots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\,dx_n$ is called an *n-fold iterated* (or *repeated*) *integral*.

In the general case the reduction of a multiple integral to an n-fold iterated integral involving successive integrations with respect to each of variables is carried out in accordance with the lemma proved below.

Let Ω be a bounded set. We shall denote by e_1 its projection on the x_1-axis. In particular, if Ω is a domain then e_1 is an open interval; if Ω is the closure of a domain then e_1 is the closed interval $[a, b]$ where $a = \min\limits_{x \in \Omega} x_1$ and $b = \max\limits_{x \in \Omega} x_1$ $(x = (x_1, \ldots, x_n))$. We also denote by $\Omega_{x_1^0}$ the section of Ω by the plane $x_1 = x_1^0$, that is the set of points of the form $(x_1^0, x_2, \ldots, x_n) \in \Omega$.

Lemma. *There holds the equality*

$$\int_{\Omega} \cdots \int f(x_1, \ldots, x_n)\,dx_1 \ldots dx_n = \int_{e_1} dx_1 \int_{\Omega_{x_1}} f(x_1, x_2, \ldots, x_n)\,dx_2 \ldots dx_n \qquad (6)$$

It is always valid when f is bounded on Ω, e_1 is a measurable one-dimensional set and the integrals $\int_{\Omega} \ldots \int$ and $\int_{\Omega_{x_1}}$ (for any $x_1 \in e_1$) make sense.

Proof. Let us embed Ω in an n-dimensional rectangle $\Delta = [a_1, b_1] \times \Delta'$ where $\Delta' = \{a_j \leq x_j \leq b_j;\ j = 2, \ldots, n\}$. This is possible since the set Ω is measurable and therefore bounded.

Let us extend the function f from Ω to Δ by putting

$$\bar{f} = \begin{cases} f & \text{on } \Omega \\ 0 & \text{on } \Delta - \Omega \end{cases}$$

($\bar{f}$ denotes the extended function).

Now we can write (see the explanations below)

$$\int_{\Omega} f(x)\,dx = \int_{\Delta} \bar{f}(x)\,dx = \int_{a_1}^{b_1} dx_1 \int_{\Delta'} \bar{f}(x_1, x_2, \ldots, x_n)\,dx_2 \ldots dx_n =$$

$$= \int_{e_1} dx_1 \int_{\Omega_{x_1}} f(x_1, x_2, \ldots, x_n)\,dx_2 \ldots dx_n$$

The first equality in this chain holds by virtue of the fact that Ω and Δ are measurable sets and $f = 0$ on $\Delta - \Omega$.

The second equality follows from Theorem 2 for the function f is not only integrable on Ω but, as function of $(x_2, \ldots, x_n)$, it is also integrable on Ω_{x_1} for any fixed admissible values of x_1, and consequently it is integrable on Δ' as well because it vanishes outside Ω_{x_1}.

The third equality holds because e_1 is a measurable set and $f = 0$ for $x_1 \overline{\in} e_1$ and also for $x_1 \in e_1$ when $(x_2, \ldots, x_n) \overline{\in} \Omega_{x_1}$.

Example 1. The area S of the ellipse W specified by the relation $\frac{x^2}{a^2} + \frac{y^2}{b^2} \leqslant 1$ $(a, b > 0$; see Fig. 12.5) can be computed as follows (see the explanations below):

$$S = \iint_{W} 1\,dx\,dy = \int_{-a}^{a} dx \int_{-b\sqrt{1-(x^2/a^2)}}^{b\sqrt{1-(x^2/a^2)}} dy = \int_{-a}^{a} 2b\sqrt{1-\frac{x^2}{a^2}}\,dx =$$

$$= 4\frac{b}{a}\int_{0}^{a} \sqrt{a^2-x^2}\,dx = 2ab\left(\arcsin\frac{x}{a} + \frac{x}{a^2}\sqrt{a^2-x^2}\right)_0^a = \pi ab$$

Here the first equality follows from the fact that W is a measurable (in the two-dimensional sense) set since its boundary is a smooth curve.

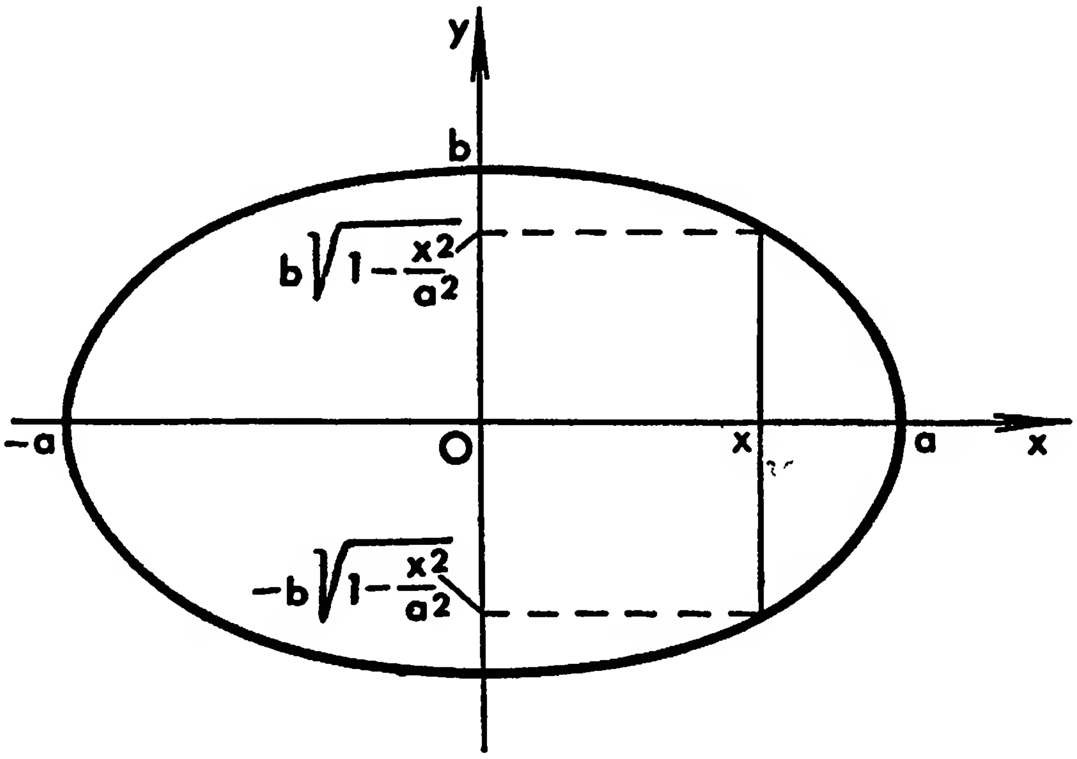

Fig. 12.5

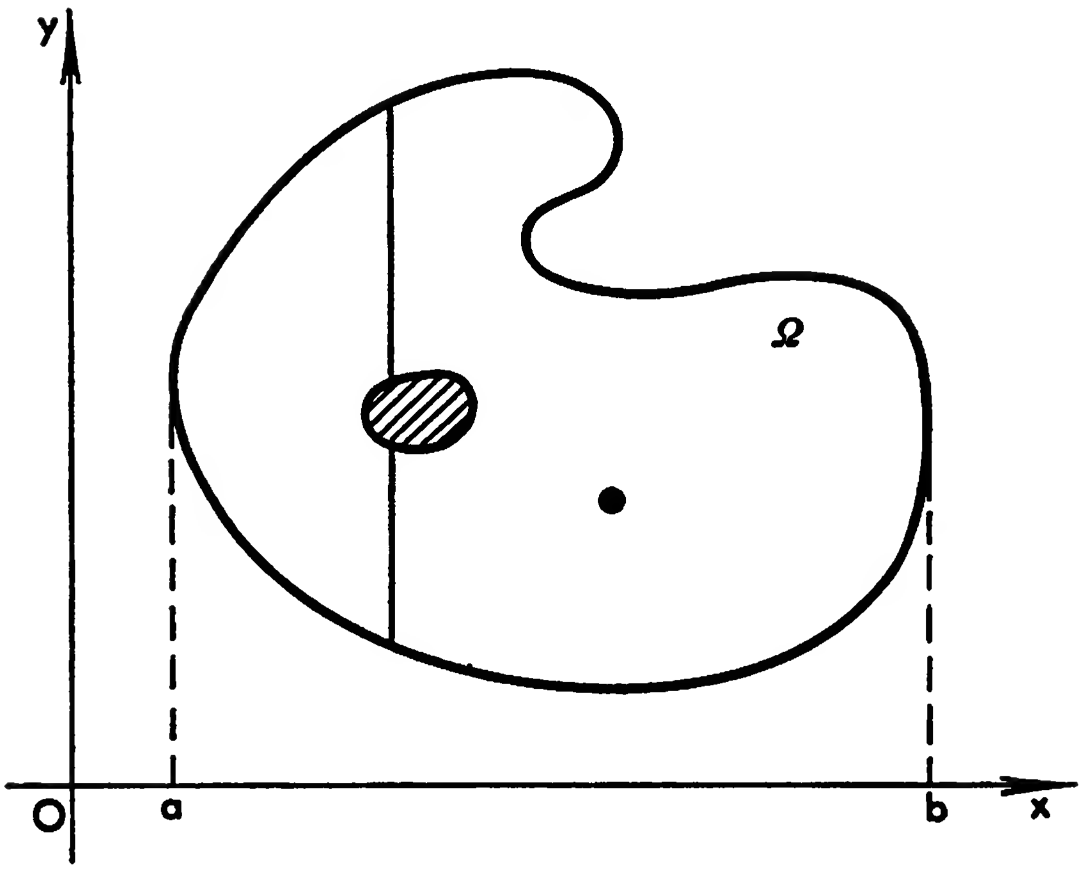

Fig. 12.6

The second equality is a consequence of the above lemma for the closed interval $[-a, a]$ is the measurable projection of W on the x-axis and the section W_x of the ellipse by the straight line parallel to the y-axis and passing through the point $x \in [a, b]$ is the closed interval $[-b\sqrt{1-(x^2/a^2)}, b\sqrt{1-(x^2/a^2)}]$, that is a measurable set in the one-dimensional sense, on which the function in question is equal to 1 and thus is integrable.

Example 2. In Fig. 12.6 we see a closed set Ω with boundary Γ consisting of two piecewise smooth closed contours and of a separate point. The set Ω is therefore measurable in the two-dimensional sense. Its projection on the x-axis is the closed interval $[a, b]$. Any section Ω_x of Ω by a line parallel to the y-axis and passing through the point $x \in [a, b]$ is a line segment or a system of two line segments or a system consisting of a finite number of points, all such sets being measurable in the one-dimensional sense. Therefore, if $f(x, y)$ is a continuous function on Ω it is integrable on Ω and also integrable over any such section Ω_x with respect to y. Consequently the above lemma applies to f, and we obtain

$$\iint\limits_{\Omega} f\,dx\,dy = \int\limits_{a}^{b} dx \int\limits_{\Omega_x} dy$$

Now we can apply this lemma to the inner integral on the right-hand side of (6).

Let $e_2(x_1)$ be the projection of the section Ω_{x_1} on the x_2-axis and $\Omega_{x_1x_2^0}$ be the section of (the $(n-1)$-dimensional set) Ω_{x_1} by the plane $x_2 = x_2^0$. Then we have

$$\int\limits_{\Omega} f(x)\,dx = \int\limits_{e_1} dx_1 \int\limits_{e_2(x_1)} dx_2 \int\limits_{\Omega_{x_1x_2}} f(x_1, x_2, x_3, \ldots, x_n)\,dx_3 \ldots dx_n$$

provided that all the sets $\Omega_{x_1x_2}$ $(x_2 \in e_2(x_1))$ are measurable in the $(n-2)$-dimensional sense and that $f(x_1, x_2, x_3, \ldots, x_n)$ as function of $(x_2, \ldots, x_n)$ is integrable on $\Omega_{x_1x_2}$.

Proceeding in this way we finally arrive at the formula

$$\int\limits_{\Omega} f\,dx = \int\limits_{e_1} dx_1 \int\limits_{e_2(x_1)} dx_2 \int\limits_{e_3(x_1,\,x_2)} dx_3 \ldots \int\limits_{e_n(x_1,\,\ldots,\,x_{n-1})} f(x_1, \ldots, x_n)\,dx_n \quad (7)$$

Here all the sets $e_1, e_2(x_1), \ldots, e_n(x_1, \ldots, x_{n-1})$ are one-dimensional and are supposed to be measurable; besides, we assume that the integral on the left-hand side of (7) and also all the inner integrals on the right-hand side exist.

Example 3. The volume $|\Omega|$ of the ellipsoid Ω determined by the inequality $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$ $(a, b, c > 0)$ can be computed as follows (see the explanations below):

$$\Omega = \iiint\limits_{\Omega} dx\,dy\,dz = \int\limits_{-a}^{a} dx \int\limits_{\Omega_x} dy\,dz =$$

$$= \int\limits_{-a}^{a} dx \int\limits_{-b\sqrt{1-(x^2/a^2)}}^{b\sqrt{1-(x^2/a^2)}} dy \int\limits_{-c\sqrt{1-(x^2/a^2)-(y^2/b^2)}}^{c\sqrt{1-(x^2/a^2)-(y^2/b^2)}} dz =$$

$$= \int\limits_{-a}^{a} dx \int\limits_{-b\sqrt{1-(x^2/a^2)}}^{b\sqrt{1-(x^2/a^2)}} 2c\sqrt{1-(x^2/a^2)-(y^2/b^2)}\,dy =$$

$$= bc \int\limits_{-a}^{a} \left[\left(1-\frac{x^2}{a^2}\right) \arcsin \frac{y}{b\sqrt{1-\frac{x^2}{a^2}}} + \frac{y}{b\left(1-\frac{x^2}{a^2}\right)} \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right]_{-b\sqrt{1-(x^2/a^2)}}^{b\sqrt{1-(x^2/a^2)}} dx =$$

$$= bc \int\limits_{-a}^{a} \pi\left(1-\frac{x^2}{a^2}\right) dx = \frac{4}{3}\pi abc$$

The set Ω is measurable since its boundary consists of two continuous surfaces

$$z = +c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \quad \text{and} \quad z = -c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}} \quad \left(\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1\right)$$

each of which is projectable in a one-to-one manner on a bounded closed set in the xy-plane.

The sections of Ω by the planes and straight lines parallel to the coordinate axes are also measurable and closed in the one- and the two-dimensional sense respectively; indeed, these sets, when they are nonempty, are ellipses or separate points for the section by the planes and are line segments or separate points for the sections by the straight lines.

Thus, the function 1 is integrable on Ω and on all indicated sections of Ω, whence follows that equality (7) is in fact applicable.

If a function $f(x, y)$ is bounded on a rectangle $\Delta = \{a \leqslant x \leqslant b, c \leqslant y \leqslant d\} = [a, b]\times[c, d]$ and if its points of discontinuity, provided there are such, form a set E of two-dimensional measure zero and, besides, $f(x, y)$ as function of y considered for any fixed x has points of discontinuity only on a set of one-dimensional measure zero, then, by virtue of Lebesgue's theorem, the conditions of Theorem 3 hold. Therefore we have the equality

$$\iint_{\Delta} f(x, y)\, dx\, dy = \int_a^b dx \int_c^b f(x, y)\, dy \tag{8}$$

where both integrals on the right-hand side (taken with respect to x and with respect to y) are understood in the Riemann sense.

In particular, these conditions hold when we deal with a function $f(x, y)$ defined and continuous on Δ except possibly on a finite or countable set of points.

If $f(x, y) = \varphi(x)\,\psi(y)$ $((x, y) \in \Delta)$ where the functions φ and ψ are bounded and continuous on the closed intervals $[a, b]$ and $[c, d]$, respectively, except possibly at points forming some sets E_1 and E_2 which are countable or, generally, are of one-dimensional measure zero, then the function $f(x, y)$ is bounded on Δ and can only have points of discontinuity belonging to the sets $E_1\times[c, d]$ and $[a, b]\times E_2$ which obviously are of (two-dimensional) measure zero. Therefore equality (6) holds for f; in this case it can be written in the form

$$\iint_{\Delta} \varphi(x)\,\psi(y)\, dx\, dy = \int_a^b dx \int_c^d \varphi(x)\,\psi(y)\, dy = \int_a^b \varphi(x)\, dx \int_c^d \psi(y)\, dy$$

The extension of these properties to many-dimensional integrals is carried out quite easily.

§ 12.13. Continuity of Integral Dependent on Parameter

Let us consider the integral

$$F(x) = F(x_1, \ldots, x_m) = \int_{\Omega}\ldots\int f(x_1, \ldots, x_m; y_1, \ldots, y_n)\, dy_1 \ldots dy_n =$$

$$= \int_{\Omega} f(x, y)\, dy \tag{1}$$

where Ω is a measurable set in the n-dimensional space of the points $y = (y_1, \ldots, y_n)$ and the function $f(x, y)$ is integrable with respect to y over Ω. Then integral (1) is a function $F(x)$ of the point x. Such an integral is spoken of as an *integral dependent on the parameters* $x_1, \ldots, x_m$.

The theorem below provides a sufficient test for the continuity of a function of the type of $F(x)$.

Theorem 1. *If a function* $f(x, y)$ *is continuous on a set of the form*

$$G\times\Omega \quad (x \in G, y \in \Omega) \tag{2}$$

lying in the $(n+m)$*-dimensional space of points* $(x,y) = (x_1, \ldots, x_m; y_1, \ldots, y_n)$ *where* G *and* Ω *are bounded closed sets belonging to the corresponding spaces of the points* x *and* y, *integral* (1) (*i.e. the function* $F(x)$) *is a continuous function of the point* $x \in G$.

Proof. Let us denote the modulus of continuity of the function f on set (2) by the symbol $\omega(\delta, f)$. This set is bounded and closed and the function f is continuous on it; consequently $\omega(\delta,f) \to 0\,(\delta \to 0)$. Hence for $x, x' \in G$, we have

$$|F(x')-F(x)| = \left|\int_\Omega [f(x', y)-f(x, y)]\, dy\right| \leqslant$$
$$\leqslant \int_\Omega |f(x', y)-f(x, y)|\, dy \leqslant \int_\Omega \omega(|x'-x|, f)\, dy =$$
$$= \omega(|x'-x|, f)\, |\Omega| \to 0 \qquad (x' \to x)$$

which proves the theorem.

Now let us consider an integral generalizing (1) in the case when y is a scalar (not vector) variable:

$$F(x) = F(x_1, \ldots, x_m) = \int_{\varphi(x_1, \ldots, x_m)}^{\psi(x_1, \ldots, x_m)} f(x_1, \ldots, x_m; y)\, dy =$$
$$= \int_{\varphi(x)}^{\psi(x)} f(x,y)\, dy \qquad (x \in G) \tag{3}$$

For an integral of type (3) we shall prove the following theorem.

Theorem 2. *If a function* $f(x, y)$ *is continuous on a set* H *of points* $(x, y) = (x_1, \ldots, x_m; y)$ *belonging to the* $(m+1)$*-dimensional space of the variables* $x_1, x_2, \ldots, x_m; y$ *such that the points of* H *satisfy the inequalities* $\varphi(x) \leqslant y \leqslant \psi(x)$ $(x \in G)$ *where* $\varphi(x)$ *and* $\psi(x)$ *are continuous functions defined on a bounded closed set* G *of points* $x = (x_1, \ldots, x_m)$ *belonging to the m-dimensional space of the variables* $x_1, \ldots, x_m$, *then the function* $F(x)$ *determined by formula* (3) *is continuous on* G.

Proof. First of all we note that if $\varphi(x)$ and $\psi(x)$ are constant functions, i.e. $\varphi(x) = a$ and $\psi(x) = b$ $(a, b = \text{const})$, then

$$F(x) = \int_a^b f(x, y)\, dy \tag{4}$$

where $f(x, y)$ is continuous on $H = G\times[a, b]$. For this case the continuity of F on G was already proved in Theorem 1.

Let us proceed to the general case. Since $f(x, y)$ is continuous on the bounded closed set H there is a constant M such that $|f(x, y)| \leqslant M$ on H.

Let $x^0 \in G$ and let us assume at present that $\varphi(x^0) < \psi(x^0)$. On setting an arbitrary number $\varepsilon > 0$ such that $\varepsilon < 1/2\,[\psi(x^0)-\varphi(x^0)]$ we can find, by the continuity of φ and ψ at the point x^0, a number $\delta > 0$ such that

$$\varphi(x^0)-\varepsilon < \varphi(x) < \varphi(x^0)+\varepsilon$$

and

$$\psi(x^0)-\varepsilon < \psi(x) < \psi(x^0)+\varepsilon$$

for all x belonging to G and satisfying the inequality $|x-x^0| < \delta$. For such x' s we can represent F as the sum

$$F(x) = F_1(x)+F_2(x)$$

where

$$F_1(x) = \int\limits_{\varphi(x)}^{\varphi(x^0)+\varepsilon} f(x, y)\, dy + \int\limits_{\psi(x^0)-\varepsilon}^{\psi(x)} f(x, y)\, dy$$

and

$$F_2(x) = \int\limits_{\varphi(x^0)-\varepsilon}^{\psi(x^0)-\varepsilon} f(x, y)\, dy$$

The term F_1 satisfies the inequality

$$|F_1(x)| \leqslant 2M\varepsilon+2M\varepsilon = 4M\varepsilon$$

for the indicated values of x, that is it can be made arbitrarily small for a sufficiently small δ. The second term F_2 is continuous at the point x^0, which follows from the remark concerning integral (4) if we put $a = \varphi(x^0)+\varepsilon$ and $b = \psi(x^0)-\varepsilon$. Therefore the function F is also continuous at the point x^0 (see Lemma 1, § 11.7).

Finally, in the case $\varphi(x^0) = \psi(x^0)$ we have

$$|F(x)-F(x^0)| = \left| \int\limits_{\varphi(x)}^{\psi(x)} f(x, y)\, dy \right| \leqslant |\psi(x)-\varphi(x)|M \to$$

$$\to |\psi(x^0)-\varphi(x^0)|\,M = 0 \qquad (x \to x^0)$$

This completes the proof.

It should be noted that if the conditions of Theorem 2 hold we have the equalities

$$\iint\limits_{H} f(x, y)\, dx\, dy = \int\limits_{G} dx \int\limits_{\varphi(x)}^{\psi(x)} f(x, y)\, dy = \int\limits_{G} F(x)\, dx$$

Indeed, the leftmost integral exists since H is a bounded closed measurable set and f is continuous on H. The first equality follows from the lemma proved in § 12.12 (6); the inner integral (with respect to y) of the continuous function $f(x, y)$ in the middle term exists for any $x \in G$. The second equality is obtained if we substitute the expression on the right-hand side of (3) for $F(x)$. If we only knew that the conditions of the above-mentioned lemma hold it would only follow that $F(x)$ is integrable (with respect to x) on G. In the case under consideration we have more information concerning the function $F(x)$, namely, we know that *it is continuous on G*.

Example. Let us consider a continuous function $f(x, y, z)$ defined in unit ball ω. The integral of the function over that ball can be written as

$$\int\limits_{\omega} f\,d\omega = \int\limits_{-1}^{+1} dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z)\,dz$$

The inner integral

$$F(x, y) = \int\limits_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x, y, z)\,dz$$

is a function F on (x, y) defined in the circle δ determined by the inequality $x^2+y^2 \leqslant 1$. It is continuous on σ. Indeed, the function f is continuous in the closed ball $\bar{\omega}$ and the surfaces $z = -\sqrt{1-x^2-y^2}$ and $z = \sqrt{1-x^2-y^2}$ $(x^2+y^2 \leqslant 1)$ bounding the ball are described by functions continuous in the circle σ. Therefore the continuity of F on σ follows from the above theorem. Hence we have

$$\int\limits_{\omega} f\,d\omega = \int\limits_{-1}^{1} dx \left(\int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} F(x, y)\,dy \right)$$

The integral

$$\Phi(x) = \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} F(x, y)\,dy$$

is in its turn a continuous function of $x \in [-1, +1]$, which follows from the same theorem. Indeed, $F(x, y)$ is continuous in the circle σ which is a bounded closed set of points (x, y), and the boundary curves $y = -\sqrt{1-x^2}$ and $y = \sqrt{1-x^2}$ $(-1 \leqslant x \leqslant 1)$ of σ are continuous. Consequently, by Theorem 2, $\Phi(x)$ is continuous on $[-1, +1]$.

§ 12.14. Geometrical Interpretation of the Sign of a Determinant

Let us take a system of rectangular coordinates (x_1, x_2) in the plane (see Figs. 12.7 and 12.8).

For definiteness, we shall suppose that the positive half-axis x_2 is obtained

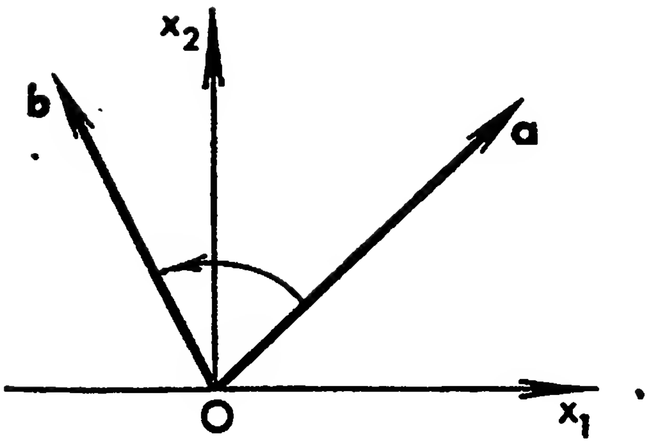

Fig. 12.7

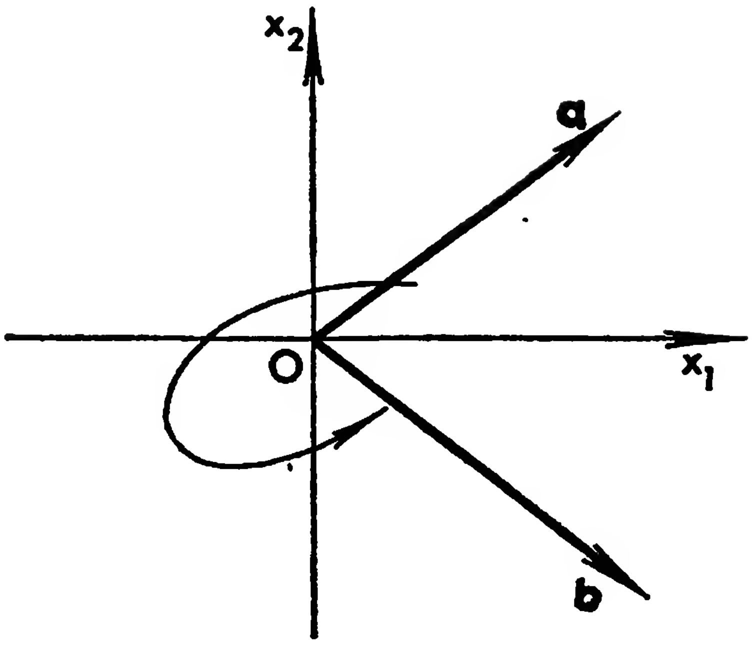

Fig. 12.8

from the positive half-axis x_1 by rotating the latter through an angle of 90° in counterclockwise direction. Let us also consider two nonzero vectors $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ issued from the origin such that the determinant formed of their components is different from zero:

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0 \tag{1}$$

If the determinant Δ is positive ($\Delta > 0$) the direction of the vector $\boldsymbol{b}$ is obtained from that of the vector $\boldsymbol{a}$ by turning the latter in the counterclockwise direction through an angle less than π; if $\Delta < 0$ it should be turned through an angle greater than π. Indeed, without violating generality, we can assume that $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors: $\boldsymbol{a} = (\cos\alpha, \sin\alpha)$ and $\boldsymbol{b} = (\cos\beta, \sin\beta)$. Then $\Delta = \sin(\beta-\alpha)$ whence $\Delta > 0$ for $0 < \beta-\alpha < \pi$ and $\Delta < 0$ for $\pi < \beta-\alpha < 2\pi$.

Now let us consider three vectors $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$ and $\boldsymbol{c} = (c_1, c_2, c_3)$, in the three-dimensional space with rectangular coordinates (x_1, x_2, x_3) (see Fig. 12.9), such that the determinant formed of their compo-

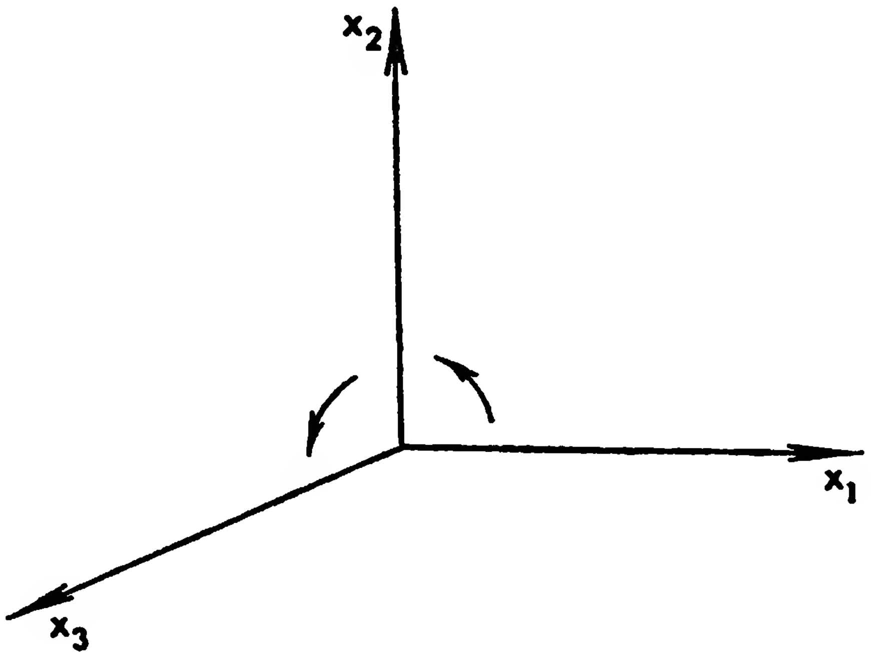

Fig. 12.9

nents is different from zero:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

The similar determinant formed for the unit vectors $\boldsymbol{i}_1 = (1, 0, 0)$, $\boldsymbol{i}_2 = (0, 1, 0)$ and $\boldsymbol{i}_3 = (0, 0, 1)$ of the axes Ox_1, Ox_2 and Ox_3 is positive:

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

If $\Delta > 0$ then it is possible to construct three continuous vector functions

$$\boldsymbol{\alpha}(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$$
$$\boldsymbol{\beta}(t) = (\beta_1(t), \beta_2(t), \beta_3(t))$$

and

$$\boldsymbol{\gamma}(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t))$$

defined on the closed interval $0 \leqslant t \leqslant 1$ such that the conditions

$$\boldsymbol{\alpha}(0) = \boldsymbol{a},\ \boldsymbol{\beta}(0) = \boldsymbol{b},\ \boldsymbol{\gamma}(0) = \boldsymbol{c},\ \boldsymbol{\alpha}(1) = \boldsymbol{i}_1,\ \boldsymbol{\beta}(1) = \boldsymbol{i}_2,\ \boldsymbol{\gamma}(1) = \boldsymbol{i}_3$$

are satisfied and the determinant of the components of these vectors is positive for any $t \in [0, 1]$:

$$\Delta(t) = \begin{vmatrix} \alpha_1(t) & \alpha_2(t) & \alpha_3(t) \\ \beta_1(t) & \beta_2(t) & \beta_3(t) \\ \gamma_1(t) & \gamma_2(t) & \gamma_3(t) \end{vmatrix} > 0 \qquad (2)$$

If $\Delta < 0$ it is impossible to construct three continuous vector functions possessing the indicated properties and such that $\Delta(t) \neq 0$ for $t \in [0, 1]$ and $\Delta(1) = 1$.

In the case $\Delta > 0$ the ordered triple of vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is said to be oriented like the triple $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ while in the case $\Delta < 0$ the triple $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ is said to have the orientation opposite to that of the triple $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$.

A similar characteristic (of the first and second cases) can be given for pairs of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in the plane (see § 13.8).

Let us prove the assertion stated. It is evident that for the interval $[0, 1/4]$ of the variation of the parameter t we can choose the vectors $\boldsymbol{\alpha}(t)$, $\boldsymbol{\beta}(t)$ and $\boldsymbol{\gamma}(t)$ so that they coincide with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for $t = 0$ and retain the directions of the given vectors $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$, respectively, for $t \in [0, 1/4]$ while their lengths vary continuously and monotonically so that $|\boldsymbol{\alpha}(1/4)| = |\boldsymbol{\beta}(1/4)| = |\boldsymbol{\gamma}(1/4)| = 1$. Next we keep the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ constant on the interval $t \in [1/4, 1/2]$ $(\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(1/4), \boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(1/4))$ while the vector $\boldsymbol{\beta}$ is rotated continuously and monotonically in the plane L of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ (or, which is the same, in the plane of the vectors $\boldsymbol{\alpha}(1/4)$ and $\boldsymbol{\beta}(1/4)$) so that for the value $t = 1/2$ of the parameter the vector $\beta(1/2)$ is perpendicular to $\alpha(1/4) = \alpha(1/2)$. Let us consider the unit vector $\boldsymbol{d}$ perpendicular to L whose terminus lies on the same side of L as the terminus of $\gamma(1/2)$. Now let us keep the vectors α and β invariable for the interval $t \in [1/2, 3/4]$ and vary only the vector γ, namely rotate it monotonically and continuously in the plane of the vectors $\gamma(1/2)$ and $\boldsymbol{d}$ so that it coincides with $\boldsymbol{d}$ for $t = 3/4$ $(\gamma(3/4) = \boldsymbol{d})$. Now we have the triple of vectors $\alpha(3/4)$, $\beta(3/4)$ and $\gamma(3/4)$ which are mutually perpendicular and have unit lengths. Finally, for the interval $[3/4, 1]$ of variation of the parameter t we regard the triple α, β, γ as a rigid tetrahedron which is rotated as a rigid body so that for $t = 1$ we have

$$\alpha(1) = \boldsymbol{i}_1,\ \beta(1) = \boldsymbol{i}_2,\ \gamma(1) = \pm \boldsymbol{i}_3 \tag{3}$$

If we have $\Delta(0) > 0$ at the beginning of the above process then we shall have $\Delta(t) > 0$ for all the values of t belonging to $[0, 1]$ because this process is such that the vectors $\alpha(t)$, $\beta(t)$ and $\gamma(t)$ are not coplanar for any value $t \in [0, 1]$ and thus $\Delta(t) \neq 0$, whence it follows, by the continuity of the function $\Delta(t)$, that $\Delta(t) > 0$ for all $t \in [0, 1]$. Consequently, if $\Delta(0) > 0$ then there must be in fact the sign "+" in the third term of (3) (i.e. $\gamma(1) = \boldsymbol{i}_3$). It is also evident that if $\Delta(0) < 0$ then $\Delta(1) < 0$ and $\gamma(1) = -\boldsymbol{i}_3$.

§ 12.15. Change of Variables in Multiple Integral. Simplest Case

We shall investigate how the integral

$$\iint_{\Omega'} f(x_1', x_2')\, dx_1'\, dx_2' \tag{1}$$

is transformed when a change of variables of the form

$$x_1' = ax_1 + bx_2, \quad x_2' = cx_1 + dx_2 \tag{2}$$

is made in it on condition that

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$$

Let us suppose that Ω' is a domain with a continuous piecewise smooth boundary curve Γ' (Fig. 12.10).

The transformation inverse to (2) specifies a mapping of Ω' on a domain Ω in the x_1x_2-plane having a piecewise smooth boundary Γ (Fig. 12.11) and determines the function

$$F(x_1, x_2) = f(ax_1 + bx_2, cx_1 + dx_2) \qquad ((x_1, x_2) \in \Omega)$$

defined on Ω.

Let us construct a rectangular network in the x_1x_2-plane with squares Δ having sides of length h. Equations (2) determine a transformation of this network into a network lying in the $x_1'x_2'$-plane which is oblique-angled in

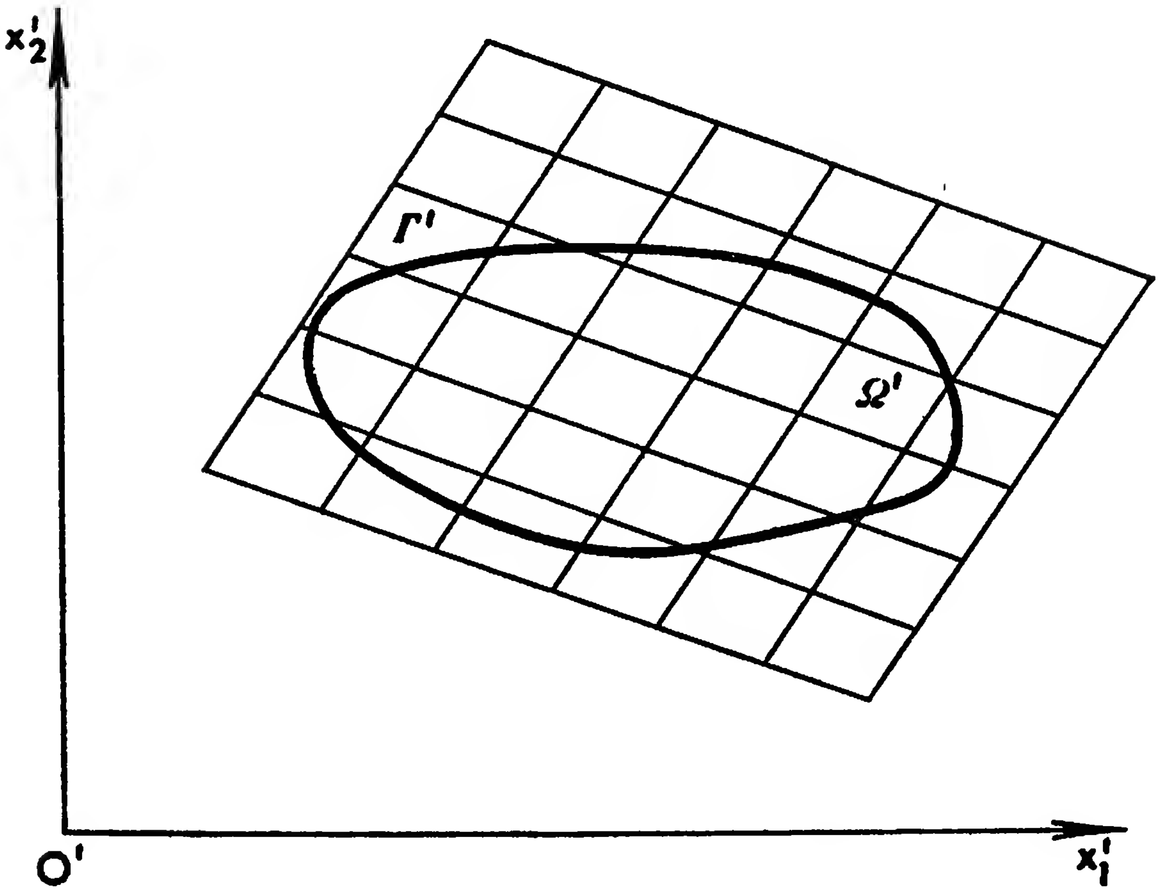

Fig. 12.10

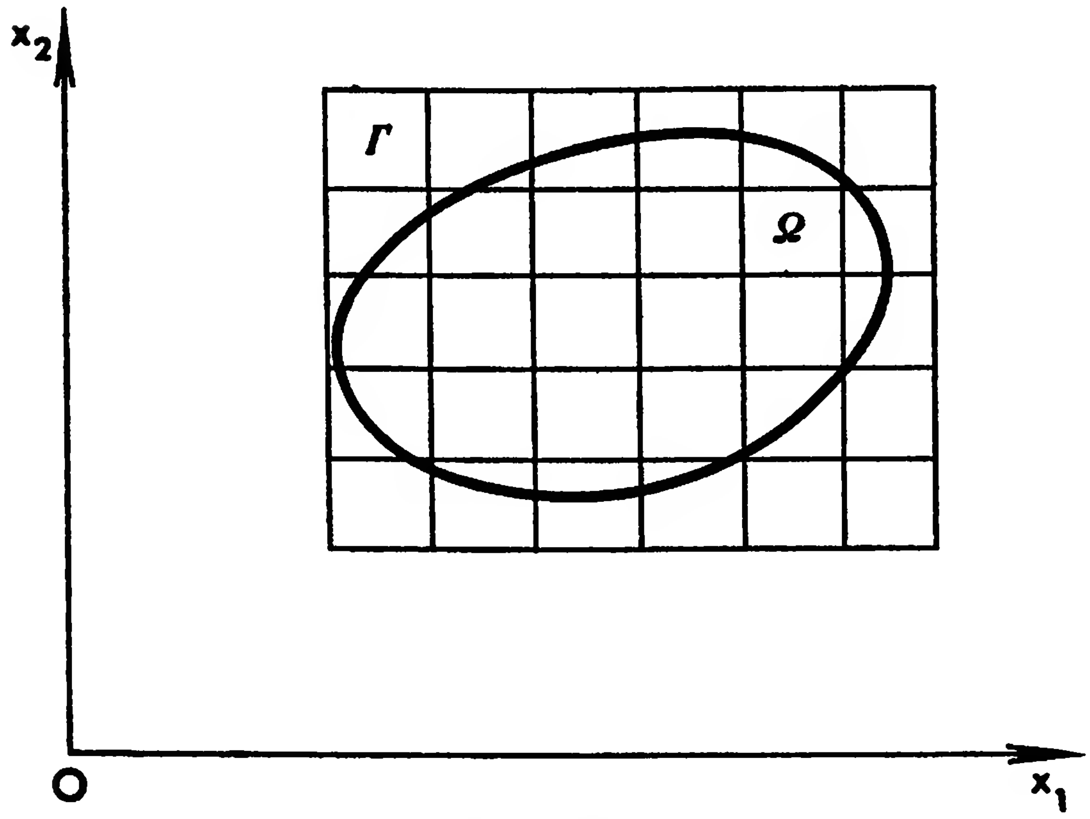

Fig. 12.11

the general case and partitions the $x_1'x_2'$-plane into congruent parallelograms (the images of Δ's) with areas

$$|\Delta'| = |D|\,|\Delta| = |D|h^2, \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{3}$$

Thus we obtain some partitions ϱ and ϱ' of the domains Ω and Ω', respectively, specified by these networks.

For the corresponding integral sums we can write

$$S_{\varrho'}(f) = \sum f(x_1', x_2')\,|\Delta'| = \sum F(x_1, x_2)\,|D|\,|\Delta| =$$
$$= S_\varrho(|D|F) \qquad ((x_1, x_2) \in \Delta) \tag{4}$$

We extend the second sum in (4) only over the entire squares $\Delta \subset \Omega$ and the first sum only over the entire parallelograms Δ' corresponding to them. On passing to the limit in (4) as $h \to 0$ we arrive at the formula

$$\iint\limits_{\Omega'} f(x_1', x_2')\,dx_1'\,dx_2' = \iint\limits_{\Omega} F(x_1, x_2)|D|\,dx_1\,dx_2 =$$

$$= |D| \iint\limits_{\Omega} F(x_1, x_2)\,dx_1\,dx_2 \tag{5}$$

In this argument we can assume that the function f is continuous on $\bar{\Omega}'$; then the function F is continuous on $\bar{\Omega}$ and both integrals in (5) exist. The fact that equality (5) holds was proved above. But it also suffices to suppose that f is integrable on Ω'; then the first sum in (4) tends to a finite limit as $d(\Delta') \to 0$, which automatically implies the existence of the limit of the second sum equal to the former when $d(\Delta) \to 0$, that is the existence of the second integral in (5) equal to the first one.

In the next section we shall derive and prove a more general formula for change of variables in multiple integrals.

§ 12.16. Change of Variables in Multiple Integral. General Case

Theorem 1. *Let Ω be a measurable domain in the n-dimensional space $R = R_n$ of the points $x = (x_1, \ldots, x_n)$ and Ω' be a measurable domain in another n-dimensional space $R' = R_n'$ of the points $x' = (x_1', \ldots, x_n')$.*

We shall suppose that transformation formulas

$$x_j' = \varphi_j(x) = \varphi_j(x_1, \ldots, x_n) \qquad (j = 1, \ldots, n; \quad x \in \Omega) \tag{1}$$

specify a mapping of $\bar{\Omega}$ on $\bar{\Omega}'$ under which the points $x \in \bar{\Omega}$ go into the points $x' \in \bar{\Omega}'$. This mapping will be written in the shortened form as

$$x' = Ax \tag{1'}$$

Let us suppose that the operator A possesses the following properties:

(i) *The mapping of Ω on Ω'* under the operator A is one-to-one:*

$$\Omega \rightleftarrows \Omega' \tag{2}$$

(we do not require that the operator A should specify a one-to-one correspondence between the points of the boundaries of Ω and Ω').

(ii) *The functions $\varphi_j(x)$ are continuous and possess continuous partial derivatives of the first order on $\bar{\Omega}$. The Jacobian** of the functions $\varphi_j(x)$ will be*

---

* The condition that Ω' is a domain may not be stated initially because it follows from the continuity of Ax on Ω and from (2) (see § 12.20, Theorem 3).

** We do not impose any conditions on the sign of $D(x)$. But the conditions of the theorem do in fact automatically imply that the inequality $D(x) \geqslant 0$ or $D(x) \leqslant 0$ is fulfilled throughout Ω (see § 12.21).

denoted as

$$D(x) = \begin{vmatrix} \frac{\partial \varphi_1}{\partial x_1} \cdots \frac{\partial \varphi_1}{\partial x_n} \\ \cdots\cdots\cdots \\ \frac{\partial \varphi_n}{\partial x_1} \cdots \frac{\partial \varphi_n}{\partial x_n} \end{vmatrix} \tag{3}$$

Further, let

$$F(x') = f(x_1', \ldots, x_n')$$

be an integrable function defined on Ω'; *substitution* (1) *transforms it into the function*

$$F(x) = f(Ax) = f[\varphi_1(x), \ldots, \varphi_n(x)] \quad (x \in \Omega) \tag{4}$$

Under these conditions there holds the formula

$$\int_{\Omega'} f(x')\, dx' = \int_{\Omega} F(x)|D(x)|\, dx \tag{5}$$

for change of variables in the multiple integral which asserts that the integral on the right-hand side of (5) *exists and is equal to the integral on its left-hand side.*

In particular, if the function $f(x')$ *is continuous on* $\bar{\Omega}'$ *then* F *is also a continuous function on* $\bar{\Omega}$ *and both integrals in* (5) *are sure to exist. In this case the meaning of formula* (5) *is that these integrals are equal.*

The most intricate part of the theorem can be stated separately as the lemma below whose proof will be given in § 12.17. Here we shall state the lemma and show immediately that equality (5) follows from it.

Lemma. 1 *Let the conditions of Theorem 1 hold and let* $\Delta \subset \Omega$ *be an arbitrary cube with edge* h *and* $\Delta' = A(\Delta)$ *be its image (lying in* Ω') *under the mapping* A. *Then there holds the relation*

$$|\Delta'| = |D(x)|\, |\Delta| + O(h^n\omega(h)) \qquad (|O(h^n\omega(h))| \leqslant C|h^n\omega(h)|) \tag{6}$$

where $D(x)$ *is the value of the Jacobian* D *at one of the points* $x \in \Delta$,

$$\omega(h) = \sup_{i, j=1, \ldots, n} \omega_{ij}(h)$$

and

$$\omega_{ij}(h) = \sup_{\substack{|x-y|<h \\ x,\, y \in \Omega}} \left| \frac{\partial \varphi_i}{\partial x_j}(x) - \frac{\partial \varphi_i}{\partial x_j}(y) \right| \tag{7}$$

(the latter expression is the modulus of continuity of $\frac{\partial \varphi_i}{\partial x_j}$ *on* $\bar{\Omega}$); *the constant* C *in* (6) *is independent of* Δ, *that is of* h *and of the location of* Δ *within* Ω.

It is important to note that since, by the condition of the theorem, the partial derivatives $\frac{\partial \varphi_i}{\partial x_j}$ are continuous on the bounded closed set $\bar{\Omega}$, their moduli of continuity tend to zero as $h \to 0$ $(\omega_{ij}(h) \to 0$ for $h \to 0)$, and therefore we also have

$$\omega(h) \to 0 \qquad (h \to 0) \tag{8}$$

Consequently the remainder term in formula (6) satisfies the inequality

$$|O(h^n\omega(h))| \leqslant Ch^n\omega(h) = o(h^n) \qquad (h \to 0) \tag{9}$$

which holds uniformly with respect to $x \in \bar{\Omega}$ because the right-hand member of (9) does not depend on $x \in \bar{\Omega}$. As to the first term on the right-hand side of (6), it can be written as

$$|D(x)|\,|\Delta| = |D(x)|\,h^n, \quad \text{i.e.} \quad |\Delta'| = |D(x)|\,h^n + o(h^n) \qquad (h \to 0)$$

In particular, this equality shows that, given any $x \in \Omega$, there holds the relation

$$\lim_{h\to 0} \frac{|\Delta'|}{|\Delta|} = |D(x)| \qquad (x \in \Omega) \tag{10}$$

which means that $|D(x)|$ *is equal, to within an infinitesimal* $o(1)$ $(h \to 0)$, *to the magnification coefficient of an elementary volume in the vicinity of the point* x *under the transformation specified by the operator* A.

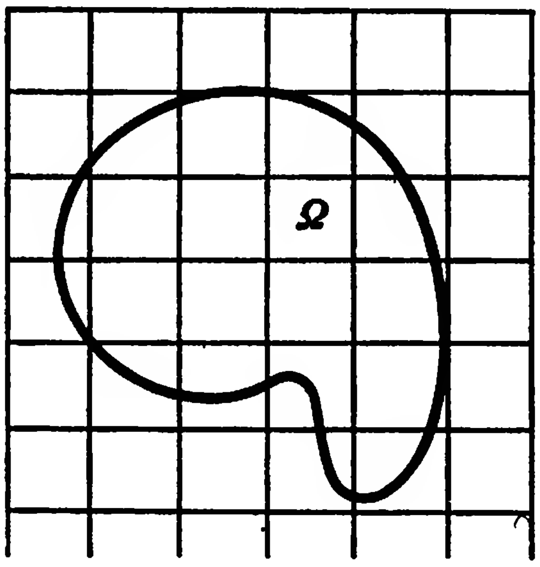

Fig. 12.12

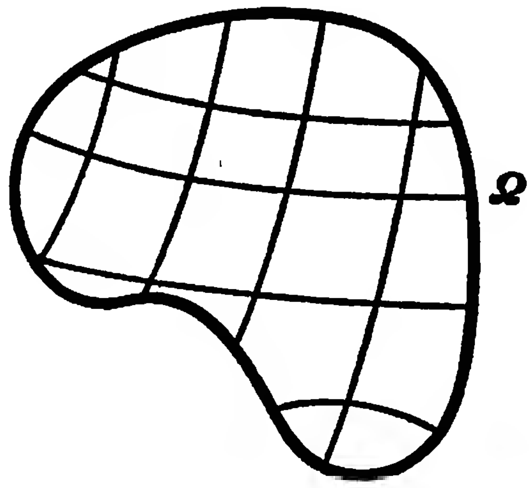

Fig. 12.13

Let us break up Ω with the aid of a network consisting of cubes with edges of length h (Fig. 12.12). The part of the network contained in Ω goes, under the mapping A, into a network of curvilinear surfaces dividing Ω' into measurable parts (Fig. 12.13); in this connection see Theorem 3, § 12.5.

Let us denote by Δ the entire cubes of the network which enter into Ω together with their boundaries. Under the mapping A the interior of Δ goes into the interior of Δ' and the boundary of Δ into the boundary of Δ' (this assertion requires a formal proof; see § 12.20, Theorem 3 and the remark to this theorem). If $h \to 0$ then the maximum diameter of the subsets forming the corresponding partition of Ω' tends to zero because the functions $\varphi_j(x)$ in transformation (1) are uniformly continuous on $\bar{\Omega}$.

According to Lemma 1, we have

$$\sum f(x')|\Delta'| = \sum F(x)\big(|D(x)|\,|\Delta| + O(\omega(h)h^n)\big) \tag{11}$$

where $\Delta' = A(\Delta)$, the sum extends over all Δ's, $f(x')$, $F(x)$ and $D(x)$ dentoe, respectively, the values of f, F and D at one of the points of Δ' or Δ. On passing to the limit in (11) as $h \to 0$ we arrive at equality (5). Indeed, the constant C (see (6)) is one and the same for all the summands on the right-hand side of (11), and therefore, taking into account the boundedness of F on Ω $(|F| < K)$, we obtain

$$\left|\sum F(x)\,O(\omega(h)h^n)\right| \le KC\omega(h)\sum|\Delta| \le KC\omega(h)|\Omega| \to 0 \qquad (h \to 0)$$

Further, by the integrability of f on Ω', we have

$$\lim_{h\to 0}\sum f(x')|\Delta'| = \lim_{\max|\Delta'|\to 0}\sum f(x')|\Delta'| = \int_{\Omega'} f(x')\,dx'$$

Consequently there exists the limit

$$\lim_{h\to 0}\sum F(x)\,|D(x)|\,|\Delta| = \int_{\Omega} F(x)\,|D(x)|\,dx$$

equal to the integral on the right-hand side of the above relation. Hence, the integral of $F(x)|D(x)|$ over Ω exists. We have thus shown that formula (5) follows from Lemma 1.

§ 12.17. Proof of Lemma 1, § 12.16

We shall carry out the proof of Lemma 1 for the case of dimension $n = 2$:

$$x_i' = \varphi_i(x_1, x_2) \qquad (i = 1, 2) \tag{1}$$

From the course of the proof and from the remarks it will readily be seen that the same argument can be applied in a completely analogous way to the n-dimensional case.

Thus, let $\Delta = \{x_i^0 < x_i < x_i^0+h;\ i = 1, 2\}$*; this set is shown as the square PA_1CA_2 in Fig. 12.4. Let Δ' (the image of Δ) be the curvilinear parallelogram $P'A_1'C'A_2'$ (see Fig. 12.15). The set Δ' is a domain whose boundary γ' is the image under A of the boundary γ of the square Δ (see § 12.20, Theorem 3). For instance, the side of Δ described by the equation $x_2 = x_2^0$ goes into its image which is the curve

$$x_1' = \varphi_1(x_1, x_2^0), \quad x_2' = \varphi_2(x_1, x_2^0) \qquad (x_1^0 \le x_1 \le x_1^0+h)$$

specified by the continuously differentiable functions $\varphi_1(x_1, x_2^0)$ and $\varphi_2(x_1, x_2^0)$ of the parameter x_1.** We do not call this curve smooth because it is not

---

* In the n-dimensional case $\Delta = \{x_i^0 < x_i < x_i^0+h;\ i = 1, \ldots, n\}$.

** In the n-dimensional case the part of the boundary of Δ' corresponding to the face $x_j = x_j^0$ is determined by the equations $x_i' = \varphi_i(x_1, \ldots, x_{j-1}, x_j^0, x_{j+1}, \ldots, x_n)$ $(i = 1, \ldots, n)$.

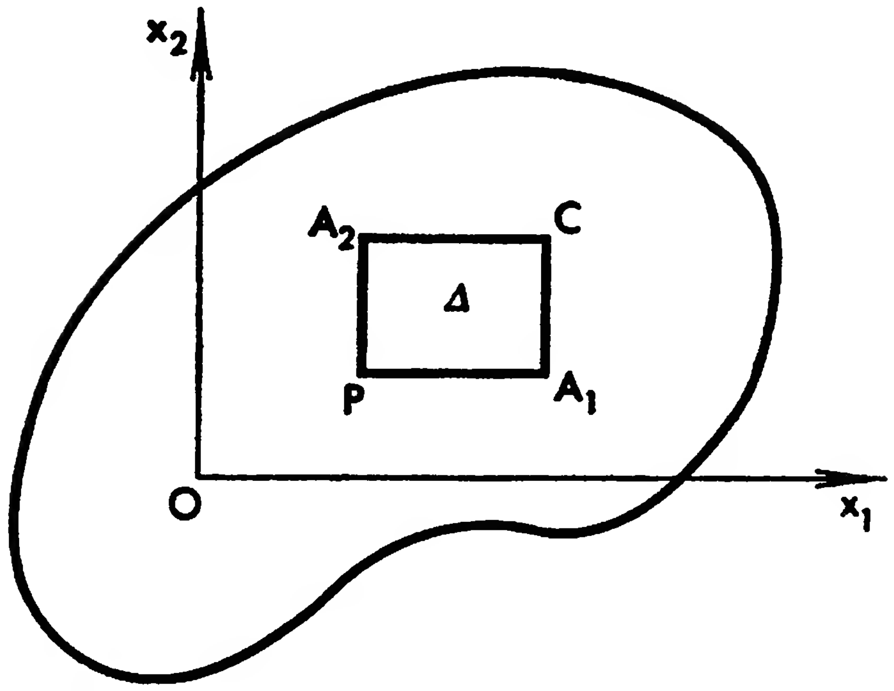

Fig. 12.14

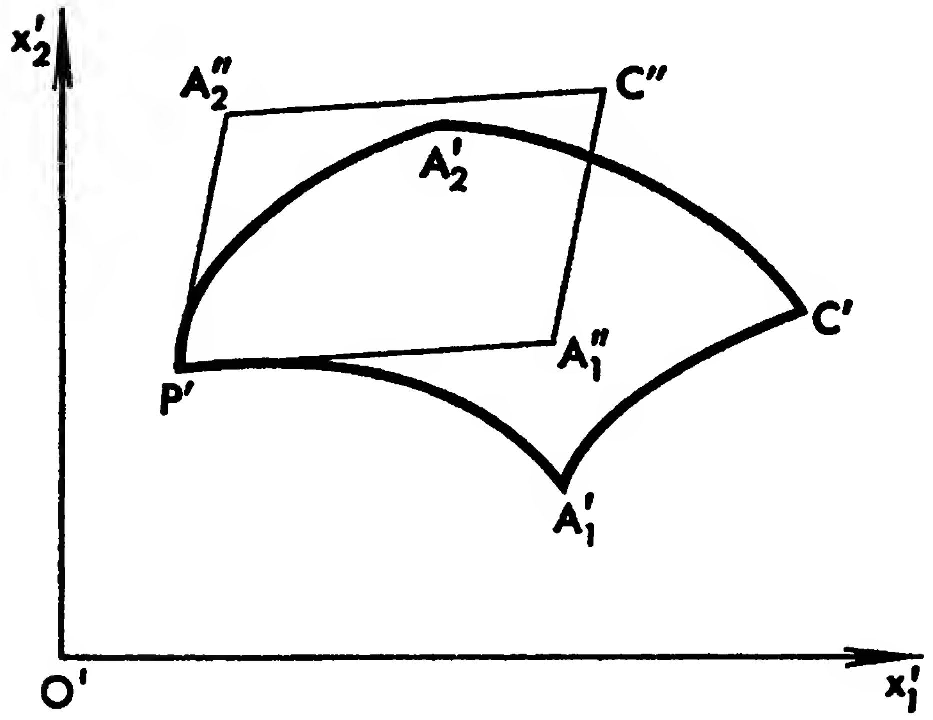

Fig. 12.15

excluded that the partial derivatives $\frac{\partial\varphi_1}{\partial x_1}(x_1, x_2^0)$ and $\frac{\partial\varphi_2}{\partial x_1}(x_1, x_2^0)$ may simultaneously vanish for some value of x_1. But nevertheless, it is of measure zero (see Theorem 3, § 12.5).

We have to prove the equality

$$|\Delta'| = |D(x^0)|\,|\Delta| + O(h^2\omega(h)) \qquad \big(|O(h^2\omega(h))| \leqslant C\,|h^2\omega(h)|\big) \tag{2}$$

where the constant C is independent of $x^0 \in \Omega$. This independence means that equality (2) remains valid when x^0 is replaced by an arbitrary point $x \in \Delta$.*

Since the functions φ_i are continuously differentiable we have

$$x_i' = x_i^{0\prime} + \left(\frac{\partial\varphi_i}{\partial x_1}\right)_1 (x_1 - x_1^0) + \left(\frac{\partial\varphi_i}{\partial x_2}\right)_1 (x_2 - x_2^0) \qquad (i = 1, 2;\; x \in \Delta) \tag{1'}$$

where $(\;)_1$ denotes the result of the substitution of a point $x = (x_1, x_2) \in \Delta$ into $(\;)$.

---

[*] By virtue of § 12.16 (7), we have $||D(x)| - |D(x^0)|| \leqslant |D(x) - D(x^0)| \leqslant |C\omega(h\sqrt{2})| \leqslant 2C\omega(h)$.

Together with mapping (1) which can also be written in equivalent form (1′) we shall consider the linear mapping $\boldsymbol{x}'' = A^*\boldsymbol{x}$ specified by the relations

$$x_i'' = x_i^{0\prime} + \left(\frac{\partial\varphi_i}{\partial x_1}\right)_0 (x_1 - x_1^0) + \left(\frac{\partial\varphi_i}{\partial x_2}\right)_0 (x_2 - x_2^0) \qquad (i = 1, 2) \tag{3}$$

with constant coefficients $\left(\frac{\partial\varphi_i}{\partial x_j}\right)_0$ $\left(\left(\frac{\partial\varphi_i}{\partial x_j}\right)_0\right.$ is the result of the substitution of the point $\boldsymbol{x}^0$ into $\left.\frac{\partial\varphi_i}{\partial x_j}\right)$. Under mapping (3) the set Δ goes into a parallelogram Δ'' with a boundary γ''. The parallelogram can be regarded as being constructed on the vectors

$$P'A_i'' = \left(h\left(\frac{\partial\varphi_1}{\partial x_i}\right)_0, \; h\left(\frac{\partial\varphi_2}{\partial x_i}\right)_0\right) \qquad (i = 1, 2)$$

(as sides) issued from the point $\boldsymbol{x}^0$. Their lengths are

$$|P'A_i''| = ha_i, \quad a_i = \sqrt{\left(\frac{\partial\varphi_1}{\partial x_i}\right)_0^2 + \left(\frac{\partial\varphi_2}{\partial x_i}\right)_0^2} \qquad (i = 1, 2) \tag{4}$$

Let us estimate the distance between the points $\boldsymbol{x}'$ and $\boldsymbol{x}''$ corresponding to one and the same point $\boldsymbol{x} \in \Delta$. From the equality

$$x_i' - x_i'' = \left[\left(\frac{\partial\varphi_i}{\partial x_1}\right)_1 - \left(\frac{\partial\varphi_i}{\partial x_1}\right)_0\right](x_1 - x_1^0) + \\ + \left[\left(\frac{\partial\varphi_i}{\partial x_2}\right)_1 - \left(\frac{\partial\varphi_i}{\partial x_2}\right)_0\right](x_2 - x_2^0) \qquad (i = 1, 2)$$

we obtain (see § 7.10 (10))*

$$|x_i'' - x_i'| \leqslant \omega(\sqrt{2}h)h + \omega(\sqrt{2}h)h \leqslant 4\omega(h)h \tag{5}$$

and

$$|x'' - x'| = \sqrt{(x_1'' - x_1')^2 + (x_2'' - x_2')^2} \leqslant 6\omega(h)h = \lambda \tag{6}$$

Thus, the point $\boldsymbol{x}'$ is inside the circle of radius λ with centre at $\boldsymbol{x}''$.

Let us describe the circles $v_{x''}$ of radius λ with centres at each point $\boldsymbol{x}'' \in \gamma''$ (γ'' is the boundary of Δ'') and denote by e the union of all $v_{x''}$ corresponding to all $\boldsymbol{x} \in \gamma$. If the acute angle of the parallelogram Δ'' is not too small the set e will have a shape like that shown in Fig. 12.16, that is it will look as a "frame" with rounded outer angles enclosing the parallelogram $\Delta'' - e$.

The circle $v_{y''}$ where $\boldsymbol{y}''$ is the centre of the parallelogram Δ'' is entirely contained within $\Delta'' - e$. Since the centre $\boldsymbol{y}$ of the square Δ goes into $\boldsymbol{y}''$ under the mapping A^* we have $\boldsymbol{y}' \in v_{y''} \subset \Delta'' - e$.

Let us denote by H_i the altitude of Δ'' perpendicular to its ith side (the length of the ith side is a_ih; $i = 1, 2$). The disposition shown in Fig. 12.16 is sure to take place if

$$H_i > 4\lambda \qquad (i = 1, 2) \tag{7}$$

---

* In the case of dimension n we should replace the numbers 4 and 6 in the right-hand sides of (5) and (6) by $n(\sqrt{n}+1)$ and $\sqrt{n^3(\sqrt{n}+1)^2}$ respectively.

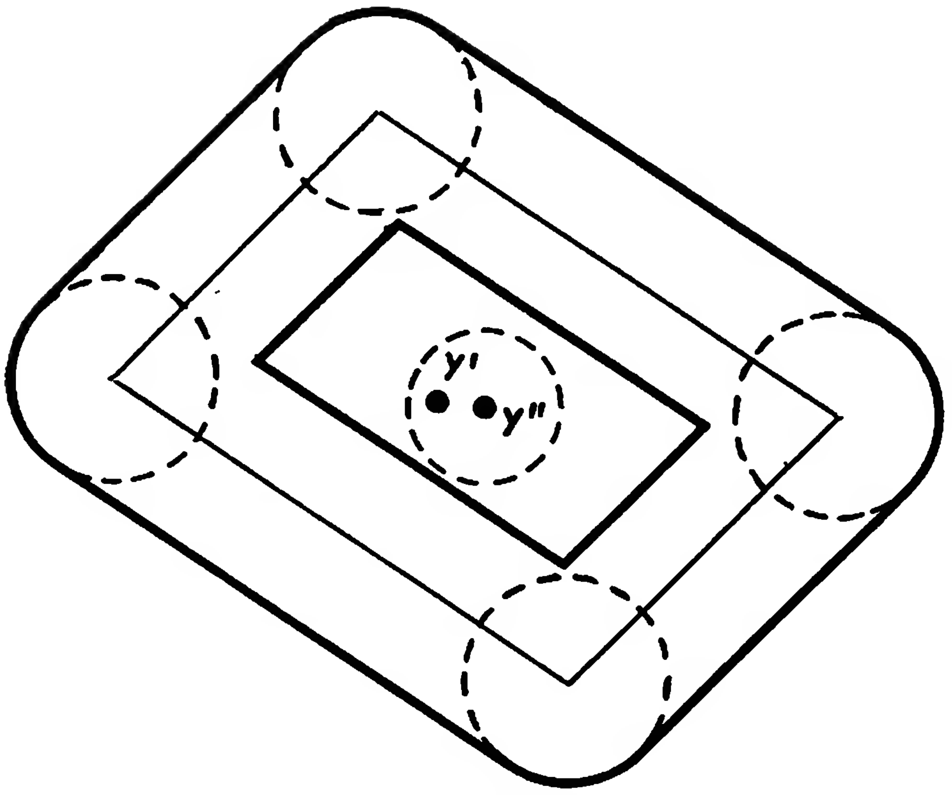

Fig. 12.16

The area of the frame e obviously does not exceed the sum of the areas of the four circles of radius λ with centres at the vertices of γ'' plus the sum of the areas of the rectangles of altitude 2λ constructed on the sides of the parallelogram Δ'' as bases.

Consequently*

$$|e| \leqslant 4\pi\lambda^2+2\sum_{i=1}^{2} 2\lambda a_i h \leqslant$$

$$\leqslant Kh^2\omega(h) \qquad (8)$$

$\left(\varphi_i \text{ and } \frac{\partial\varphi_i}{\partial x_j} \text{ are bounded on } \bar{\Omega} \text{ because so are } a_i \text{ and } \omega(h)\right)$ where K is a constant independent of h and of the location of Δ within Ω.

Since γ' is contained in the set e and Δ' is a bounded set (because the functions φ_i are continuous and therefore bounded on $\bar{\Omega}$) it is intuitively clear that

$$\Delta''-e \subset \Delta' \subset \Delta''+e \qquad (9)$$

Below we present a rigorous deduction of relation (9), and now we shall use it to estimate $|\Delta'|$.

It follows from (9) that $|\Delta'| = |\Delta''|+\theta|e|$ $(-1 \leqslant \theta \leqslant 1)$.

Consequently, by virtue of (8), and taking into account that the area of the parallelogram Δ'' is equal to the absolute value of the determinant formed of the components of the vectors serving as its sides, we see that

$$|\Delta'| = |D(x^0)|\,h^2+O(h^2\omega(h)), \qquad |O(h^2\omega(h))| \leqslant Ch^2\omega(h)$$

where the constant C is independent of x^0 and h.

Let us prove (9). The inclusion relation $\Delta''-e \subset \Delta'$ follows from the fact that $\Delta''-e$ necessarily contains at least one point $y' \in \Delta'$ and does not contain any boundary point of Δ' since all the boundary points of Δ' belong to e. If $\Delta''-e$ contained a point z not belonging to Δ' then the line segment $\overline{y'z}$ (joining the point $y' \in \Delta'$ with the point $z \,\bar{\in}\, \Delta'$) would contain a boundary point of Δ'.

The inclusion relation $\Delta' \subset \Delta''+e$ can be established as follows. Suppose that there is a point $z' \in \Delta'-(\Delta''+e)$. Let us issue from the centre of Δ'' a ray passing through z' and watch the variable point moving along that ray from the centre of Δ'' to infinity. Since Δ' is a bounded set (because the functions φ_1 and φ_2 are bounded on $\bar{\Delta}$) this variable point must necessarily pass through a point belonging to γ' (γ' is the boundary of Δ'), which is impossible since $\gamma' \subset e$.

---

* In the case of dimension n we shall have $K\omega(h)h^n$ on the right-hand side.

It now remains to consider the case when there holds the inequality

$$H_i \leqslant 4\lambda \tag{10}$$

for some $i = 1, 2$ (in particular, this is the case when $D(x^0) = 0$ and the parallelogram Δ'' degenerates).

In this case, by (8), (4), (5) and (6), we have

$$D(x^0)h^2 = |\Delta''| = a_i h H_i = O(h^2\omega(h))$$

and

$$|\Delta'| = |e| + |\Delta''| \dot{=} O(h^2\omega(h)) + O(h^2\omega(h))$$

which means that equality (2) is valid because the difference between its left-hand member and the first summand on its right-hand side is of order $O(h^n\omega(h))$.

Thus, in all the possible cases we have relation (2), and the constant C is independent of h and of $x^0 \in \Omega$. The remarks made in the course of the proof show that the general case of an arbitrary dimension n can be treated in just the same way.

§ 12.18. Double Integral in Polar Coordinates

The formulas

$$x = \varrho \cos\theta, \quad y = \varrho \sin\theta \tag{1}$$

describe the transformation of polar coordinates in the plane into Cartesian coordinates. The right-hand sides of (1) are continuously differentiable functions with Jacobian

$$D = \frac{D(x, y)}{D(\varrho, \theta)} = \begin{vmatrix} \cos\theta & \sin\theta \\ -\varrho\sin\theta & \varrho\cos\theta \end{vmatrix} = \varrho \geqslant 0 \tag{2}$$

Let us consider an auxiliary plane with Cartesian coordinates (ϱ, θ) and the domain

$$\Lambda = \{\varrho > 0, 0 < \theta < 2\pi\} \tag{3}$$

lying in this plane. It is obvious that formulas (1) specify a one-to-one mapping of Λ onto the xy-plane with the positive half-axis x (i.e. the ray $\theta = 0$ in the xy-plane) deleted. We shall denote this plane (with the deleted ray) as Λ'. The Jacobian D is positive on Λ ($D > 0$).

Now let us consider an arbitrary measurable (in the two-dimensional sense) domain in the xy-plane and a continuous function $f(x, y)$ defined in the closure of that domain. On deleting from this domain the points belonging to the ray $\theta = 0$ (provided there are such) we denote the remaining set as Ω'. Let us suppose that Ω' is a domain or a union of a finite number of pairwise nonintersecting domains. Formulas (1) specify a mapping $\Omega \rightleftarrows \Omega'$ of the set Ω' on a set Ω (we shall additionally suppose that Ω is measurable). Under these conditions there holds the equality

$$\iint_{\Omega'} f(x, y)\, dx\, dy = \iint_{\Omega} f(\varrho \cos\theta, \varrho \sin\theta)\varrho\, d\varrho\, d\theta \qquad (4)$$

for the conditions stated imply that all the requirements of the theorem on change of variables in a double integral are fulfilled (that is the function f is continuous on $\bar{\Omega}'$, transformation (1) is continuously differentiable on Ω, its Jacobian is equal to $\varrho > 0$ and $\Omega \rightleftarrows \Omega'$).

In the resultant formula (4) we can now replace Ω and Ω' by their closures $\bar{\Omega}$ and $\bar{\Omega}'$, respectively, because this only adds to the domains of integration some sets of measure zero.

If the domain Ω has the shape of a sector bounded by two rays $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2 \leqslant \theta_1 + 2\pi)$ and by a continuous curve $\varrho = \psi(\theta)$ then formula (4) yields

$$\iint_{\Omega'} f(x, y)\, dx\, dy = \int_{\theta_1}^{\theta_2} d\theta \int_{0}^{\psi(\theta)} f(\varrho \cos\theta, \varrho \sin\theta)\varrho\, d\varrho \qquad (5)$$

Formula (4) can also be derived on the basis of geometrical considerations without resorting to the auxiliary plane with Cartesian coordinates (ϱ, θ).

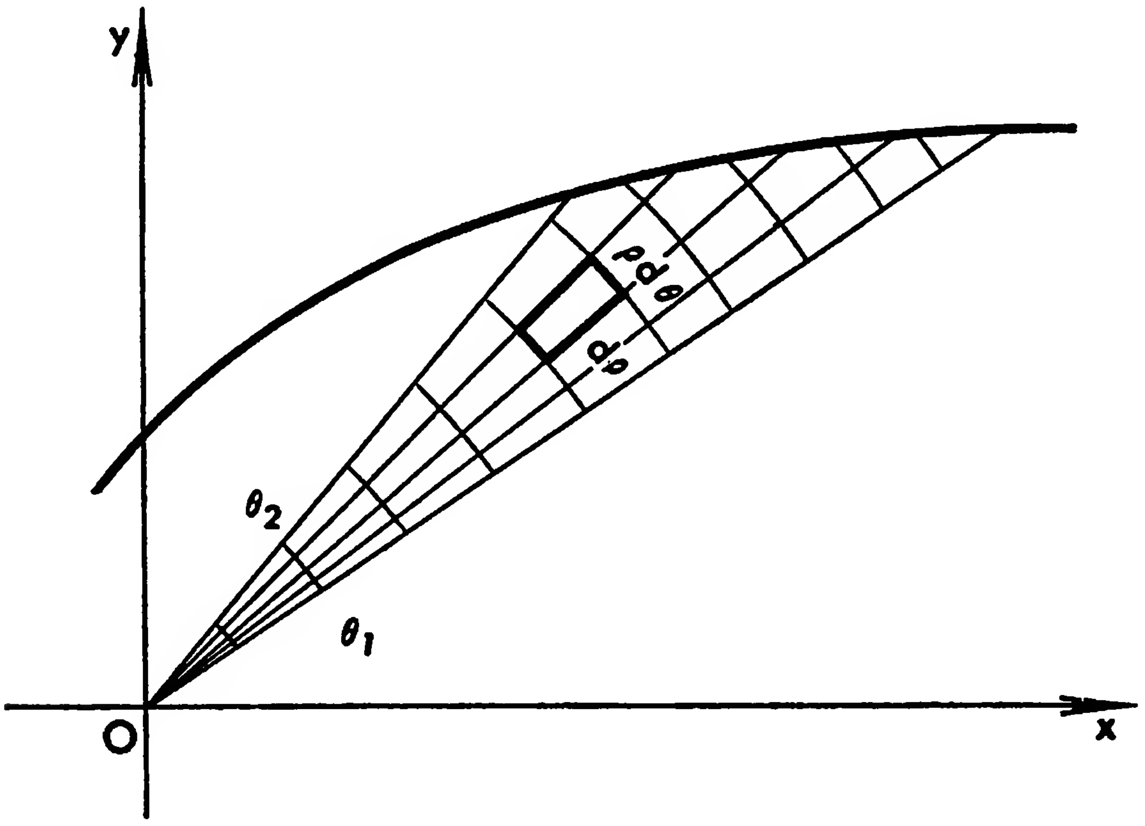

Fig. 12.17

To this end we break up the xy-plane into small cells with the aid of concentric circles with centre at the origin and rays issued from the origin (see Fig. 12.17). The area ΔS of a small curvilinear cell of this partition located in the vicinity of a point (ϱ, θ) (or, as we say, the element of area in polar coordinates) is equal, to within infinitesimals of higher order, to $\varrho\, d\varrho\, d\theta$: $\Delta S \sim \varrho\, d\varrho\, d\theta$. Therefore, if we proceed from such a partition and pass to the limit in the corresponding integral sums we arrive at

$$\lim_{\Delta\varrho,\,\Delta\theta\to 0} \sum f_j \varrho_j \Delta\varrho \Delta\theta = \iint_{\Omega} f(\varrho, \theta)\varrho \, d\varrho \, d\theta$$

Example:

$$\iint_{x^2+y^2\leq R^2} e^{(x^2+y^2)} \, dx \, dy = \int_0^R \int_0^{2\pi} e^{\varrho^2} \varrho \, d\varrho \, d\theta = \pi(e^{R^2}-1)$$

Remark. The mapping (or, synonymously, operator or transformation) determined by formulas (1) is continuous on the closure $\bar{\Lambda}$ of the domain Λ and establishes a one-to-one correspondence $\Lambda \rightleftarrows \Lambda'$. However, this does not necessarily mean that there is a one-to-one correspondence between the points of the boundaries of Λ and Λ' (see Theorem 1, § 12.16).

§ 12.19. Triple Integral in Spherical Coordinates

The formulas

$$x = \varrho \cos\theta \cos\varphi, \quad y = \varrho \cos\theta \sin\varphi, \quad z = \varrho \sin\theta \tag{1}$$

describe the transformation from spherical (polar) coordinates in space to Cartesian coordinates (see Fig. 12.18). Here ϱ is the distance from the variable point $P(x, y, z)$ to the origin, θ is the angle between the radius vector ϱ of the point P and its projection on the xy-plane and φ is the angle between this projection and the positive x-axis. The angle φ is reckoned in the direction of the shortest rotation of the x-axis about the z-axis making the former coincident with the y-axis.

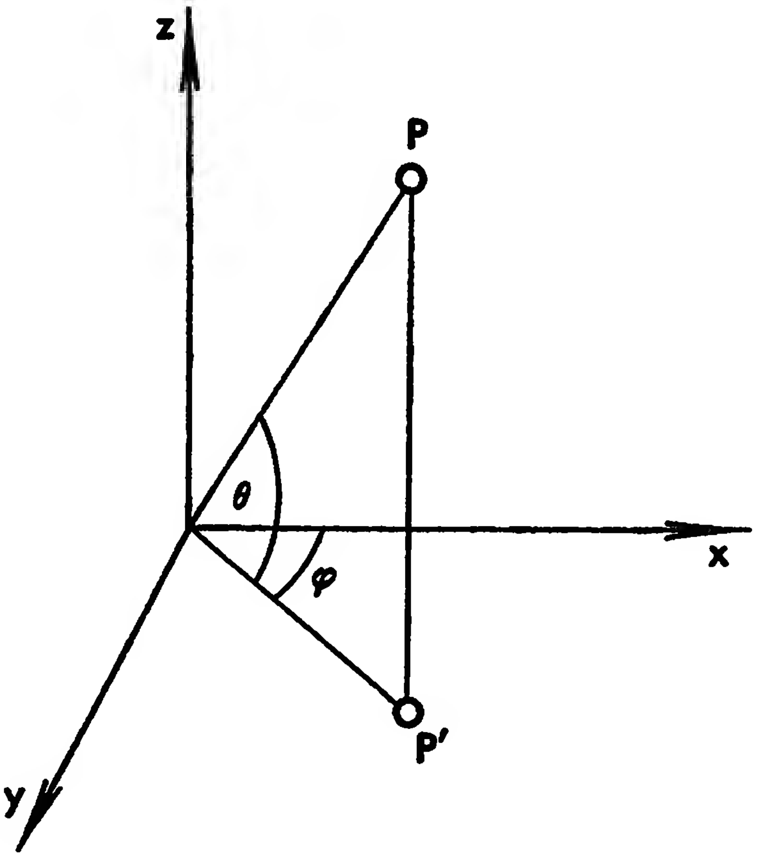

Fig. 12.18

The functions on the right-hand sides of (1) are continuously differentiable, their Jacobian being

$$D = \frac{D(x, y, z)}{D(\varrho, \theta, \varphi)} = \varrho^2 \cos\theta \tag{2}$$

Let us consider an auxiliary three-dimensional space with Cartesian coordinates $(\varrho, \theta, \varphi)$ and the open set

$$\Lambda = \left\{0 < \varrho, \; -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \; 0 < \varphi < 2\pi\right\} \tag{3}$$

lying in this space.

Transformation (1) establishes a one-to-one correspondence between the points of Λ and the points of the three-dimensional space (x, y, z) with the half-plane $\varphi = 0$ (i.e. the set of the points $(x, 0, z)$, $x \geqslant 0$) deleted; denoting this space with the deleted half-plane by Λ', we can write

$$\Lambda \rightleftarrows \Lambda' \tag{4}$$

Now let us consider an arbitrary measurable three-dimensional domain in the *xyz*-space and a continuous function $f(x, y, z)$ defined in the closure of that domain. On deleting from this domain the points belonging to the half-plane $\varphi = 0$ (provided that it contains such points) we denote by Ω' the remaining point set. Let us suppose additionally that Ω' is a domain or a finite sum of pairwise nonintersecting domains. The image of Ω' under mapping (4) is a set Ω; we shall suppose that Ω is also measurable.

Under the conditions imposed there holds the equality

$$\iint\limits_{\Omega'} f(x, y, z)\, dx\, dy\, dz = \iiint\limits_{\Omega} F(\varrho\ \theta, \varphi)\, \varrho^2 \cos\theta\, d\varrho\, d\theta\, d\varphi \tag{5}$$

where

$$F(\varrho, \theta, \varphi) = f(\varrho\cos\theta\cos\varphi,\ \varrho\cos\theta\sin\varphi,\ \varrho\sin\theta) \tag{6}$$

Indeed, the above conditions mean that all the requirements of the theorem on change of variables in a multiple integral are fulfilled.

Now we can replace, if necessary, Ω and Ω' in (5) by $\bar{\Omega}$ and $\bar{\Omega}'$ respectively.

Suppose that σ is a surface described in spherical coordinates by a function $\varrho = \psi(\theta, \varphi)$ ($(\theta, \varphi) \in \omega$ where ω is a domain) which is continuous on the closure of ω. Let Ω be the measurable three-dimensional domain in the *xyz*-space bounded by the surface σ and by the conical surface whose vertex is at the origin and whose elements pass through the edge of σ. Then if $f(x, y, z)$ is a continuous function on $\bar{\Omega}$ we have

$$\iiint\limits_{\Omega} f\, dx\, dy\, dz = \iint\limits_{\omega} d\theta\, d\varrho \int\limits_{0}^{\psi(\theta, \varphi)} F\varrho^2 \cos\theta\, d\varrho$$

In particular, if ω corresponds to the entire unit sphere the above integral is equal to $\int\limits_{-\pi/2}^{\pi/2} \cos\theta\, d\theta \int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{\psi(\theta, \varphi)} F\varrho^2\, d\varrho$.

To derive the expression of the element of volume in spherical coordinates in a visual way let us partition the *xyz*-space into small cells with the aid of a family of concentric spherical surfaces with centre at the origin O (i.e. at the point $\varrho = 0$), a family of planes passing through the *z*-axis and a family of circular surfaces having Oz as axis. It is readily seen that the volume Δv of a small cell thus obtained lying in the vicinity of a point $(\varrho, \theta, \varphi)$ equals, to within infinitesimals of higher order, to $\varrho^2 \cos\theta\, d\varrho\, d\theta\, d\varphi$: $\Delta v \sim \sim \varrho^2 \cos\theta\, d\varrho\, d\theta\, d\varphi$.

Remark. Operator (1) is continuous on the closure $\bar{\Lambda}$ of the domain Λ and sets up a one-to-one correspondence $\Lambda \rightleftarrows \Lambda'$. However, this does not necessarily mean that there is a one-to-one correspondence between the points of the boundaries of Λ and Λ'.

§ 12.20. General Properties of Continuous Operators

In this section we shall discuss some general properties of an operator A specified by relations of the form

$$x' = Ax, x = (x_1, \ldots, x_n) \in \Omega \subset R, x' = (x', \ldots, x'_n) \in \Omega' \subset R'$$

establishing a one-to-one correspondence $\Omega \rightleftarrows \Omega'$ between the points of some sets $\Omega \subset R$ and $\Omega' \subset R'$. It is thus assumed that the inverse operator A^{-1} ($x = A^{-1}x'$, $x' \in \Omega'$) exists.

We shall use the following notation. If a point belonging to Ω' is denoted x' this will mean that $x' = Ax$ ($x \in \Omega$). By e' we shall denote the image of $e \subset \Omega$ under A: $e' = Ae$. Finally, σ_x and $\sigma_{x'}$ will, respectively, denote some balls in R and R' with centres at $x \in R$ and $x' \in R'$.

Theorem 1. *If the operator A is continuous and one-to-one on a bounded closed set F then the set F' (the image of F under A) is also bounded and closed, and the inverse operator A^{-1} is continuous on F'.*

Indeed, let $x'_l \in F'$ $(l = 1, 2, \ldots)$, $y_0 \in R'$ and $x'_l \to y_0$. Then there is a subsequence x_{l_k} and a point $x_0 \in F$ such that $x_{l_k} \to x_0$ (because F is bounded and closed) and, by the continuity of A on F, we have $x'_{l_k} = Ax_{l_k} \to Ax_0$. Consequently, $y_0 = Ax_0$ which shows that F' is a closed set. The set F' is bounded because, if otherwise, there would exist a sequence of points x'_l with $|x'_l| \to \infty$, which is impossible because then there would exist a subsequence x_{l_k} and a point $x_0 \in F$ such that $Ax_{l_k} \to Ax_0$.

Now we must prove the continuity of the inverse operator. Let $x'_l, x'_0 \in F'$ $(l = 1, 2, \ldots)$ and $x'_l \to x'_0$. If we supposed that the point x_l did not tend to x_0 then there would exist a subsequence x_{l_k} and a point $x_* \neq x_0$ such that $x_{l_k} \to x_*$ $(k \to \infty)$ since the set F is bounded and closed. It would follow that $x'_{l_k} \to Ax_*$ and, since $x'_{l_k} \to Ax_0$, we would obtain $Ax_0 = Ax_*$. Since the operator A is one-to-one this would imply that $x_0 = x_*$, and we would arrive at a contradiction.

Theorem 2. *Let the conditions of Theorem 1 hold. Then the image $(\sigma_x)'$ of any ball $\sigma_x \subset F$ under the mapping A contains a ball $\sigma_{x'}$, that is if $g \subset F$ is an open set (or a domain), the set g' is also open (or, respectively, is a domain).*

This important theorem was proved by L. E. J. Brouwer; the proof which we do not present here can be found, for instance, in the book by W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, 1941.

For the special case when the operator A is continuously differentiable and the corresponding Jacobian does not vanish on g the proof was given in § 7.18.

Theorem 3. *Let Ω be a domain and let a continuous operator A ($Ax = x'$) defined on Ω establish a one-to-one correspondence*

$$\Omega \rightleftarrows \Omega' \qquad (x \in \Omega, x' \in \Omega')$$

between the points of the sets Ω and Ω'.

Then

(i) *if $g \subset \Omega$ is a bounded domain and $\bar{g} \subset \Omega$, then the boundary γ of g is a nonempty set, g' is also a domain and its boundary is γ' (the image of γ under A);*

(ii) *if $g \subset \Omega$ is an arbitrary domain then g' is also a domain; in particular, the set Ω' is also a domain. However, the correspondence between the boundaries of Ω and Ω' may not be one-to-one* (see the remarks at the ends of §§ 12.18 and 12.19).

By the condition of assertion (i), the bounded closed set $g+\gamma = \bar{g}$ goes into the set $g'+\gamma'$ under the continuous operator A, the correspondence between the points being one-to-one. Therefore, by the foregoing theorem, the set $g'+\gamma'$ is closed and g' is a domain since it is the image of the domain $g \subset g+\gamma = \bar{g}$. Therefore γ' is closed and, moreover, it is the boundary of g'; the latter property is implied by the fact that under A and, consequently, under A^{-1} as well, a domain is mapped onto a domain.

Now let us suppose that $g \subset \Omega$ is an arbitrary domain (it is not required that it should be bounded and that $\bar{g}$ should belong to the domain Ω; see the condition of assertion (ii)). Let $x_0' \in g'$ and let $\sigma_{x_0} \subset g$ be an open ball (which is a bounded domain) such that $\bar{\sigma}_{x_0} \subset g$. As was proved above, $(\sigma_{x_0})'$ is a domain and consequently it contains a ball σ_{x_0}', which means that g' is an open set. The connectedness of g' follows from the connectedness of g since the operator A $(Ax = x')$ is continuous.

Remark. Suppose that the conditions of the theorem on change of variables in a multiple integral are fulfilled (§ 12.16). Then the following assertions hold:

(i) If Δ is a (closed) cube contained in Ω then its interior goes into the interior of Δ' under the mapping A while the boundary of Δ goes into the boundary of Δ' (see Theorem 3).

(ii) There is no cube contained in the set $\Omega_0 = \{D(x) = 0\}$. For, if there were a cube $\Delta \subset \Omega_0$, its interior would be mapped by the operator A on a nonempty open set Δ' and this would imply that $|\Delta'| > 0$. On the other hand, $|\Delta'| = \int\limits_{\Delta} |D(x)|\, dx = 0$, which leads to a contradiction.

§ 12.21. More on Change of Variables in Multiple Integral

We shall show in this section that *if the conditions of the theorem on change of variables* (§ 12.16) *are fulfilled, then there cannot exist a pair of points $y, z \in \Omega$ such that $D(y) > 0$ and $D(z) < 0$.*

We shall prove this assertion by contradiction. Suppose that there is such a pair of points; then there also exists a cube $\Delta \subset \Omega$ within which the Jacobian D changes sign. Let us prove this auxiliary assertion.

To this end we join y and z with a continuous curve $C \subset \Omega$. Each point of C can be covered by an open cube, with edges parallel to the coordinate axes and centre at that point, lying within Ω. This system of cubes contains a finite subsystem as $\Delta_1, \Delta_2, \ldots, \Delta_N$ so that they follow one after another along the curve C regarded as being oriented from y to z. One of these cubes must necessarily satisfy the indicated property. Indeed, if a change of sign takes place in Δ_1 then this property obviously holds. If this is not the case let us assume, for definiteness, that $D(x) \geqslant 0$ on Δ_1 and denote by k the largest of the indices j for which $D(x) \geqslant 0$ on Δ_j ($j = 1, \ldots, k$). Then either $D(x)$ changes sign on Δ_{k+1} or $D(x) \leqslant 0$ on Δ_{k+1}. But the latter is impossible because this would mean that $D(x) = 0$ on the nonempty rectangle $\Delta_k \Delta_{k+1}$ (see assertion (ii) in the remark in § 12.20). We have thus proved the existence of a cube Δ within which D changes sign.

If Δ is a rectangle in which D changes sign, that is if there are interior points y and z of Δ such that $D(y) > 0$ and $D(z) < 0$, we can always assume that both points are in one plane $x_j = \alpha$ for some α.

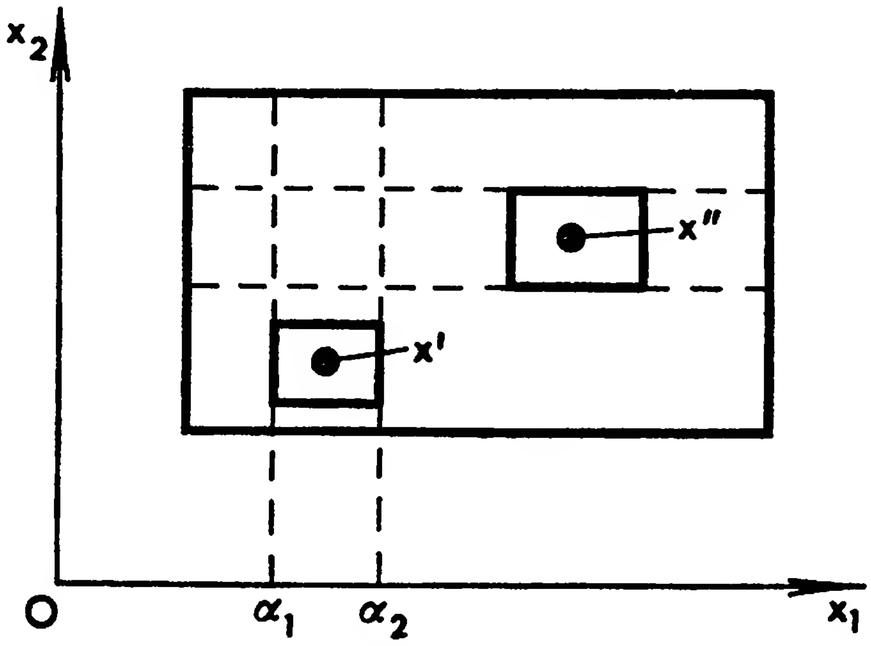

Fig. 12.19

Indeed, let us construct cubes Δ' and Δ'' consisting of interior points of Δ with centres at y and z respectively, such that $D(x) > 0$ on Δ' and $D(x) < 0$ on Δ''. Let the faces of Δ' perpendicular to x_1-axis be $x_1 = \alpha_1$ and $x_1 = \alpha_2$ ($\alpha_1 < \alpha_2$). If we suppose that the last assertion does not hold this would mean that $D(x) \geqslant 0$ on the rectangle $\Delta_{\alpha_1\alpha_2}$ consisting of all the points $x \in \Delta$ for which $\alpha_1 \leqslant x_1 \leqslant \alpha_2$. Besides, we would have $D(x) \leqslant 0$ for all the points $x \in \Delta$ belonging to the rectangle Π constructed on Δ'' as base with lateral faces parallel to the x_1-axis. Therefore we would have $D(x) = 0$ on the nonempty intersection $\Pi\Delta_{\alpha_1\alpha_2}$ containing a cube, which is impossible (see assertion (ii) in the remark in § 12.20). See Fig. 12.19 where the two-dimensional case is shown; in the figure x' corresponds to y and x'' to z.

Now we proceed to show that the change of sign we spoke of cannot take place.

We begin with the case when Δ is an open two-dimensional rectangle (Fig. 12.20).

The oriented line segment $\overline{ab}$ divides the rectangle Δ into two open rectangles Δ_1 and Δ_2. Under the operator A the line segment $\overline{ab}$ goes into the oriented segment $a'b'$ of a continuous curve* cutting Δ' into two domains Λ and Ω (Fig. 12.21).

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* The curvilinear segment $a'b'$ is determined by $x_1' = \varphi(x_1, x_2)$, $x_2' = \psi(x_1, x_2)$ as functions of x_2 which are continuous together with their first-order derivatives. These derivatives may simultaneously vanish at some points x_2 but this does not affect the further course of our argument.

Thus, we have either (i) $\Delta_2' = \Delta$ or (ii) $\Delta_2' = \Omega$. If we assume that (i) takes place this would contradict the inequality $D(z) < 0$ since the latter indicates that the points of Δ_2 lying sufficiently close to z must go, under the operator A, into points lying to the left of the oriented curve $a'b'$ (when moving from a' to b'), which implies (ii).

Finally, assumption (ii) contradicts the inequality $D(y) > 0$ which indicates that the points of Δ_2 lying sufficiently close to z go, under A, into points lying to the right of the curve $a'b'$.

We have completed the proof for the case of dimension 2.

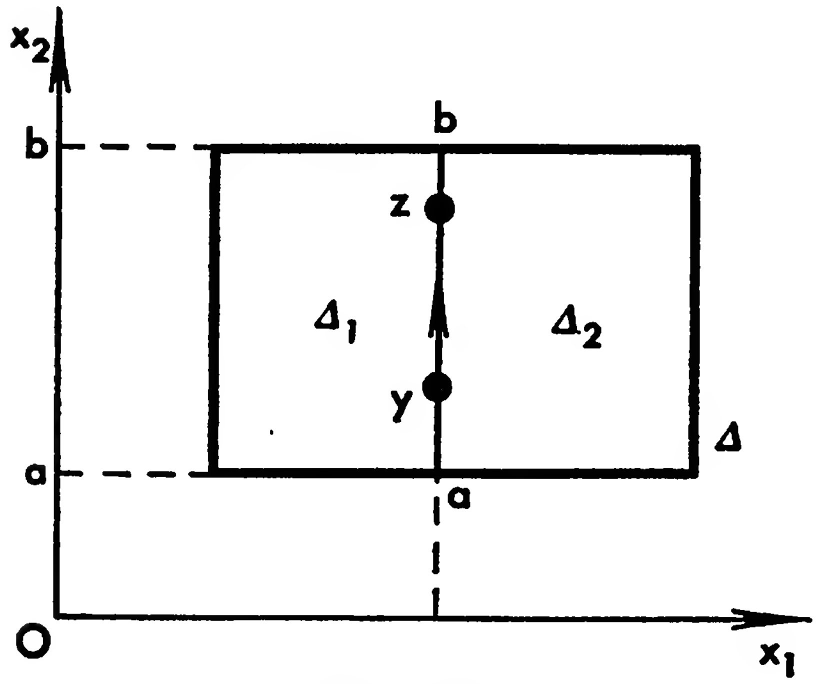

Fig. 12.20

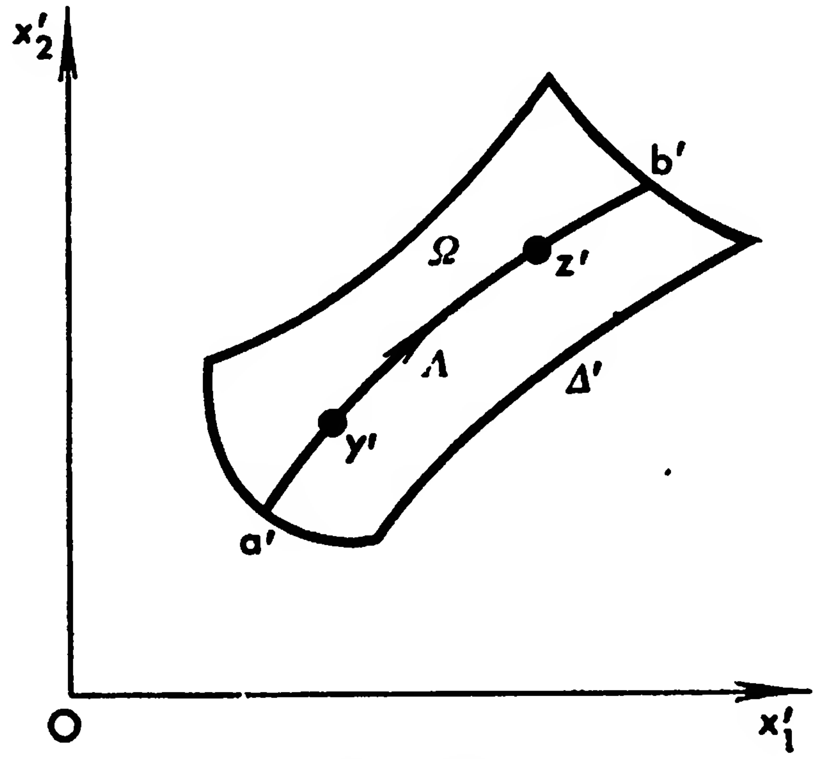

Fig. 12.21

In the three-dimensional case Δ is a three-dimensional rectangle (i.e. a rectangular parallelepiped). In this case the role of the oriented line segment $\overline{ab}$ is played by the rectangular plane area σ which is cut out of Δ by the plane $x_1 = \alpha$. The surface σ divides Δ into two parts with interiors Δ_1 and Δ_2. Let us choose a certain orientation of σ. Then every closed smooth contour $\Gamma \subset \sigma$ receives the orientation coherent with that of σ, that is a positive direction of traversing this contour will be defined. In this sense the image σ' of σ will also be accordingly oriented. But it should be noted that at the points of σ where $D(x) = 0$ the normal to σ may not exist and therefore in this sense our definition of orientation is not completely applicable to σ. Suppose that $D(y) > 0$ and $D(z) < 0$ where y and z are interior points of Δ belonging to σ. On the normals drawn to σ', at the points $y', z' \in \sigma'$ we now take some points y'' and z'', respectively, belonging to Δ_1' so that the vectors $\overrightarrow{y'y''}$ and $\overrightarrow{z'z''}$ (with their origins deleted) are entirely contained in Δ'. These vectors together with small oriented areas $\sigma_{y'}', \sigma_{z'}' \subset \sigma'$ containing the points y' and z' form oriented elements. The operator A^{-1} transforms the element corresponding to y' into an element oriented in the same sense $\big(D(y) > 0\big)$, while the element corresponding to z' is transformed into an element oriented in the opposite sense $\big(D(z) < 0\big)$. This means that there are points of Δ_1 lying on the opposite sides of σ in the vicinity of y and z, respectively, which is impossible.

§ 12.22. Improper Integral with Singularities on the Boundary of the Domain of Integration. Change of Variables

Lemma 1. *Let* $\Omega_1, \Omega_2, \Omega_3, \ldots$ *be a sequence of open sets. Then the set* $\Omega = \sum_1^\infty \Omega_k$ *is also open.*

Indeed, if a point x^0 belongs to Ω, there is k such that $x^0 \in \Omega_k$. Since Ω_k is an open set there is a ball V with centre at x^0 contained in Ω_k and, consequently, in Ω.

Lemma 2. *If the conditions of Lemma 1 are fulfilled and if, in addition,* $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \ldots$, *then for any bounded closed set* $F \subset \Omega$ *there is* k *such that* $F \subset \Omega_k$.

Indeed, if we assume the contrary, then for every $k = 1, 2, \ldots$ there is a point x_k belonging to F and not belonging to Ω_k. Since F is a bounded closed set, the sequence of points $\{x_k\}$ belonging to it contains a subsequence $\{x_{k_j}\}$ convergent to a point $x_0 \in F$ $(x_{k_j} \to x_0)$. But we have $x_0 \in F \subset \Omega$, and consequently $x_0 \in \Omega_{k_0}$ for some k_0. The set Ω_{k_0} being open, there is a ball V with centre at x_0 belonging to Ω_{k_0}. The points x_{k_j} belong to the ball V for sufficiently large j. Let us take one of these points with $k_j > k_0$; for this point we have $x_{k_j} \in V \subset \Omega_{k_0} \subset \Omega_{k_j}$, and thus arrive at a contradiction.

Lemma 3. *If the conditions of Lemma 2 hold and, besides, the sets* Ω *and* Ω_k *are measurable then*

$$\lim_{k\to\infty} |\Omega_k| = |\Omega| \tag{1}$$

Proof. It is obvious that $|\Omega_k| \leqslant |\Omega_{k'}| \leqslant |\Omega|$ $(k \leqslant k')$. On the other hand, given any $\varepsilon > 0$, there is a closed measurable set $F \subset \Omega$ such that $|F| > |\Omega| - \varepsilon$. For instance, we can take as F a figure consisting of cubes $\Delta \subset \Omega$ belonging to a sufficiently refined network. According to Lemma 2, there is k_0 such that $F \subset \Omega_{k_0}$. For this k_0 we have $|\Omega| - \varepsilon < |F| < |\Omega_{k_0}| < |\Omega_k|$ $(k \geqslant k_0)$, and equality (1) is thus proved.

Definition. Let a function $f(x)$ be defined and continuous (but not bounded) in a domain Ω (which is not necessarily measurable). If the limit

$$\lim_{k\to\infty} \int_{\Omega_k} f(x)\, dx = \int_\Omega f\, dx \tag{2}$$

exists where $\{\Omega_k\}$ is an arbitrary sequence of measurable domains possessing the properties

$$\bar{\Omega}_k \subset \Omega, \qquad \Omega_1 \subset \Omega_2 \subset \ldots \quad \text{and} \qquad \Omega = \sum_{k=1}^\infty \Omega_k \tag{3}$$

(we also suppose that Ω_k have smooth or piecewise smooth boundaries) we call (2) the *improper integral of* $f(x)$ *on* Ω.

The existence of limit (2) is understood in the sense that it is independent of the choice of such a sequence $\{\Omega_k\}$. If the function $f(x)$ is nonnegative on Ω then the existence of limit (2) for one such sequence $\{\Omega_k\}$ automatically implies the existence of the limit, equal to the former, for any other sequence $\{\Omega'_k\}$. For, given any k, there is, by Lemma 2, a number $l = l(k)$ such that $\Omega_k \subset \Omega'_l$, and therefore

$$\int_{\Omega_k} f\,dx = \int_{\bar{\Omega}_k} f\,dx \leq \int_{\Omega'_l} f\,dx$$

whence it follows that limit (2) for sequence $\{\Omega'_k\}$ exists $\left(\text{since the number sequence } \left\{\int_{\Omega'_l} f\,dx\right\} \text{ is monotone}\right)$ and is not less than limit (2) for sequence $\{\Omega_k\}$. Reversing the argument we conclude that, conversely, limit (2) for $\{\Omega'_k\}$ is not greater than limit (2) for $\{\Omega_k\}$, whence follows the desired equality of the limits.

Theorem. *Let relations*

$$x = \varphi(u, v), y = \psi(u, v) \qquad ((u, v) \in \Omega) \tag{4}$$

specify a transformation continuously differentiable on the closure $\bar{\Omega}$ of Ω (which is a measurable domain) such that the corresponding Jacobian is different from zero on Ω:

$$D(u, v) = \begin{vmatrix} \dfrac{\partial \varphi}{\partial u} & \dfrac{\partial \varphi}{\partial v} \\ \dfrac{\partial \psi}{\partial u} & \dfrac{\partial \psi}{\partial v} \end{vmatrix} \neq 0 \qquad ((u, v) \in \Omega)$$

Let the mapping of Ω on Ω' under (4) *be one to one:*

$$\Omega \rightleftarrows \Omega'$$

If $f(x, y)$ is a continuous (but not bounded) function defined on Ω' such that the function

$$F(u, v)\,|D(u, v)| = f[\varphi(u, v), \psi(u, v)]\,|D(u, v)|$$

is uniformly continuous on Ω (and therefore can be continuously extended to $\bar{\Omega}$) then

$$\iint_{\Omega'} f(x, y)\,dx\,dy = \iint_{\Omega} F(u, v)\,|D(u, v)|\,du\,dv \tag{5}$$

where the integral on the left-hand side is understood in the improper sense defined above.

Proof. Let us take an arbitrary sequence of domains $\{\Omega'_k\}$ satisfying conditions (3) (in which the letter Ω must be everywhere replaced by Ω'). To this sequence there corresponds a sequence of domains $\{\Omega_k\}$ which, by the bicontinuity (i.e. the continuity in both directions) of transformation (4), also satisfies conditions (3).

By the basic theorem on change of variables (which is applicable since f is continuous on $\bar{\Omega}'_k$), we have

$$\iint\limits_{\Omega'_k} f(x, y)\, dx\, dy = \iint\limits_{\Omega_k} F(u, v)\, |D(u, v)|\, du\, dv \qquad (k = 1, 2, \ldots) \quad (6)$$

Using Lemma 3 we write

$$\left| \iint\limits_{\Omega} F(u, v)\, |D(u, v)|\, du\, dv - \iint\limits_{\Omega_k} F(u, v)\, |D(u, v)|\, du\, dv \right| =$$

$$= \left| \iint\limits_{\Omega-\Omega_k} F(u, v)\, |D(u, v)|\, du\, dv \right| \leqslant M\, |\Omega-\Omega_k| \to 0, \qquad k \to \infty,$$

$$M \geqslant |F(u, v)\, D(u, v)| \qquad ((u, v) \in \bar{\Omega})$$

and therefore the right-hand member of (6) converges to the right-hand member of (5). Consequently the left-hand member of (6) converges to one and the same number irrespective of the choice of the sequence $\{\Omega'_k\}$. The theorem has been proved.

It should be noted that, according to the properties of transformation (4) (whose Jacobian is different from zero on Ω), the smooth (piecewise smooth) boundary of Ω'_k goes, under operator (4), into the smooth (piecewise smooth) boundary of Ω_k, whence follows the measurability of Ω_k.

Examples of such integrals see in § 12.23 (in particular, Example 1).

§ 12.23. Surface Area

Let us consider the three-dimensional space R with rectangular coordinates (x, y, z) and a surface S described by an equation

$$z = f(x, y) \qquad ((x, y) \in \bar{G}) \qquad (1)$$

We shall suppose that G is a measurable open domain and that the function f possesses the continuous partial derivatives

$$p = \frac{\partial f}{\partial x} \quad \text{and} \quad q = \frac{\partial f}{\partial y}$$

in $\bar{G}$ (see § 7.11).

According to the definition stated in § 7.19, S is a smooth surface element projectable in a one-to-one manner on the plane $z = 0$.

Let us break up G into a finite number of measurable (in the two-dimensional sense) parts $G_1, \ldots, G_N$ ($G = G_1 + G_2 + \ldots + G_N$) any two of which either do not intersect or intersect only along some parts of their boundaries. Let (x_j, y_j) be an arbitrary point of G_j ($j = 1, \ldots, N$). To this point there corresponds the point $P_j \in S$ with coordinates (x_j, y_j, f_j) where $f_j = f(x_j, y_j)$. Let us draw through the point P_j the tangent plane L_j to S. The cylindrical surface Γ_j with the boundary of G_j as directrix and elements parallel to the z-axis cuts a portion e_j out of the tangent plane L_j. Let $|e_j|$ denote the area of that portion.

The *area of the surface* S is defined as the limit

$$|S| = \lim_{\max (G_j)\to 0} \sum_{j=1}^{N} |e_j|$$

The cosine of the acute angle between the normal n_j to S at the point P_j and the z-axis (see § 7.5 (13)) is equal to $\cos (n_j, z) = 1/\sqrt{1+p_j^2+q_j^2}$ where the square root is taken with the sign "+" and p_j and q_j are the results of the substitution of x_j and y_j into p and q. It is evident that Ω_j is the projection of e_j on the xy-plane and, consequently, the measure of G_j is $|G_j| = |e_j| \times \times \cos (n_j, z)$ where $|e_j| = |G_j| \sqrt{1+p_j^2+q_j^2}$ $(j = 1, \ldots, N)$ and

$$|S| = \lim \sum_{j=1}^{N} |e_j| = \lim_{\max d(G_j)\to 0} \sum_{j=1}^{N} \sqrt{1+p_j^2+q_j^2}\,|G_j| =$$

$$= \iint_G \sqrt{1+p^2+q^2}\, dx\, dy \tag{2}$$

We have derived the formula for the area of a surface represented explicitly in form (1).

Let us consider the transformation of integral (2) with the aid of a substitution.

$$x = \varphi(u, v),\ y = \psi(u, v) \qquad ((u, v) \in \bar{\Omega}) \tag{3}$$

establishing a one-to-one correspondence between the measurable domain G and a measurable domain Ω on condition that the functions φ and ψ are continuously differentiable on $\bar{\Omega}$ and their Jacobian does not vanish on Ω:

$$\frac{D(x, y)}{D(u, v)} \neq 0 \text{ on } \Omega \tag{4}$$

Let us put $z = f(\varphi, \psi) = \chi(u, v)$. On the basis of the theorem on change of variables in a multiple integral (see § 7.26 (4)) we obtain the equality

$$\iint_\Omega \sqrt{1+p^2+q^2}\, dx\, dy = \iint_\Omega \sqrt{1+\left(\frac{D(y, z)}{D(u, v)}\Big/\frac{D(x, y)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)}\right)\Big/\left(\frac{D(x, y)}{D(u, v)}\right)^2} \times$$

$$\times \left|\frac{D(x, y)}{D(u, v)}\right| du\, dv$$

whence it follows that the area of the surface S is given by the formula

$$|S| = \iint_\Omega \sqrt{\left(\frac{D(x, y)}{D(u, v)}\right)^2 + \left(\frac{D(y, z)}{D(u, v)}\right)^2 + \left(\frac{D(z, x)}{D(u, v)}\right)^2}\, du\, dv = \iint_\Omega |\dot{r}_u \times \dot{r}_v|\, du\, dv \tag{5}$$

Formula (5) makes it possible to define the notion of the area for a surface represented parametrically which is not necessarily projectable in a one-to-one manner on one of the coordinate planes.

Let us take a smooth surface S represented by a vector equation

$$\boldsymbol{r} = \varphi(u, v)\boldsymbol{i}+\psi(u, v)\boldsymbol{j}+\chi(u, v)\boldsymbol{k} \quad (|\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v| > 0,\ (u, v) \in \Omega \rightleftarrows S) \tag{6}$$

where Ω is a measurable set in the uv-plane and φ, ψ and χ possess continuous partial derivatives of the first order on $\bar{\Omega}$. As usual, the symbolic relation $\Omega \rightleftarrows S$ indicates that equality (6) sets up a one-to-one correspondence between the points of Ω and S.

Since the set Ω is measurable it is bounded and therefore has a nonempty boundary γ. The boundary γ goes, under the mapping specified by equality (6), into the edge $\Gamma = \bar{S}-S$ of the surface in question. It is not required that the mapping of γ on Γ should also be one-to-one. There are many important examples in which such a condition does not hold (see examples below).

By definition, by the *area* of S (or of $\bar{S}$) is meant the number

$$|S| = \iint_{\Omega} |\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|\, du\, dv \tag{7}$$

We shall enumerate a number of properties of integral (7) demonstrating the expedience of this definition of area.

(i) The magnitude $|S|$ is invariant with respect to the admissible transformations of the parameters, that is if

$$u = \lambda(u', v'),\ \ v = \mu(u', v') \qquad ((u', v') \in \Omega' \rightleftarrows \Omega)$$

where λ and μ are continuously differentiable on $\bar{\Omega}'$ and

$$\frac{D(\lambda, \mu)}{D(u', v')} \neq 0 \qquad ((u', v') \in \Omega')$$

then

$$\iint_{\Omega} |\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|\, du\, dv = \iint_{\Omega} \left\{\left(\frac{D(x, y)}{D(u, v)}\right)^2+\left(\frac{D(y, z)}{D(u, v)}\right)^2+\left(\frac{D(z, x)}{D(u, v)}\right)^2\right\}^{1/2} du\, dv =$$

$$= \iint_{\Omega'} \left\{\left(\frac{D(x, y)}{D(u', v')}\right)^2+\left(\frac{D(y, z)}{D(u', v')}\right)^2+\left(\frac{D(z, x)}{D(u', v')}\right)^2\right\}^{1/2}\left|\frac{D(u', v')}{D(u, v)}\right|\cdot\left|\frac{D(u, v)}{D(u', v')}\right| du'\, dv' =$$

$$= \iint_{\Omega'} |\dot{\boldsymbol{r}}_{u'}\times\dot{\boldsymbol{r}}_{v'}|\, du'\, dv'$$

(ii) Let the surface S be projectable in a one-to-one manner on the plane $z = 0$ and let the equalities

$$x = \varphi(u, v),\ \ y = \psi(u, v) \qquad (\Omega \rightleftarrows G \ni (x, y)) \tag{8}$$

set up a one-to-one correspondence between the measurable domains Ω and G with the Jacobian

$$\left|\frac{D(x, y)}{D(u, v)}\right| \neq 0 \tag{9}$$

Let the partial derivatives $p = \frac{\partial f}{\partial x}$ and $q= \frac{\partial f}{\partial y}$ of the function $z=f(x,y) =$

$= \chi(u(x,y), v(x,y))$ $((x,y) \in G)$ where $u(x,y)$ and $v(x,y)$ is the solution of equations (8) be not only continuous on G (this follows form the theorem on implicit functions) but also uniformly continuous. Then there holds the equality

$$|S| = \iint_{\Omega} |\dot{r}_u \times \dot{r}_v|\, du\, dv = \iint_{G} \sqrt{1+p^2+q^2}\, dx\, dy \tag{10}$$

which was already proved above (see (5)).

We see that the new definition of the area of a surface is equivalent to the old definition if the conditions of the latter hold.

It should also be noted that if the condition of uniform continuity of p and q on G did not hold, that is if p and q were only continuous and bounded on G, equality (10) would nevertheless hold because in that case all the requirements guaranteeing the possibility of the change of variables in the integral would be fulfilled.

Moreover, if p and q are continuous but not bounded on G (while the functions φ and ψ possess continuous partial derivatives on $\bar{\Omega}$) equality (10) nevertheless remains valid if the integral on its right-hand side is understood in the improper sense (see § 12.22).

(iii) If ω is an arbitrary measurable open set belonging to Ω $(\omega \subset \Omega)$, the part $S(\omega)$ of the smooth surface in question corresponding to it which is determined by the equalities

$$\boldsymbol{r}(u, v) = \varphi \boldsymbol{i} + \psi \boldsymbol{j} + \chi \boldsymbol{k} \qquad ((u, v) \in \omega \rightleftarrows S(\omega)) \tag{11}$$

is in its turn a smooth surface whose area is given by the formula

$$|S(\omega)| = \iint_{\omega} |\dot{r}_u \times \dot{r}_v|\, du\, dv \tag{12}$$

The integral on the right-hand side of (12) also makes sense when ω is an arbitrary measurable subset of $\bar{\Omega}$. It is natural to regard the value of the integral as the area of the part $S(\omega)$ of the smooth surface S described by vector-function (11).

It is obvious that

$$|S(\bar{\omega})| = |S(\omega)|$$

and, in particular,

$$|\bar{S}| = |S(\bar{\Omega})| = |S(\Omega)| = |S| \tag{13}$$

Thus, we can speak of the parts $S(\omega)$ of the surface (set) $\bar{S}$ which, in accordance with equality (11), correspond to all the possible measurable subsets $\omega \subset \Omega$. To each such part (which is a subset of the set $\bar{S}$) we can assign the nonnegative number $|S(\omega)|$ specified by integral (12). This dependence (of the number on the subsets) possesses the additive property

$$|S(\omega_1+\omega_2)| = |S(\omega_1)| + |S(\omega_2)|$$

if ω_1 and ω_2 either do not intersect or intersect only along some parts of their boundaries.

The magnitude $|S(\omega)|$ is a concrete example of the notion of an *additive set function* playing an important role in mathematics. Another such magnitude which we dealt with is the measure of a (measurable) set.

The expression $dS = |\dot{r}_u \times \dot{r}_v|\, du\, dv$ is called the *differential of area of the surface S*. The area of the part of S corresponding to the variations of u from u to $u+du$ and of v from v to $v+dv$ is equal to

$$\Delta S = \int_u^{u+du} \int_v^{v+dv} |\dot{r}_u \times \dot{r}_v|\, du\, dv = |\dot{r}_u \times \dot{r}_v|_{\substack{u=\xi\\ v=\eta}}\, du\, dv =$$

$$= (|\dot{r}_u \times \dot{r}_v| + \varepsilon)\, du\, dv = dS + o(du\, dv) \qquad (du, dv \to 0)$$

In the second equality we have used the mean value theorem and in the third equality the point (ξ, η) at which the value of $|\dot{r}_u \times \dot{r}_v|$ was originally taken is replaced by (u, v), the discrepancy being compensated for by the addition of the term ε which tends to zero as $du, dv \to 0$ since the function $|\dot{r}_u \times \dot{r}_v|$ is continuous.

Hence, dS can be defined as a (uniquely determined) expression of the form $A\, du\, dv$ where A is independent of du and dv such that the difference between $A\, du\, dv$ and ΔS is $o\,(du\, dv)$ $(du, dv \to 0)$.

Example 1. Let us consider the area of a spherical surface. The equations

$$x = \cos\theta \cos\varphi, \quad y = \cos\theta \sin\varphi, \quad z = \sin\theta \ \ (|\dot{r}_\theta \times \dot{r}_\varphi| = \cos\theta) \quad (14)$$

$$\Omega = \{0 < \varphi < 2\pi, \ -\pi/2 < \theta < \pi/2\}$$

specify a smooth surface S which is the part of the sphere of unit radius with centre at the origin obtained from the latter by deleting the meridian $\varphi = 0$, $|\theta| \leq \pi/2$. Conditions (6) obviously hold in this case. In particular, there is a one-to-one correspondence $\Omega \rightleftarrows S$. However, equations (14) do not establish a one-to-one correspondence between $\gamma = \bar{\Omega} - \Omega$ and $\Gamma = \bar{S} - S$. The edge Γ of the surface S is the above-mentioned meridian. By virtue of equality (14), to each of its end points there corresponds the infinite set of points of γ composed of two opposite sides of Ω and to every other point of Γ there corresponds a pair of points belonging to γ which lie on the other two opposite sides of Ω.

The surface of unit sphere is the closure $\bar{S}$ of the surface S described by parametric equations (14). By formula (7), we have

$$|\bar{S}| = |S| = \iint_\Omega |\cos\theta|\, d\theta\, d\varphi = 2\pi \cdot 2 \int_0^{\pi/2} \cos\theta\, d\theta = 4\pi$$

It should be noted that the area of the unit sphere $\bar{S}$ can be regarded as the sum of the areas of the eight congruent parts cut out of $\bar{S}$ by the coordinate planes. One of them is the part σ which lies in the first coordinate trihedral and is described by the function $z = +\sqrt{1-x^2-y^2}$ $(x, y \geq 0;\ x^2+y^2 \leq 1)$

possessing the unbounded partial derivatives $p = -x/\sqrt{1-x^2-y^2}$ and $q = -y/\sqrt{1-x^2-y^2}$.

As was mentioned, it is possible to evaluate the area of σ with the aid of formula (2) for the computation of the area of a surface in Cartesian coordinates but the integral should be understood in the improper sense (see the end of § 13.13, Remark 1):

$$|\sigma| = \lim_{\varrho\to 1} \iint\limits_{x^2+y^2\leq\varrho^2<1} \sqrt{1+p^2+q^2}\, dx\, dy = \lim_{\varrho\to 1} \iint\limits_{x^2+y^2\leq\varrho^2<1} \frac{dx\, dy}{\sqrt{1-x^2-y^2}}$$

$$x > 0,\ y > 0$$

The formula for the area of a small element of a spherical surface of radius R can be derived on the basis of geometrical considerations. The network of meridians and parallels drawn close to each other divides the spherical surface S in question into small parts. The area ΔS of such a part lying in the vicinity of a point $A = (R, \theta, \varphi)$ $(\theta > 0)$ can obviously be estimated by the equalities

$$R \cos (\theta+d\theta)\, d\varphi\, R\, d\theta < \Delta S < R \cos \theta\, d\varphi\, R\, d\theta$$

whence follows

$$\Delta S = R^2 \cos \theta'\, d\varphi\, d\theta = R^2 \cos \theta\, d\varphi\, d\theta + d\varphi\, o(d\theta)$$

$$(\theta < \theta' < \theta + d\theta,\ d\theta \to 0)$$

Example 2. Let us compute the surface area of the torus specified by the equations

$$\begin{aligned} x &= (b+a\cos\theta)\cos\varphi, \quad y = a\sin\theta \qquad (0 < a < b),\\ z &= (b+a\cos\theta)\sin\varphi \ \ (|\dot{r}_u \times \dot{r}_v| = a(b+a\cos\theta) > 0) \end{aligned} \tag{15}$$

To apply the above considerations we should regard this surface as the closure $\overline{T}$ of the smooth surface T described by equations (15) with (θ, φ) ranging over the domain $\Omega = \{0 < \theta, \varphi < 2\pi\}$.

In this case relations (6) and the corresponding conditions of continuous differentiability hold when T is taken as S, and therefore

$$|\overline{T}| = |T| = \int_0^{2\pi} d\varphi \int_0^{2\pi} a(b+a\cos\theta)\, d\theta = 2\pi a(2\pi b+0) = 4\pi^2 ab$$

Example 3. Let us consider a circular cylinder of radius R and altitude H. Let σ denote its lateral surface and $|\sigma|$ the area of that surface. On dividing σ into parts with the aid of a system of equidistant planes perpendicular to the axis of the cylinder with step (the distance between the neighbouring planes) equal to H/N^3, we denote by $C_0, C_1, \ldots, C_{N^3}$ the circles appearing in the sections of the surface σ thus obtained. Let these circles be numbered consecutively in the upward direction of the axis of the cylinder. Next we partition the circle C_0 with the aid of equidistant points into $2N$ equal parts.

These points will be consecutively numbered along C_0. Let us draw the elements of the cylindrical surface σ through these points of division; each such element will intersect every circle C_k at one of its points. The points on C_k thus obtained will be numbered according to the rule that the points of division of the circles C_k lying in a common element of the cylindrical surface receive the same index. The further step of this construction (known as the *example of Schwarz**) is to delete from the circles C_k with even indices k the points of division with odd indices and from the circles C_k with odd indices k the points with even indices. Then there remains a finite system of points marked on the surface σ. On constructing on each triple of neighbouring points of this system a triangle Δ (with these points as vertices) we obtain a polyhedral surface σ_N inscribed in σ. Two points of each such triple are neighbouring points belonging to some C_k while the third point is either on C_{k+1} or on C_{k-1}, the element of the cylinder to which the third point belongs lying in the smallest dihedral angle with the axis of the cylinder as edge and faces passing through the element of the cylinder to which the former two points belong.

The number of the triangles Δ is obviously equal to $2N \cdot N^3 = 2N^4$, and the area of Δ is

$$|\Delta| = \frac{1}{2}\, 2 \sin\frac{\pi}{N}\, R \sqrt{R^2\left(1-\cos\frac{\pi}{N}\right)^2 + \left(\frac{H}{N^3}\right)^2} \sim \frac{\pi}{N}\, RR\, \frac{1}{2}\left(\frac{\pi}{N}\right)^2 > cN^{-3}$$

$$(N \to \infty,\ c > 0)$$

Therefore $|\sigma_N| > c_1 N^4 N^{-3} = c_1 N$ ($c_1 = \text{const} > 0$) despite the fact that the diameter of Δ's tends to zero as $N \to \infty$.

We see that the area of a surface cannot be simply defined as the limit of the area of a polyhedral surface inscribed in the former when the maximum diameter of its faces tends to zero. Such a definition is inapplicable even to the case of a very simple surface such as the cylinder we have considered.

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* H. A. Schwarz (1843-1921), a German mathematician.

CHAPTER 13

Scalar and Vector Fields. Differentiation and Integration of Integral with Respect to Parameter. Improper Integrals

§ 13.1. Line Integral of the First Type

Let us consider the three-dimensional (geometrical) space E with rectangular coordinates (x, y, z) in which is defined a continuous piecewise smooth curve Γ described by parametric equations

$$x = \varphi(s),\quad y = \psi(s),\quad z = \chi(x) \qquad (0 \leqslant s \leqslant \Lambda) \tag{1}$$

where the parameter s is the arc length. This means that the functions φ, ψ and χ are continuous on the closed interval $[0, \Lambda]$ and that this interval can be split into a finite number of parts with the aid of points of division $0 = s_0 < s_1 < \ldots < s_N = \Lambda$ so that the functions φ, ψ and χ possess continuous derivatives on each (closed) subinterval $[s_j, s_{j+1}]$ which satisfy the condition

$$\varphi'(s)^2+\psi'(s)^2+\chi'(s)^2 = 1 \tag{2}$$

The values of $\varphi'(s)$, $\psi'(s)$ and $\chi'(s)$ at the end points s_j and s_{j+1} are understood as the corresponding one-sided (i.e. right-hand or left-hand) derivatives. The partition of the interval $[0, \Lambda]$ induces the corresponding partition of the curve Γ which splits into the finite number N of smooth parts Γ_j: $\Gamma = \sum\limits_1^N \Gamma_j$. Let us also consider a function $F(x, y, z)$ defined at the points of Γ or on a wider set containing Γ which is continuous on each smooth part Γ_j; this condition means that the points of discontinuity of the function $F(\varphi(s), \psi(s), \chi(s))$ (provided there are such) can only be at some of the points s_j and are of the first kind.

The line integral of the first type of the function F over the curve Γ is defined as the expression

$$\int\limits_\Gamma F(x, y, z)\, ds = \int\limits_0^\Lambda F(\varphi(s), \psi(s), \chi(s))\, ds \tag{3}$$

The left-hand member of (3) is simply the notation of the line integral of the first type while the right-hand member is an ordinary Riemann integral; this integral shows how the line integral can be computed.

If, for instance, Γ is a "material curve" whose mass is distributed with linear density assuming the values $F(x, y, z)$ at the points $(x, y, z) \in \Gamma$, the total mass of the curve is equal to integral (3).

The curve Γ can also be specified by the parametric equations $x = \varphi(\Lambda - s)$, $y = \psi(\Lambda - s)$ and $z = \chi(\Lambda - s)$ $(0 \leqslant s \leqslant \Lambda)$ but this does not alter the value of integral (3):

$$\int_0^\Lambda F(\varphi(\Lambda-s), \psi(\Lambda-s), \chi(\Lambda-s))\, ds = -\int_\Lambda^0 F(\varphi(s'), \psi(s'), \chi(s'))\, ds' =$$

$$= \int_0^\Lambda F(\varphi(s), \psi(s), \chi(s))\, ds$$

§ 13.2. Line Integral of the Second Type

Let us consider an oriented continuous piecewise smooth curve Γ with initial point A_0 and terminal point A_1 lying in the space E with rectangular coordinates (x, y, z). If the curve Γ is closed, the point A_1 coincides with A_0. Let

$$x = \varphi(s), \;\; y = \psi(s), \;\; z = \chi(s) \qquad (0 \leqslant s \leqslant \Lambda) \tag{1}$$

be parametric equations specifying Γ where s is the arc length reckoned along Γ (see § 10.3). It is meant that to the value $s = 0$ of the parameter s there corresponds the point A_0 and to the value $s = \Lambda$ the point A_1 and that the direction of increase of s is coherent with the orientation of Γ.

At each interior point A of every smooth part of Γ (such a point A must not be a corner point of Γ) is uniquely determined unit tangent vector $\boldsymbol{\tau}$ to Γ (drawn in the direction of increase of s).

We shall also suppose that there is a vector (or, more precisely, a continuous vector field)

$$\boldsymbol{a} = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$$

defined on Γ or on an open set Ω containing Γ where P, Q and R are continuous functions defined on Γ (or on Ω).

The line integral of the vector $\boldsymbol{a}$ over the oriented curve Γ is defined as the magnitude

$$\int_\Gamma (\boldsymbol{a}\, d\boldsymbol{s}) = \int_\Gamma (P\, dx + Q\, dy + R\, dz) = \int_\Gamma (\boldsymbol{a\tau})\, ds = \int_0^\Lambda (\boldsymbol{a\tau})\, ds \tag{2}$$

The first three members in (2) are symbols denoting the line integral of $\boldsymbol{a}$ over the oriented curve Γ while the fourth member is an ordinary Riemann integral defining the line integral and showing how it can be computed. In the general case the function $(\boldsymbol{a\tau})$ can be piecewise smooth, its discontinuities being of the first kind and lying at some of the corner points of Γ.

If the same curve Γ is oriented oppositely its unit tangent vector is equal to $-\boldsymbol{\tau}$; denoting the contour as Γ_- we thus have $\int\limits_{\Gamma_-}(\boldsymbol{a}\,d\boldsymbol{s}) = -\int\limits_{\Gamma}(\boldsymbol{a}\,d\boldsymbol{s})$. Since $\cos(\tau, x) = \varphi'(s)$, $\cos(\tau, y) = \psi'(s)$ and $\cos(\tau, z) = \chi'(s)$, line integral (2) can also be written in the form

$$\int\limits_{\Gamma}(\boldsymbol{a}\,d\boldsymbol{s}) = \int\limits_{\Gamma}\{P\cos(\boldsymbol{\tau}, x)+Q\operatorname{ccs}(\boldsymbol{\tau}, y)+R\cos(\boldsymbol{\tau}, z)\}\,ds =$$

$$= \int\limits_{0}^{\Lambda}[P(\varphi(s), \psi(s), \chi(s))\,\varphi'(s)+Q(\varphi(s), \psi(s), \chi(s))\,\psi'(s)+ + R(\varphi(s), \psi(s), \chi(s))\,\chi'(s)]\,ds \quad (3)$$

where the right-hand side is an ordinary definite integral.

The oriented smooth curve Γ can also be specified with the aid of an arbitrary parameter t by means of equations

$$x = \varphi_1(t),\quad y = \psi_1(t),\quad z = \chi_1(t) \qquad (t_0 \leqslant t \leqslant T_0) \quad (4)$$

where $t = \lambda(s)$ is a function possessing a continuous derivative $\lambda'(s) > 0$ on $[0, \Lambda]$. Then integral (2) is computed according to the formula

$$\int\limits_{\Gamma}(\boldsymbol{a}\,d\boldsymbol{s}) = \int\limits_{t_0}^{T_0}[P(\varphi_1(t), \psi_1(t), \chi_1(t))\,\varphi_1'(t)+Q(\varphi_1(t), \psi_1(t), \chi_1(t))\,\psi_1'(t)+ + R(\varphi_1(t), \psi_1(t), \chi_1(t))\,\chi_1'(t)]\,dt \quad (5)$$

We have performed the change of variable $t = \lambda(s)$ in the definite integral on the right-hand side of (3). According to this change we have

$$\varphi'(s)\,ds = \left(\varphi_1'(t)\frac{dt}{ds}\right)\left(\frac{ds}{dt}\,dt\right) = \varphi_1'(t)\,dt$$

The second expression in (2) is a convenient symbol for the notation of the line integral of $\boldsymbol{a}$ over the oriented curve Γ. It is also called *the line integral of the second type*. This expression not only symbolizes the integral but also shows in which way it can be computed. Namely, it means that the curve Γ should be represented by equations (4) with a parameter t *the direction of whose increase should be coherent with the orientation of* Γ, after which x, y, z, dx, dy and dz are replaced, respectively, by $\varphi_1(t)$, $\psi_1(t)$, $\chi_1(t)$, $\varphi_1'(t)\,dt$, $\psi_1'(t)\,dt$ and $\chi_1'(t)\,dt$ and the definite integral of the resultant function of the variable t over the closed interval $[t_0, T_0]$ is computed.

The oriented curve Γ can be represented (in infinitely many ways) as a sum of two oriented curves Γ_1 and Γ_2 specified by the same equations (1) (or (3)) and corresponding to the variation of the parameter s over closed intervals $[0, s_*]$ and $[s_*, \Lambda]$, $0 < s_* < \Lambda$. Then, obviously,

$$(P\,dx+Q\,dy+R\,dz) = \int\limits_{\Gamma_1}(P\,dx+Q\,dy+R\,dz)+\int\limits_{\Gamma_2}(P\,dx+Q\,dy+R\,dz)$$

If the oriented curve Γ is a closed contour the line integral of $\boldsymbol{a}$ taken round it is also called the *circulation of the vector (vector field)* $\boldsymbol{a}$ *over* Γ.

It is sometimes convenient to regard as the contour of integration the union $C = C_1 + \ldots + C_m$ of several disjoint oriented contours $C_1, \ldots C_m$, the contour C being also understood as an oriented curve. Then, by definition, the integral of $\boldsymbol{a}$ over C is equal to the sum of the integrals of $\boldsymbol{a}$ over the constituent contours C_k:

$$\int_C = \sum_1^m \int_{C_k}$$

Formula (5) not only applies to a smooth but also to a piecewise smooth continuous curve (4) for in this case Γ is a finite union of smooth oriented parts Γ_j corresponding to intervals $[s_j, s_{j+1}]$ (or $[t_j, t_{j+1}]$) of variation of the arc length s (or of the parameter t), and consequently

$$\int_\Gamma (\boldsymbol{a}\, d\boldsymbol{s}) = \sum_1^N \int_{\Gamma_j} (\boldsymbol{a}\, d\boldsymbol{s}) = \sum_1^N \int_{t_j}^{t_{j+1}} = \int_0^{T_0}$$

where by the element of integration in the integral on the right-hand side is meant the same expression as the one in the integral on the right-hand side of (5).

Finally, it should be noted that if $\boldsymbol{a}$ is a field of force, the line integral of $\boldsymbol{a}$ over Γ is obviously equal to the work of the field $\boldsymbol{a}$ performed in the motion along the oriented path Γ.

§ 13.3. Potential of a Vector Field

An important special case of a vector field $\boldsymbol{a}$ is the one for which there is a function $U(x, y, z)$ defined in the domain G where the field is considered and possessing continuous partial derivatives for which the equalities

$$\frac{\partial U}{\partial x} = P, \quad \frac{\partial U}{\partial y} = Q \quad \text{and} \quad \frac{\partial U}{\partial z} = R \qquad (\text{on } G)$$

are fulfilled. Such a function U (determined to within an arbitrary constant) is called a *potential* (or a *potential function*) *of the vector* $\boldsymbol{a}$. We see that in this case *the vector* $\boldsymbol{a}$ *is the gradient* (see § 7.6) *of the function* U:

$$\operatorname{grad} U = \frac{\partial U}{\partial x}\boldsymbol{i} + \frac{\partial U}{\partial y}\boldsymbol{j} + \frac{\partial U}{\partial z}\boldsymbol{k} = \boldsymbol{a}$$

Let us prove the following theorem.

Theorem 1. *Let a continuous vector field* $\boldsymbol{a}$ *be defined in a domain* $G \subset E$. *Then the following properties are equivalent:*

(i) *There is a one-valued function* $U = U(x, y, z)$ *possessing continuous partial derivatives for which the equality* grad $U = \boldsymbol{a}$ *holds at each point of* G.

(ii) *The integral of* $\boldsymbol{a}$ *over any closed continuous piecewise smooth contour C lying in G is equal to zero:*

$$\int_C (\boldsymbol{a}\, d\boldsymbol{s}) = 0$$

(iii) *If* A_0 *is a definite fixed point in G, the integral* $\int\limits_{C_{A_0A}} (\boldsymbol{a}\, d\boldsymbol{s})$ *over any oriented piecewise smooth curve* $C_{A_0A} \subset G$ *with initial point* A_0 *and terminal point A depends solely on* A_0 *and A and is independent of the shape of the curve. This means that*

$$\int\limits_{C_{A_0A}} (\boldsymbol{a}\, d\boldsymbol{s}) = V(A) = V(x, y, z)$$

for any fixed point A_0.

The function $V(x, y, z)$ *is a potential function of the vector* $\boldsymbol{a}$ *on G, that is it differs from U by a constant.*

Proof. Implication (i) → (iii). Let there be a function U on G serving as a potential of $\boldsymbol{a}$.

Denoting by $A_0 = (x_0, y_0, z_0)$ a definite point of G and by $A = (x, y, z)$ the variable point we join A_0 to A by a continuous piecewise smooth curve $C = C_{A_0A}$ specified by equations

$$x = \varphi(\tau),\quad y = \psi(\tau),\quad z = \chi(\tau) \qquad (t_0 \leqslant \tau \leqslant t)$$

Thus, to the values t_0 and t of the parameter τ there correspond the points A_0 and A respectively.

On replacing x, y and z in U by the functions φ, ψ and χ respectively we obtain a continuous piecewise smooth function of the variable τ (for which we retain the notation U). By virtue of the theorem on the derivative of a composite function, we have

$$\frac{\partial U}{d\tau} = \frac{\partial U}{\partial x}\frac{d\varphi}{d\tau} + \frac{\partial U}{\partial y}\frac{d\psi}{d\tau} + \frac{\partial U}{\partial z}\frac{d\chi}{d\tau}$$

at the points of smoothness (i. e. at the points at which the tangent exists) of the curve C. It follows that

$$\int_C (P\,dx + Q\,dy + R\,dz) = \int_{t_0}^{t} \frac{dU}{d\tau}\,d\tau = U(\varphi(t), \psi(t), \chi(t)) - U(\varphi(t_0), \psi(t_0), \chi(t_0)) =$$

$$= U(x, y, z) - U(x_0, y_0, z_0) = U(A) - U(A_0) = V(A)$$

which shows that, for a fixed initial point A_0, the line integral is dependent solely on the position of the point $A \in G$ and is independent of the shape of the path connecting A_0 and A.

Implication (iii) → (i). Let us choose a point $A_0 \in G$ and fix it. Suppose that it is known that the given field of a vector $\boldsymbol{a}$ is such that the line integral over any piecewise smooth curve connecting A_0 with an arbitrary point $A \in G$ is independent of the shape of the curve and is dependent solely on the point A. This means that there is a one-valued function $V(A)$ such that

$$\int_{C_{A_0A}} (P\,dx+Q\,dy+R\,dz) = V(A) = V(x, y, z)$$

In order to prove that $\frac{\partial V}{\partial x} = P$ at an arbitrary point $A_1 = (x_1, y_1, z_1)$ belonging to G we make the following construction. Through the point A_1 is drawn a line segment A_2A_1 lying within G and parallel to the x-axis.

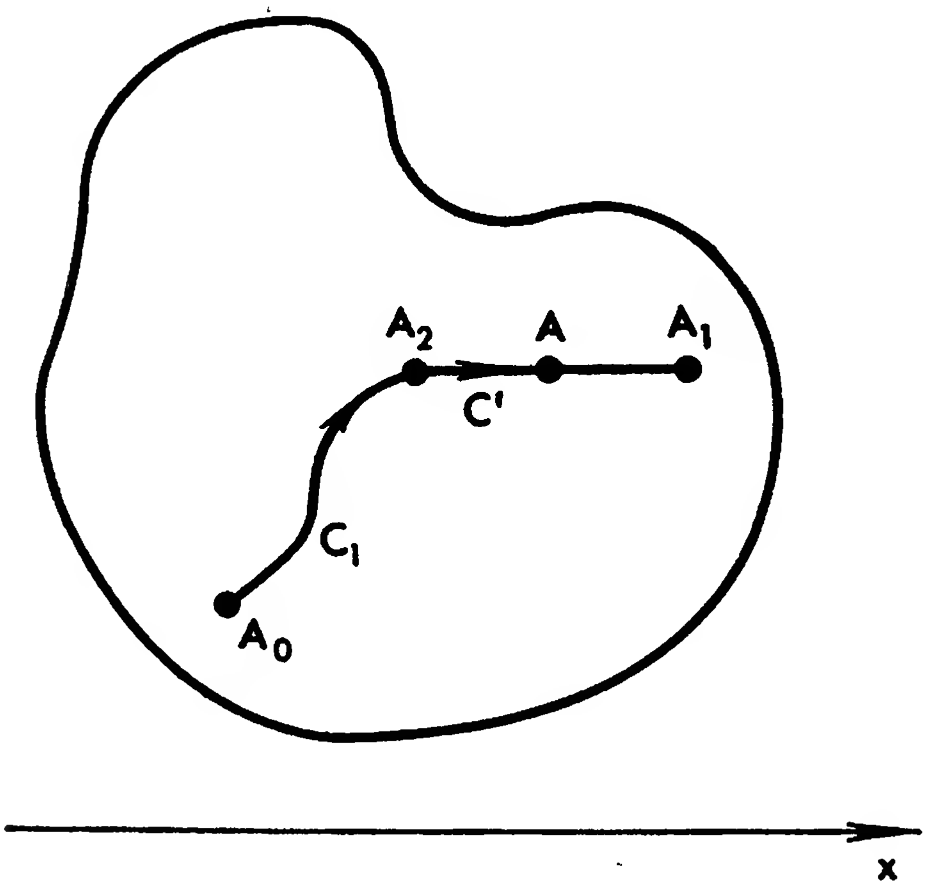

Fig. 13.1

For definiteness, let it be the line segment $y = y_1$, $z = z_1$, $x_2 \leqslant x \leqslant x_1$. Thus, $A_2 = (x_2, y_1, z_1)$. Next, the point A_0 is connected with $A_2 = (x_2, y_1, z_1)$ by an arbitrary oriented continuous piecewise smooth curve C_1 going from A_0 to A_2. Let C' denote the line segment A_2A oriented in the direction from A_2 to $A \in A_2A_1$. Then $C = C_1+C'$ (Fig. 13.1) and

$$V(x, y_1, z_1) = \int_{C_1} (P\,dx+Q\,dy+R\,dz)+\int_{C'} P\,dx \tag{1}$$

since we obviously have $\int_{C'} Q\,dy = \int_{C'} R\,dz = 0$.

In the argument below, the curve C_1 remains unchanged and therefore the first integral on the right-hand side of (1) can be regarded as a constant number; we shall denote it by K. Hence,

$$V(x, y_1, z_1) = K+\int_{x_2}^{x} P(t, y_1, z_1)\,dt$$

The function P is continuous and, in particular, continuous with respect to

x for any fixed y and z; therefore $\frac{\partial V}{\partial x} = P(x, y_1, z_1)$. This equality applies, in particular, to the point $x = x_1$, which proves the desired equality. The equalities

$$\frac{\partial V}{\partial y} = Q \quad \text{and} \quad \frac{\partial V}{\partial z} = R$$

are proved similarly by constructing curves connecting the points A_0 and A_1 whose terminal parts adjoining A_1 are, respectively, line segments parallel to the y-axis and to the z-axis. To complete the proof of (i) it only remains to put $U = V$.

The equivalence of (ii) and (iii) is quite obvious. Indeed, let (ii) take place and let $C' = C'_{A_0A}$ and $C'' = C''_{A_0A}$ be two paths lying within G and joining the points A_0 and A. Then $C'+C''_-$ is a closed contour and we have

$$0 = \int_{C'} + \int_{C''_-} = \int_{C'} - \int_{C''}$$

that is (iii) holds. Conversely, if (iii) takes place and $C \subset G$ is a closed contour we can represent C in the form $C = C'+C''$ where C' and C'' are some contours and write

$$\int_{C} = \int_{C'} + \int_{C''} = \int_{C'} - \int_{C''_-} = 0$$

because C' and C''_- join the same pair of points.

If a vector field

$$\boldsymbol{a} = P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k}$$

defined in an open set G is not only continuous but also possesses continuous partial derivatives we can consider the vector

$$\text{rot } \boldsymbol{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

which is called the *rotation* (or *curl*) *of the vector* $\boldsymbol{a}$.

If the vector $\boldsymbol{a}$ possesses one of the properties (ii) and (iii) mentioned in the foregoing theorem then, according to that theorem, there is a one-valued function $U(x,y,z)$ (a potential function of $\boldsymbol{a}$) on G possessing continuous partial derivatives such that $\frac{\partial U}{\partial x} = P$, $\frac{\partial U}{\partial y} = Q$ and $\frac{\partial U}{\partial z} = R$. In this case if the functions P, Q and R themselves have continuous partial derivatives in G, the function U possesses continuous partial derivatives of the second order and the following equalities hold:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial^2 U}{\partial x\, \partial y} - \frac{\partial^2 U}{\partial y\, \partial x} = 0$$

$$\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = \frac{\partial^2 U}{\partial y\, \partial z} - \frac{\partial^2 U}{\partial z\, \partial y} = 0$$

$$\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = \frac{\partial^2 U}{\partial z\, \partial x} - \frac{\partial^2 U}{\partial x\, \partial z} = 0$$

We have thus arrived at the following theorem.

Theorem 2. *If a field of a vector* $\boldsymbol{a}$ *having continuous partial derivatives in an open set* G *possesses the property that*

$$\int\limits_{C} (\boldsymbol{a}\, d\boldsymbol{s}) = 0 \tag{2}$$

for any oriented peicewise smooth closed contour $C \subset G$ *then*

$$\operatorname{rot} \boldsymbol{a} = \boldsymbol{0} \quad \text{on} \quad G \tag{3}$$

The converse of this theorem does not hold for an arbitrary open set G even when the set G is connected. On the other hand, it is sure to hold when G is a rectangular parallelepiped of the form $G = \{a_1 \leqslant x \leqslant b_1,\ a_2 \leqslant y \leqslant b_2,\ a_3 \leqslant z \leqslant b_3\}$. Indeed, in this case, given a continuously differentiable vector field $\boldsymbol{a}$ defined in G for which $\operatorname{rot} \boldsymbol{a} = \boldsymbol{0}$, the potential function U is effectively constructed with the aid of the formula

$$U(x, y, z) = \int\limits_{x_0}^{x} P(u, y_0, z_0)\, du + \int\limits_{y_0}^{y} Q(x, v, z_0)\, dv +$$

$$+ \int\limits_{z_0}^{z} R(x, y, v)\, dv + U(x_0, y_0, z_0) \qquad ((x, y, z) \in G) \tag{4}$$

where $(x_0, y_0, z_0) \in G$ is an arbitrary fixed point and $U(x_0, y_0, z_0)$ is an arbitrary constant. To verify this we differentiate (4) with respect to x to obtain (see the explanation below)

$$\frac{\partial U}{\partial x} = P(x, y_0, z_0) + \int\limits_{y_0}^{y} \frac{\partial Q}{\partial x}(x, v, z_0)\, dv + \int\limits_{z_0}^{z} \frac{\partial R}{\partial x}(x, y, w)\, dw =$$

$$= P(x, y_0, z_0) + \int\limits_{y_0}^{y} \frac{\partial P}{\partial y}(x, v, z_0)\, dv + \int\limits_{z_0}^{z} \frac{\partial P}{\partial z}(x, y, w)\, dw =$$

$$= P(x, y_0, z_0) + [P(x, y, z_0) - P(x, y_0, z_0)] + [P(x, y, z) - P(x, y, z_0)] =$$

$$= P(x, y, z)$$

Here we have applied the Newton-Leibniz formula, and property (iii); besides, we have performed the differentiation under the integral sign. The latter operation is legitimate in the case under consideration, which will be proved in § 13.12. The relations $\frac{\partial U}{\partial y} = Q$ and $\frac{\partial U}{\partial z} = R$ are proved analogously. Thus, $\operatorname{grad} U = \boldsymbol{a}$, and consequently equality (2) holds for any oriented (closed) contour $C \subset G$.

Note that the right-hand side of (4) taken without the last term is the line integral of the vector $\boldsymbol{a}$ over the polygonal line with vertices at the points (x_0, y_0, z_0), (x, y_0, z_0), (x, y, z_0) and (x, y, z).

There also holds the more general

Theorem 3. *Let G be a simply connected domain, that is such that any closed piecewise smooth contour belonging to it can be contracted to a point $P^0 \subset G$ so that it always remains within G during the contraction process.*[*] *Then if a continuously differentiable vector field* $\boldsymbol{a}$ *defined in G satisfies the condition* rot $\boldsymbol{a} = \mathbf{0}$, *equality* (2) *holds for any oriented closed contour $C \subset G$.*

The proof of the theorem will be given at the end of the section.

Thus, the conditions of Theorem 3 imply the existence of a one-valued potential function of the vector $\boldsymbol{a}$ in G.

As an example of a domain satisfying the condition of the theorem we can take the one lying between two concentric spherical surfaces. The domain obtained from the entire space by deleting the points of the z-axis does not satisfy this condition. In the latter case it is possible to construct an example (see below) of a field of a vector for which Theorem 3 does not hold.

Later in this chapter we shall prove the theorems of Green and Stokes. They provide a plausible explanation of the validity of Theorem 3 (see the remark at the end of § 13.11) which cannot however be considered a rigorous proof.

All the notions and theorems we have dealt with can readily be extended to the case of the plane. Let E be the xy-plane and let

$$x = \varphi(t), \quad y = \psi(t) \qquad (a \leqslant t \leqslant b;\ a < b)$$

be equations of an arbitrary oriented piecewise smooth curve C belonging to a given domain $G \subset E$. Also let a vector field $\boldsymbol{a} = P(x, y)\boldsymbol{i}+Q(x, y)\boldsymbol{j}$ be defined in G.

The line integral of $\boldsymbol{a}$ over the curve C is defined in exactly the same way as in the three-dimensional case. It can be considered as a special case of the integral defined in § 13.2 (3) if we put

$$R \equiv 0, P = P(x, y) \quad \text{and} \quad Q = Q(x, y)$$

Thus,

$$\int_C (\boldsymbol{a}\, d\boldsymbol{s}) = \int_C (P\,dx+Q\,dy) = \int_a^b = \{P[\varphi(t), \psi(t)]\,\varphi'(t)+Q[\varphi(t), \psi(t)]\,\psi'(t)\}\,dt$$

In this case a potential function U of the vector $\boldsymbol{a}$, provided it exists, is a one-valued function $U = U(x, y)$ of the two variables x and y defined in G. Its gradient is the vector

$$\text{grad } U = \frac{\partial U}{\partial x}\boldsymbol{i}+\frac{\partial U}{\partial y}\boldsymbol{j} = \boldsymbol{a}$$

Example 1. The vector $\boldsymbol{a}$ in the xy-plane having the components

$$P(x, y) = -\frac{y}{x^2+y^2} \quad \text{and} \quad Q(x, y) = \frac{x}{x^2+y^2}$$

---

[*] A rigorous mathematical description of such a "contraction" of a contour to a point is given at the end of the present section

possesses continuous partial derivatives in the domain G obtained from the xy-plane by deleting the origin. It can also be easily checked that rot $\boldsymbol{a} = 0$ in G. The domain G does not satisfy the conditions of Theorem 3 and the theorem itself fails to hold in this case.

Indeed, let G^* denote the domain obtained from the xy-plane by deleting the negative half-axis x, that is the ray $x < 0$. According to Theorem 3, there is a function U defined on G^* such that grad $U = \boldsymbol{a}$. For instance, it can be defined at the moving point (x, y) as the line integral of $\boldsymbol{a}$ over an arbitrary path $C \subset G^*$ connecting a fixed point, say $(1, 0)$, with (x, y):

$$U(x, y) = \int\limits_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

However, *this function cannot be extended from G^* to the whole xy-plane so that it remains one-valued and continuous.*

Indeed, the value of $U(x, y)$ at an arbitrary point $(\cos\theta, \sin\theta) \in G^*$ lying on the circle of unit radius with centre at the origin is equal to

$$U(\cos\theta, \sin\theta) = \int\limits_0^\theta d\theta = \theta$$

The variable point can be moved along the unit circle from the initial point $(1, 0)$ to the point $(-1, 0)$ (lying on the deleted ray) so that θ increases from 0 to π or decreases from 0 to $-\pi$. In the first case the limiting value of U is equal to π and in the second case to $-\pi$, which shows that the function U cannot be extended to the whole plane in the desired way.

Since any function serving as a potential of $\boldsymbol{a}$ in the domain G only differs by a constant from the function $U(x, y)$ we have constructed, our consideration means that there exists no one-valued function defined on G which is a potential of the vector $\boldsymbol{a}$ in question everywhere on G.

Here we have used a comparatively lengthy argument which can in fact be replaced by a shorter one; namely, we can simply show that there exist closed smooth contours lying within G such that the integrals of the vector $\boldsymbol{a}$ taken round them are different from zero. For example, as a contour of this kind we can take the circle C of radius 1 with centre at the origin; for this contour we have $\int\limits_C (\boldsymbol{a}\,ds) = \int\limits_0^{2\pi} d\theta = 2\pi$. This means that there cannot exist a one-valued function U defined in G serving as a potential function of $\boldsymbol{a}$ (everywhere on G) because its existence would contradict Theorem 1.

Proof of Theorem 3. The proof below is based on the fact that the theorem holds in the case when G is a cube.

Let us consider an arbitrary closed piecewise smooth contour $\Gamma \subset G$ specified by equations

$$x = \varphi(u), \quad y = \psi(u), \quad z = \chi(u), \qquad 0 \leqslant u \leqslant 1$$

$$\varphi(0) = \varphi(1), \quad \psi(0) = \psi(1), \quad \chi(0) = \chi(1)$$

Here the parameter u runs over the closed interval $[0, 1]$, which obviously does not violate the generality. The fact that the contour Γ can be contracted to a point (see the conditions of Theorem 3) is described as follows: there is a surface $S \subset G$ represented by functions

$$\left.\begin{aligned} &x = \varphi(u, t),\ \ y = \psi(u, t),\ \ z = \chi(u, t)\\ &\{0 \leqslant u \leqslant t,\ \ 0 \leqslant t \leqslant 1\} = \Delta\\ &\varphi(0, t) = \varphi(t, t),\ \ \psi(0, t) = \psi(t, t),\ \ \chi(0, t) = \chi(t, t) \end{aligned}\right\} \tag{5}$$

which are continuous in the triangle Δ, piecewise smooth relative to u on $[0, t]$ and such that

$$\varphi(u, 1) = \varphi(u),\ \ \psi(u, 1) = \psi(u),\ \ \chi(u, 1) = \chi(u)$$

Since the surface S is bounded and closed and the set G is open and $S \subset G$, there is a number $d > 0$ such that, given an arbitrary point $A \in S$, a cube with edge of length d covering A belongs to G.

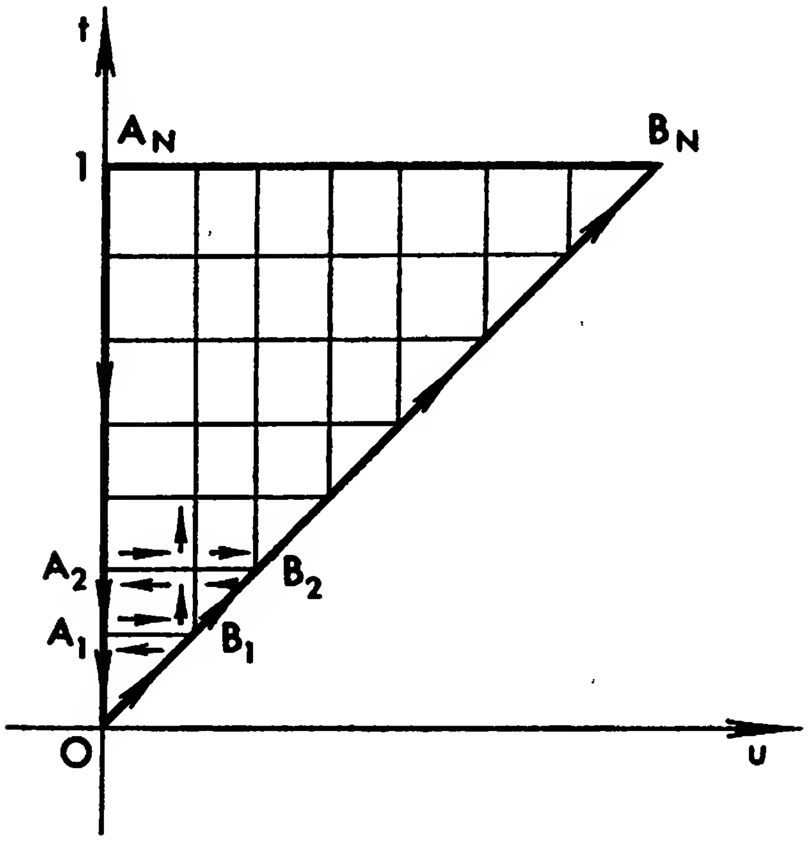

Fig. 13.2

Let us denote by σ the cubes belonging to G.

Given a set $e \subset \Delta$, we shall regard as the image of e the set $e' \subset S$ consisting of all the points obtained by the mapping of the points of e with the aid of the three functions (5).

Let us partition Δ with the aid of a rectangular network (Fig. 13.2) so that the images of the cells obtained in the partition can be embedded in cubes σ with edge d.

The image of the lowermost triangle A_1B_1O belongs to some σ. Since the images of the points A_1 and B_1 coincide, that is they are at one and the same point of S, the image of the line segment A_1B_1 is a closed piecewise smooth curve $C_{A_1B_1}$ belonging to the cube σ; therefore

$$\int\limits_{C_{A_1B_1}} (\boldsymbol{a}\, d\boldsymbol{s}) = 0 \tag{6}$$

Let us supply all the cells lying between A_1B_1 and A_2B_2 (there are in fact exactly two of them; see Fig. 13.2) with coherent orientations. The integral over the image of the contour of this complex is equal to the sum of the integrals taken over the images of the boundaries of the constituent cells, each of these integrals being equal to zero. The integrals over the line segments along which neighbouring cells adjoin each other mutually cancel out and, besides, there hold (6) and the equality

$$\int_{C_{A_1A_2}} = \int_{C_{B_1B_2}}$$

since the images of A_1A_2 and B_1B_2 coincide. Therefore $\int_{C_{A_2B_2}} = 0.$

Arguing in a similar way we conclude, by induction, that

$$\int_{C_{A_kB_k}} = 0 \qquad (k = 1, \ldots, N)$$

and, since $C_{A_NB_N} = \Gamma$, it follows that $\int_{\Gamma} (\mathbf{a}\, d\mathbf{s}) = 0$, which is what we set out to prove.

§ 13.4. Orientation of a Domain in the Plane

There are two different types of rectangular coordinates in a plane such as those shown in Figs. 13.3 and 13.4. The essential distinction between them is that these systems, regarded as rigid bodies, cannot be made coincident with the aid of a motion in the plane so that their positive x- and y-axes go in the same directions respectively.

Let us construct in both coordinate systems circles with centres at the origins (see the figures). Next, for each of the circles we choose a positive directions of traversing its circumference so that the variable point covers the shortest path when moving from the positive x-axis to the positive y-axis (that is it covers a quarter circle, not three quarters). This means that the circle is traversed in the counterclockwise direction in the case of Fig. 13.3 and in the clockwise direction in the case of Fig. 13.4.

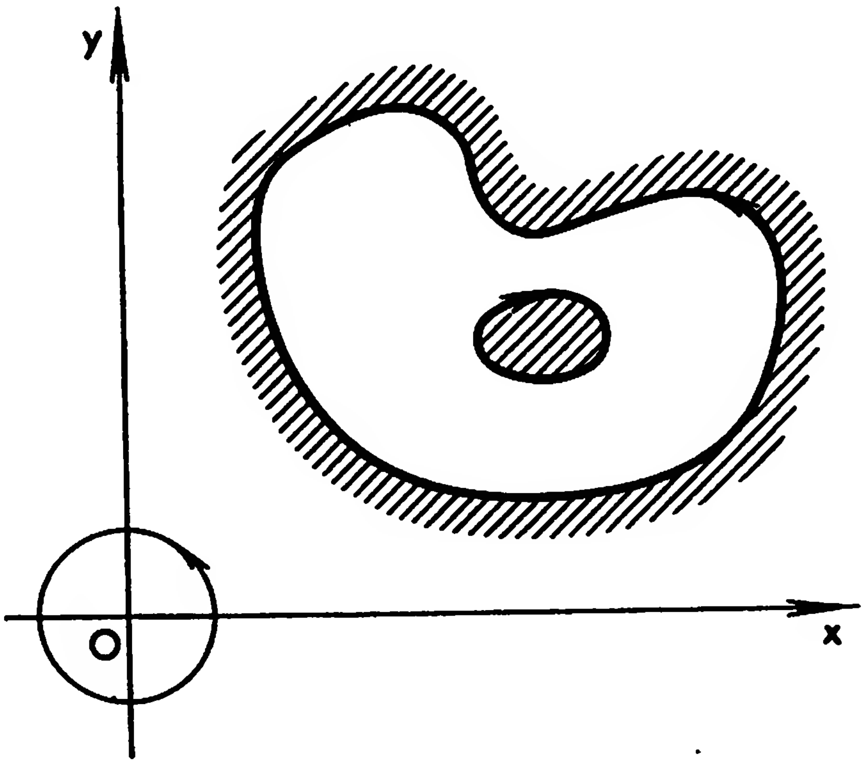

Fig. 13.3.

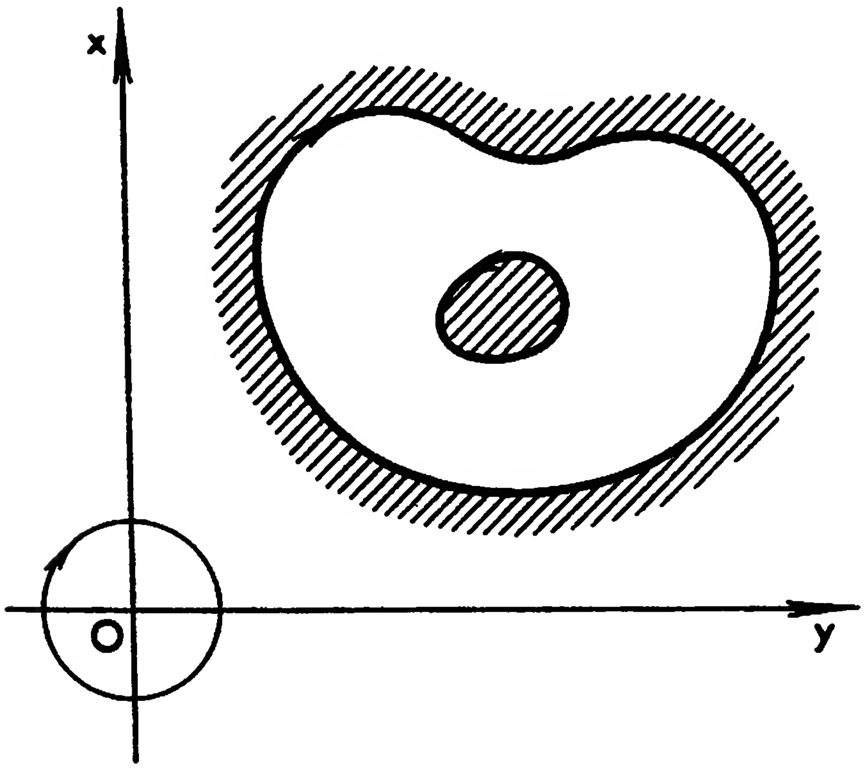

Fig. 13.4

In the former case the interior of the circle is kept on the left of a person walking round its circumference in the positive direction and in the latter case it is kept on the right. From these cases we now proceed to further generalizations.

Let us consider a domain Ω with a piecewise smooth boundary C which may consist of a finite number of closed contours not intersecting themselves such that Ω is contained inside one of them and outside all the others. Let us choose a direction of motion along C such that when we describe C in this direction the domain Ω is kept on the left (see Fig. 13.3). This direction will be called *positive* in the case of a coordinate system of the type shown in Fig. 13.3 and *negative* in the case shown in Fig. 13.4. In the latter case the positive direction of describing the boundary C of the domain Ω is the one for which Ω is kept on the right.

§ 13.5. Green's* Formula. Computing Area with the Aid of Line Integral

In this section we shall prove *Green's formula*

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\Gamma} (P\, dx + Q\, dy) \tag{1}$$

which applies to any domain Ω in the plane (with a positively oriented boundary Γ) satisfying some general conditions. It is supposed that the functions $P, Q, \frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are continuous on $\bar{\Omega}$.

We shall begin with the derivation of Green's formula for a rectangular domain

$$\Delta = \{a < x < b;\ c < y < d\}$$

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* G. Green (1793-1841), a British mathematician. An alternative derivation of Green's formula will be presented in § 13.10.

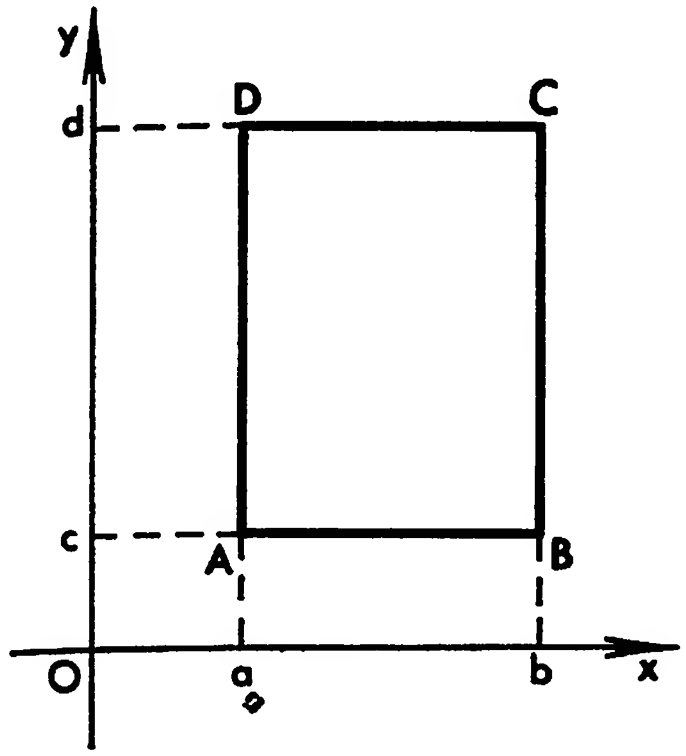

Fig. 13.5

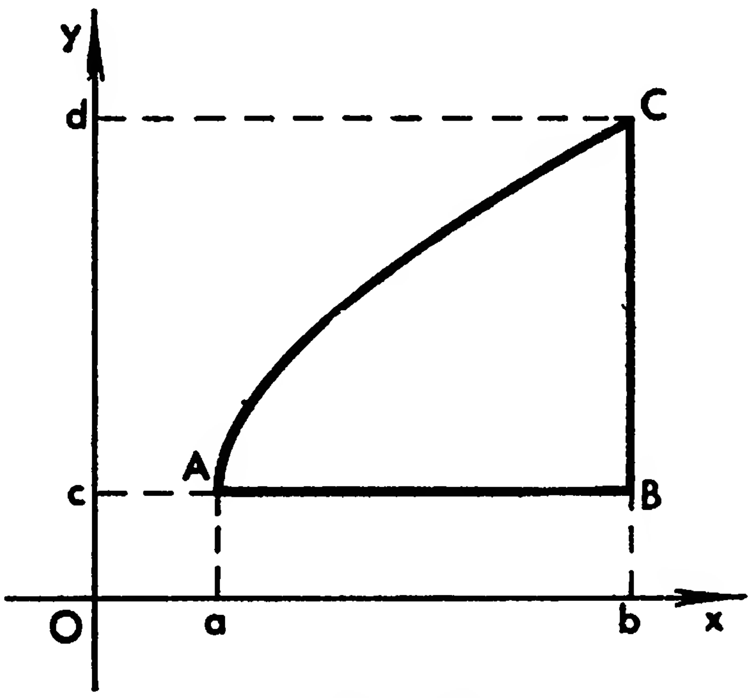

Fig. 13.6

(see Fig. 13.5). In this case we have

$$\iint_{\Delta} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d\int_a^b \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b,y)-Q(a,y)]\,dy =$$
$$= \left(\int_{\Gamma_{BC}} + \int_{\Gamma_{DA}}\right) Q(x,y)\,dy = \int_{\Gamma} Q(x,y)\,dy$$

and

$$-\iint_{\Delta} \frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b [P(x,d)-P(x,c)]\,dx = \left(\int_{\Gamma_{CD}} + \int_{\Gamma_{AB}}\right) P(x,y)\,dx =$$
$$= \int_{\Gamma} P(x,y)\,dx$$

which completes the proof of formula (1) for the domain Δ.

Now we shall prove (1) for the domain ω depicted in Fig. 13.6 where the arc $\overset{\frown}{AC}$ is described by a function $y = \lambda(x)$ continuous and strictly increasing on $[a, b]$. The inverse function of $y = \lambda(x)$ will be denoted $x = \mu(y)$ $(c \leqslant y \leqslant d)$.

We have

$$\iint_{\omega} \frac{\partial Q}{\partial x}\,dx\,dy = \int_c^d [Q(b,y)-Q(\mu(y),y)]\,dy = \left(\int_{\Gamma_{BC}} - \int_{\Gamma_{AC}}\right) Q(x,y)\,dy =$$
$$= \int_{\Gamma} Q(x,y)\,dy$$

and

$$-\iint_{\omega} \frac{\partial P}{\partial y}\,dx\,dy = -\int_a^b \int_c^{\lambda(x)} \frac{\partial P}{\partial y}\,dy\,dx = -\int_a^b [P(x,\lambda(x))-P(x,c)]\,dx =$$
$$= \left(\int_{\Gamma_{CA}} + \int_{\Gamma_{AB}}\right) P(x,y)\,dx = \int_{\Gamma} P(x,y)\,dx$$

whence follows (1).

On turning the domain ω as a rigid body about the origin through angles of $\pi/2$, π and $3\pi/2$ radians (the coordinate system remains unchanged) we obtain three more domains which, together with the original domain ω, will be referred to as *sets of the type of* ω. Any rectangle will be called a *domain of the type of* ω. By analogy with the above, it can readily be proved that Green's formula is valid for any set of type ω.

Now we proceed to the general case.

Theorem. *Let a domain* Ω *with a continuous piecewise smooth boundary* Γ *possess the property that its closure* $\bar{\Omega}$ *can be cut with the aid of straight lines parallel to the coordinate axes* Ox *and* Oy *into a finite number of parts each of which is a set of the type of* ω. *Then Green's formula holds for* Ω.

Proof. Let $\bar{\Omega} = \sum_{1}^{N} \omega_k$ be a partition of $\bar{\Omega}$ into parts of the type of ω and let Γ_k $(k = 1, \ldots, N)$ be the positively oriented contour bounding the part ω_k. Then, since Green's formula applies to each of the domains ω_k, we obtain

$$\iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \sum_{k=1}^{N} \iint_{\omega_k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \sum_{k=1}^{N} \int_{\Gamma_k} (P\, dx + Q\, dy) =$$

$$= \int_{\Gamma} (P\, dx + Q\, dy)$$

The last of these equalities is explained as follows. The union C of the boundaries of all the parts ω_k consists of Γ and of the sum of a finite number of line segments each of which belongs to Ω and serves as a boundary of two adjoining domains of type ω. Every such line segment is traversed twice in the opposite directions; therefore the line integrals corresponding to these paths of integration mutually cancel out and thus only the integral over Γ remains.

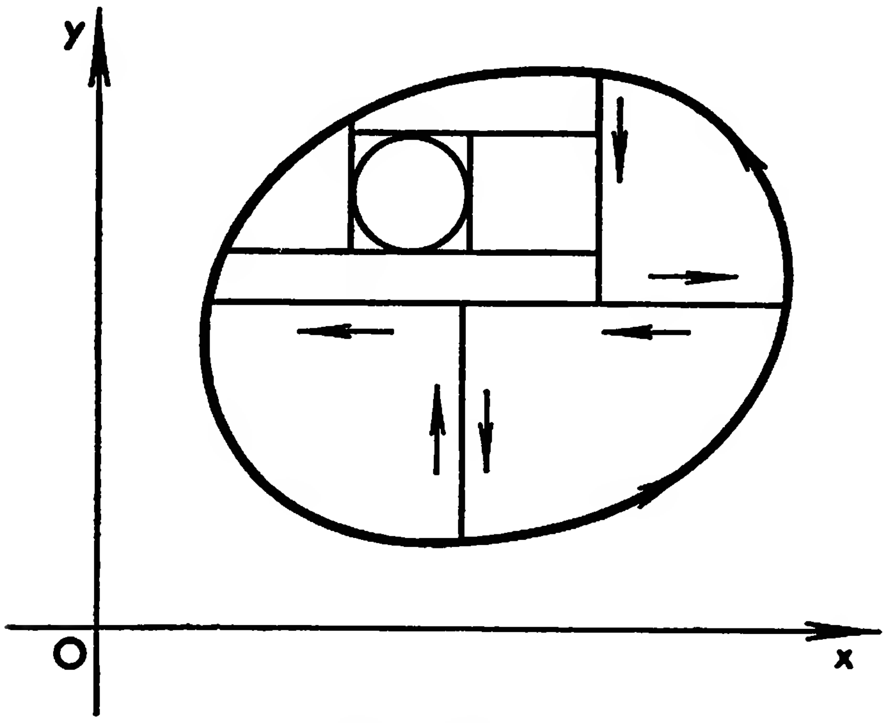

Fig. 13.7

See Fig. 13.7 which shows a (doubly connected) domain split into a finite number of domains of the type of ω.

Remark. In practical problems Green's formula is sometimes applied when the functions P and Q are continuous on $\bar{\Omega}$ while their partial derivatives $\frac{\partial Q}{\partial x}$ and $\frac{\partial P}{\partial y}$ are only continuous on Ω. In such cases Green's formula (1) usually continues to hold if the double integral on its left-hand side is understood in the improper sense. For example, let Ω be the interior of the circle $x^2+y^2=1$. On denoting by Ω_ε the interior of the circle $x^2+y^2=$ $=(1-\varepsilon)^2$ $(\varepsilon>0)$ with boundary Γ_ε (the circle of radius $1-\varepsilon$) we can write, by virtue of what has already been proved, the relation

$$\iint\limits_{\Omega_\varepsilon}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int\limits_{1-\varepsilon}^{-1+\varepsilon}P\left(x,\sqrt{(1-\varepsilon)^2-x^2}\right)dx+$$

$$+\int\limits_{-1+\varepsilon}^{1-\varepsilon}P\left(x,-\sqrt{(1-\varepsilon)^2-x^2}\right)dx+\int\limits_{-1+\varepsilon}^{1-\varepsilon}Q\left(\sqrt{(1-\varepsilon)^2-y^2},y\right)dy+$$

$$+\int\limits_{1-\varepsilon}^{-1+\varepsilon}Q\left(-\sqrt{(1-\varepsilon)^2-y^2},y^2\right)dy\qquad(\varepsilon>0)\qquad(2)$$

Since $P(x,y)$ and $Q(x,y)$ are uniformly continuous on Ω the right-hand side of (2) tends to a limit, as $\varepsilon\to 0$, which is equal to the result of the substitution of $\varepsilon=0$ into it, and therefore the left-hand side tends to the same limit which is thus the improper double integral (with singularities on Γ; see Remark 1 in § 13.13). Hence we obtain

$$\iint\limits_{\Omega}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\int\limits_{\Gamma}P\,dx+Q\,dy$$

Let Ω be a domain in the xy-plane to which Green's formula is applicable and let Γ be its positively oriented boundary. Then the area $|\Omega|$ of Ω is expressed by the formula

$$\frac{1}{2}\int\limits_{\Gamma}(x\,dy-y\,dx)=\frac{1}{2}\iint\limits_{\Omega}(1+1)\,dx\,dy=|\Omega|\qquad(3)$$

which follows directly from Green's formula if we put $P=-y$ and $Q=x$.

It is also evident that there holds the equality

$$\int\limits_{\Gamma_\pm}x\,dy=\pm|\Omega|\qquad(4)$$

in which the plus sign corresponds to the case of the positive orientation of the contour Γ $(\Gamma=\Gamma_+)$ and the minus sign to its negative orientation $(\Gamma=\Gamma_-)$.

Example. The area of the ellipse (more precisely, of the interior of the ellipse) represented by the parametric equations $x=a\cos\theta$, $y=b\sin\theta$ $(0\leqslant\theta<2\pi)$ can easily be found using formula (4):

$$|\Omega|=\frac{1}{2}\int\limits_{0}^{2\pi}[a\cos\theta\,b\cos\theta-b\sin\theta(-a\sin\theta)]\,d\theta=\frac{ab}{2}\int\limits_{0}^{2\pi}d\theta=\pi ab$$

§ 13.6. Surface Integral of the First Type

Let S be a smooth surface represented by a vector equation

$$r(u, v) = \varphi i+\psi j+\chi k \qquad ((u, v) \in \Omega \rightleftarrows S, |\dot{r}_u \times \dot{r}_v| > 0) \tag{1}$$

where Ω is a measurable domain and φ, ψ and χ are continuously differentiable functions on $\bar{\Omega}$.

Suppose that there is a continuous function $F(x, y, z)$ defined on $\bar{S}$ or in a neighbourhood of $\bar{S}$. Let us break up Ω into measurable parts Ω_j so that every two of them can only intersect along some parts of their boundaries. To each part Ω_j there corresponds a definite part S_j of the surface S. Let $A_j = (x_j, y_j, z_j)$ $(j = 1, \ldots, N)$ be an arbitrary point belonging to S_j. On forming the (integral) sum

$$\Pi_N = \sum_{j=1}^{N} F(A_j)\,|S_j|$$

where $|S_j|$ is the area of S_j (see § 12.23) we take the limit

$$\lim_{\max d(\Omega_j)\to 0} \sum_{j=1}^{N} F(A_j)\,|S_j| = \int_S F(x, y, z)\, dS \tag{2}$$

which is called *the surface integral (of the first type) of the function F over the surface S* (provided it exists and is independent of the partition of Ω into the parts Ω_j and of the choice of the points A_j).

If, for instance, S is a "material surface" over which a mass is distributed with (surface) density F, the integral of F over S is equal to the total mass of S.

Let us prove that under the conditions stated integral (2) exists and is independent of the parametric representation of S.

Let x_j, y_j and z_j be the coordinates of A_j $\big(A_j = (x_j, y_j, z_j)\big)$ and

$$x_j = \varphi(u_j, v_j),\ \ y_j = \psi(u_j, v_j),\ \ z_j = \chi(u_j, v_j)$$
$$(u_j, v_j) \in \Omega_j \qquad (j = 1, \ldots, N)$$

Then

$$\Pi_N = \sum_{j=1}^{N} f(A_j) \int_{\Omega_j} |\dot{r}_u \times \dot{r}_v|\, du\, dv = \sum_{j=1}^{N} f(A_j)\, \mu_j\, |\Omega_j| =$$
$$= \sum_{j=1}^{N} f(A_j)\, |\dot{r}_u \times \dot{r}_v|_j\, |\Omega_j| + \varepsilon_N \to \iint_{\Omega} f(\varphi, \psi, \chi)\, |\dot{r}_u \times \dot{r}_v|\, du\, dv$$
$$\max d\,(\Omega_j) \to 0$$

where the symbol $|\ |_j$ indicates that the expression $|\ |$ is taken for the values $x = x_j$, $y = y_j$ and $z = z_j$ of the coordinates of the point A_j and μ_j $(j = 1, \ldots, N)$ is a number appearing in the application of the mean value theorem ($m_j \leqslant \mu_j \leqslant M_j$; see below).

It is evident that

$$|\varepsilon_N| = \left|\sum_j^N f(A_j)\left(\mu_j - |\dot{r}_u \times \dot{r}_v|_j\right)\right| |\Omega_j| \leqslant K \sum_{j=1}^{N} (M_j - m_j)|\Omega_j| \underset{\max d(\Omega_j) \to 0}{\longrightarrow} 0,$$

$$M_j = \sup_{(u,\, v) \in \Omega_j} |\dot{r}_u \times \dot{r}_v|, \quad m_j = \inf_{(u,\, v) \in \Omega_j} |\dot{r}_u \times \dot{r}_v| \qquad (K > |f(A)|)$$

whence follows what we intended to prove.

We have not only proved the existence of the integral of F over S but also established the equality

$$\int\limits_S F(x, y, z)\, ds = \int\limits_\Omega F[\varphi, \psi, \chi]\, |\dot{r}_u \times \dot{r}_v|\, du\, dv \tag{3}$$

reducing the computation of the surface integral to the evaluation of an ordinary double integral. It is readily seen that expression (3) is invariant with respect to the admissible parametric representations (see § 7.20 (3) and (7)) of S.

If a smooth surface S is determined by an equation $z = f(x, y)$ $\left((x, y) \in G\right)$ where f is a function continuous together with its first-order partial derivatives on $\bar{G}$, we can write its parametric representation with the aid of the parameters x and y

$$x = x, \quad y = y, \quad z = f(x, y) \tag{4}$$

whence

$$|\dot{r}_x \times \dot{r}_y| = \sqrt{1 + p^2 + q^2} \qquad \left(p = \frac{\partial f}{\partial x},\ q = \frac{\partial f}{\partial y}\right)$$

and, consequently,

$$\int\limits_S F(x, y, z)\, ds = \int\limits_G F\left(x,\, y,\, f(x, y)\right) \sqrt{1 + p^2 + q^2}\, dx\, dy \tag{5}$$

Remark. If we deal with a smooth surface described by parametric equation (1) with functions φ, ψ and χ possessing the properties indicated above and if, at the same time, S admits of a description by an equation of form (4), it often occurs that the function $z = f(x, y)$ is continuous on $\bar{G}$ while its partial derivatives are only continuous on G and become unbounded when the variable point (x, y) approaches the boundary of G. For instance, this is the case when we compute the integral of F over the upper hemisphere. In such cases formula (5) for the integral of F over S often continues to hold if the integral on its right-hand side is understood in the improper sense (see the remark in § 13.13).

§ 13.7. Orientation of a Surface

In the three-dimensional space there are two essentially different types of rectangular coordinates shown in Figs. 13.8 and 13.9. The distinction between these types is that neither of the coordinate systems of the first type can be moved so that its origin O and the positive semi-axes Ox, Oy and Oz are made coincident with the origin O and the positive semi-axes Ox, Oy and Oz of a system belonging to the second type.

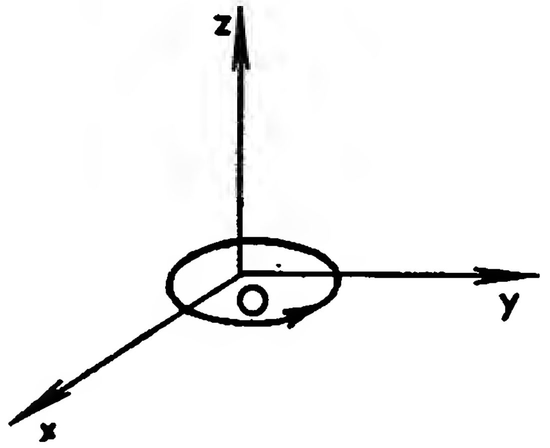

Fig. 13.8

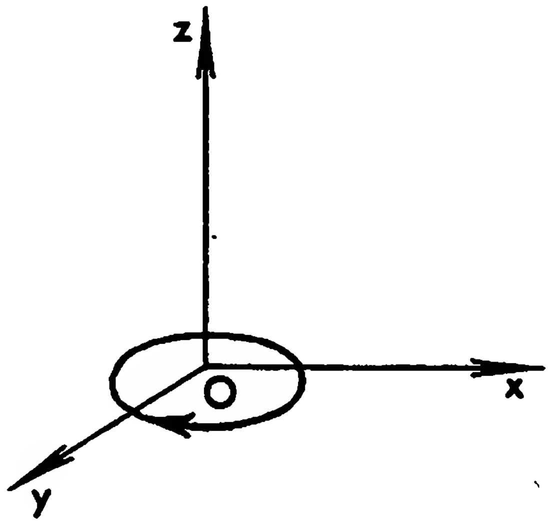

Fig. 13.9

A coordinate system of the first type (Fig. 13.8) is called *right-handed* while a system of the second type (Fig. 13.9) is called *left-handed*. The shortest rotation (in the xy-plane) of the positive x-axis making it coincident with the positive y-axis, when looked at along the positive direction of the z-axis, is seen in the clockwise direction in the case of Fig. 13.8 and in the counterclockwise direction in the case of Fig. 13.9. These properties express, respectively, the so-called *right-hand* and *left-hand screw rules*.

It appears natural to attribute to each of the two coordinate systems shown in Figs. 13.8 and 13.9 a "screw" understood as an object consisting of the unit vector going in the positive direction of the z-axis and of a circle (the "screw head") whose plane is perpendicular to the z-axis and whose boundary is oriented in the direction of the shortest rotation from the positive x-axis to the positive y-axis.

If the z-axis is regarded as the "screw axis" and the indicated circle as the head of a (real) right-hand screw (rigidly connected to its head) then if the head is rotated in the direction of the arrow shown in Fig. 13.8 the screw is driven in the positive direction of the z-axis. In the case of Fig. 13.9 we have the same effect when Oz is regarded as the axis of a left-hand screw.

The head of the screw (understood in this sense) can have a "warped" shape in the sense that it can be a part of a smooth surface, not necessarily plane, but such that the z-axis is the normal to that surface at the point O. In this case the object consisting of such a screw head with a chosen direction of traversing its boundary and of the unit normal vector is also regarded as a (right-hand or left-hand) screw associated with the given type of rectangular coordinates.

Finally, we can speak of such a (right-hand or left-hand) screw whose normal vector goes in an arbitrary direction, not necessarily coinciding with the z-axis.

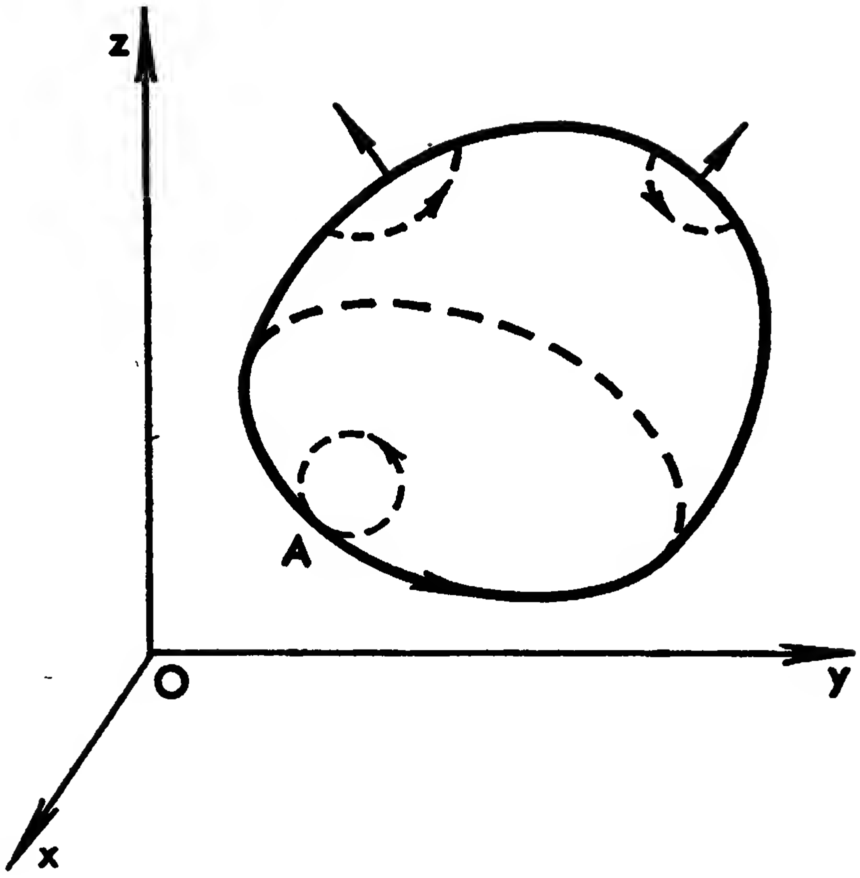

Fig. 13.10

For our further aims the following construction is useful. Let us consider the three-dimensional space with a chosen (right-handed or left-handed) rectangular coordinate system in which an oriented surface S is specified. This means that it is possible to draw a unit normal vector $\boldsymbol{n}(P)$ at each point $P \in S$ so that $\boldsymbol{n}(P)$ is a continuous function of P. A ball $V(P)$ of a sufficiently small radius with centre at P cuts out of the surface S a connected part $\sigma(P)$ containing point P. Let us choose the direction of traversing the contour (the edge) $\gamma(\mathrm{P})$ of that part so that the vector $\boldsymbol{n}(P)$ and the surface element $\sigma(P)$ form a screw whose orientation is coherent with that of the coordinate system taken, that is let the screw be right-hand or left-hand depending on whether the coordinate system is right-handed or left-handed.

If the surface S in question has an edge Γ the above construction leads in a natural way to a definite direction of traversing Γ (see Fig. 13.10). For instance, consider the point A belonging to the contour Γ. At this point coincide the directions in which Γ and the boundary γ of a small portion of the surface S adjoining the point A are traversed.

If the given surface were oppositely oriented while the coordinate system remained the same the directions of traversing Γ and the contours γ we have spoken of should be replaced by the opposite ones.

In Fig. 13.11 we see an oriented surface whose edge consists of two closed smooth curves Γ_1 and Γ_2.

Here we also note the following. Let an oriented smooth surface S be cut into two surfaces S_1 and S_2 by a smooth arc h (Fig. 13.12), the orientations of S_1 and S_2 being coherent with that of S. Then along the arc h the boundaries of S_1 and S_2 are described in the opposite directions. This remark will be essential for the proper understanding of the notion of an oriented piecewise smooth surface to whose definition we now proceed.

A piecewise smooth surface S is said to be oriented if each of its smooth elements is oriented and if the corresponding directions in which the contours of the elements are described are coherent in the sense that they are opposite to each other along every common arc of any two such contours.

As an instance, see the cube depicted in Fig. 13.13 whose boundary surface is oriented with the aid of its outer normal.

It is convenient to consider small elements of an oriented surface as vectors. Let S be an oriented smooth surface. This means that at each point

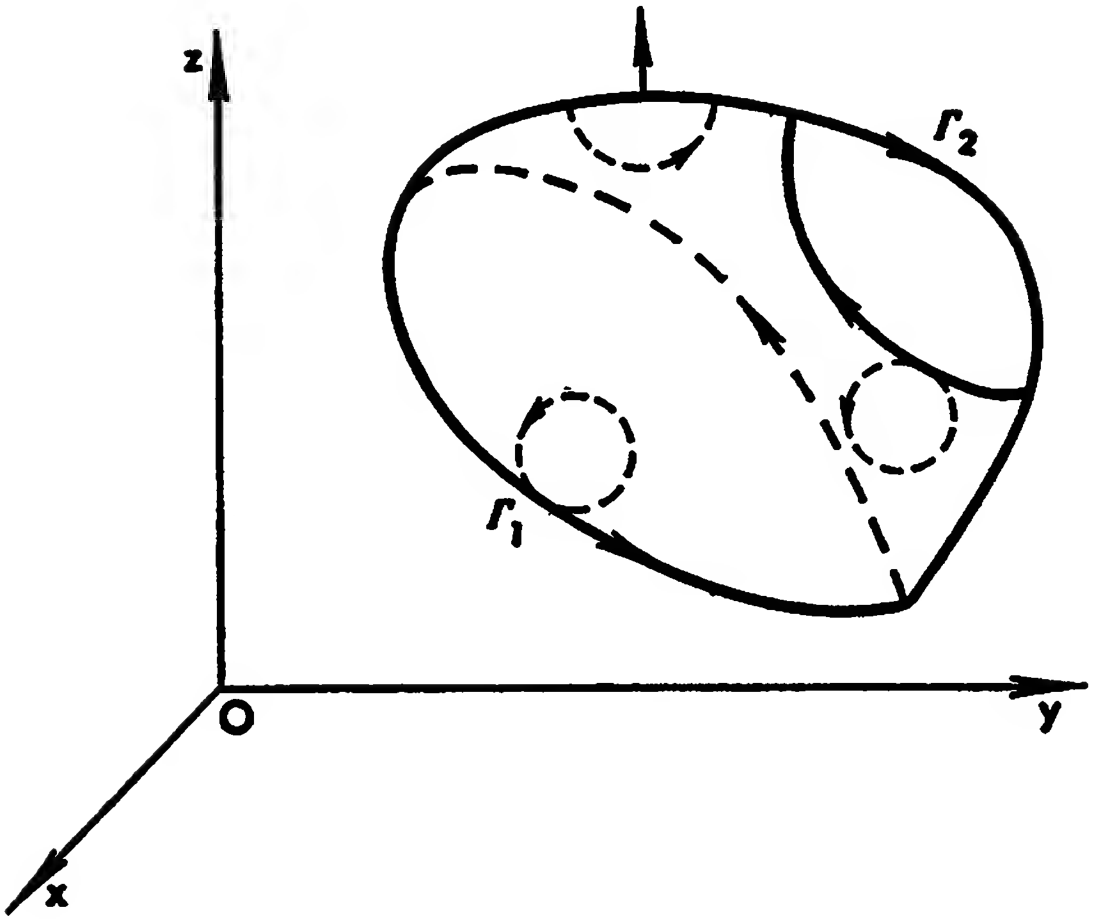

Fig. 13.11

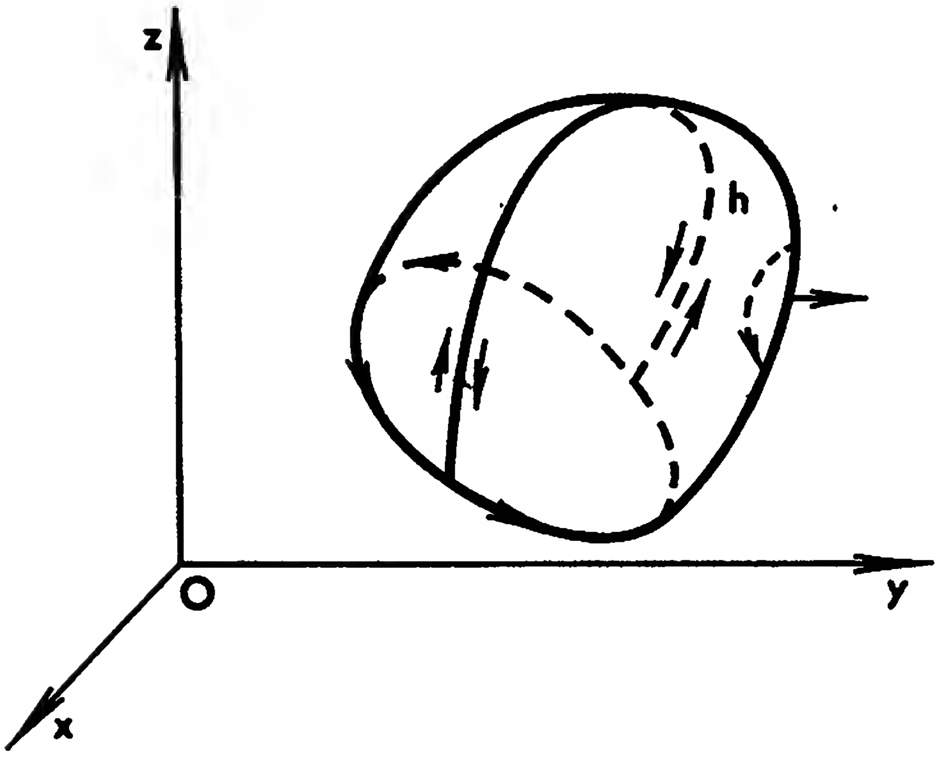

Fig. 13.12

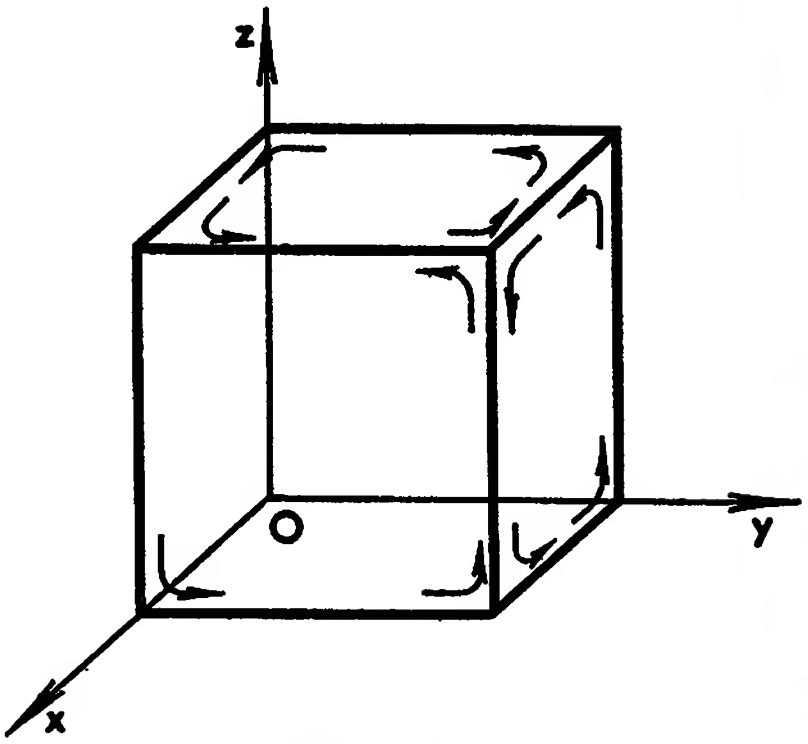

Fig. 13.13

$A \in S$ a unit normal $\boldsymbol{n}(A)$ to S is drawn so that $\boldsymbol{n}(A)$ is a continuous function of A. Let σ be a small smooth element of S. To represent σ as a vector we construct the vector $\boldsymbol{\sigma}$ whose length is equal to the area $|\sigma|$ of σ and whose direction coincides with that of the vector $\boldsymbol{n}(A)$ where A is an arbitrary point belonging to σ. Thus, $\boldsymbol{\sigma} = |\sigma| \boldsymbol{n}(A)$.

This construction does not of course define the vector $\boldsymbol{\sigma}$ uniquely. But if the diameter $d(\sigma)$ of σ is small the vectors $\boldsymbol{n}(A)$ $(A \in \sigma)$ lie within a small cone, and if σ is a variable small element containing permanently a fixed point A_0 then obviously $\boldsymbol{n}(A) \rightarrow \boldsymbol{n}(A_0)$ as $d(\sigma) \rightarrow 0$ independently of the choice of the point $A \in \sigma$ for each σ.

By the *vector of the differential of area of the surface S at its point $A \in S$* is naturally meant the vector $d\boldsymbol{S} = \boldsymbol{n}(A)\, ds$ which is equal to the product of the differential of area of S at the point A by the unit normal vector $\boldsymbol{n}(A)$ specifying the orientation of S.

If the surface S is represented by an equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \varphi\boldsymbol{i}+\psi\boldsymbol{j}+\chi\boldsymbol{k} \qquad ((u, v) \in \bar{G})$$

then $\boldsymbol{n}(A)$ is determined by one of the two equalities

$$\boldsymbol{n}(A) = \pm\frac{\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v}{|\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|} \tag{1}$$

and $dS = |\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|\,du\,dv$ whence

$$d\boldsymbol{S} = \pm(\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v)\,du\,dv \tag{2}$$

For definiteness, in what follows we shall choose the sign "+" in formulas (1) and (2). This can always be achieved by interchanging, if necessary, the roles of the parameters u and v.

We can thus assert that *if we are given a definitely oriented smooth surface S it can always be represented by a vector function $\boldsymbol{r} = \boldsymbol{r}(u, v)$ such that the unit normal $\boldsymbol{n}(A)$ $(A \in \dot{S})$ specifying the orientation of S is expressed by the equality*

$$\boldsymbol{n}(A) = \frac{\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v}{|\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|} \tag{3}$$

and, accordingly,

$$d\boldsymbol{S} = (\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v)\,du\,dv \tag{4}$$

For the transformation from the old parameters (u, v) to some new parameters (u', v') not to lead to the appearance of the minus sign in the above formulas it is sufficient to require that the Jacobian $\frac{D(u, v)}{D(u', v')}$ of the transformation should be positive. Indeed, this follows from the relation

$$\boldsymbol{n}(A) = \frac{\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v}{|\dot{\boldsymbol{r}}_u\times\dot{\boldsymbol{r}}_v|} = \frac{\frac{D(y, z)}{D(u, v)}\boldsymbol{i}+\frac{D(z, x)}{D(u, v)}\boldsymbol{j}+\frac{D(x, y)}{D(u, v)}\boldsymbol{k}}{\sqrt{\left(\frac{D(y, z)}{D(u, v)}\right)^2+\left(\frac{D(z, x)}{D(u, v)}\right)^2+\left(\frac{D(x, y)}{D(u, v)}\right)^2}} =$$

$$= \frac{\left(\frac{D(y, z)}{D(u', v')}\boldsymbol{i}+\frac{D(z, x)}{D(u', v')}\boldsymbol{j}+\frac{D(x, y)}{D(u', v')}\boldsymbol{k}\right)}{\sqrt{\left(\frac{D(y, z)}{D(u', v')}\right)^2+\left(\frac{D(z, x)}{D(u', v')}\right)^2+\left(\frac{D(x, y)}{D(u', v')}\right)^2}}\cdot\frac{\frac{D(u', v')}{D(u, v)}}{\left|\frac{D(u', v')}{D(u, v)}\right|} =$$

$$= \frac{\dot{\boldsymbol{r}}_{u'}\times\dot{\boldsymbol{r}}_{v'}}{|\dot{\boldsymbol{r}}_{u'}\times\dot{\boldsymbol{r}}_{v'}|}\cdot\operatorname{sgn}\frac{D(u', v')}{D(u, v)}$$

Thus, formula (3) (with the plus sign) for unit normal $\boldsymbol{n}(A)$ and, together with it, formula (4) *are only invariant with respect to the transformations of the parameters having positive Jacobians*. Therefore it is advisable to use the transformations of that type.

However, there are problems in which we have to consider transformations with negative Jacobians; then we should carefully choose the proper signs in the formulas.

For a formal treatment of the orientation of surfaces (manifolds) and their edges we refer the reader to §§ 17.1 and 17.2.

§ 13.8. Integral over an Oriented Domain in the Plane

Let us consider a domain G with a piecewise smooth boundary Γ lying in the xy-plane. We shall suppose that a definite direction of describing the contour Γ is chosen. *The domain G thus oriented will be denoted G_+ or G_- depending on whether the orientation of the contour Γ is positive or negative.*

Let $f(x, y)$ be a function defined and integrable on G. Let us introduce the notion of the *integral of f over the oriented domain G;* to this end we put

$$\int_{G_+} f\,dx\,dy = \int_{G} f\,dx\,dy = -\int_{G_-} f\,dx\,dy$$

The expedience of this definition can be demonstrated by the following fact. Let us take two planes with rectangular coordinates x, y and x', y' respectively having the same orientation. Let G be an oriented domain with piecewise smooth (oriented) boundary Γ lying in the xy-plane and let a continuously differentiable transformation

$$x' = \varphi(x, y),\ \ y' = \psi(x, y) \qquad ((x, y) \in \bar{G}) \tag{1}$$

specify a one-to-one mapping of the domain G onto a domain G' in the $x'y'$-plane and of the boundary Γ of G onto the boundary Γ' of the domain G'. We shall suppose that the Jacobian of transformation (1) is different from zero:

$$D = \frac{D(x', y')}{D(x, y)} \neq 0 \qquad ((x', y') \in G')$$

When the variable point (x, y) traverses Γ, transformation (1) induces a definite direction in which the corresponding point (x', y') describes Γ', and therefore G' can be regarded as a definitely oriented domain.

If $D > 0$ the contours Γ and Γ' are of the same orientation. If $D < 0$ then Γ and Γ' are oppositely oriented. Indeed, let us suppose that the functions φ and ψ are doubly continuously differentiable. Then (see the explanations below) we have

$$\int_{\Gamma} x'\,dy' = \int_{\Gamma} \varphi\left(\frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy\right) = \iint_{G}\left[\frac{\partial}{\partial x}\left(\varphi\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\varphi\frac{\partial\psi}{\partial x}\right)\right]dx\,dy =$$

$$= \iint_{G} D\,dx\,dy = D_1\iint_{G} dx\,dy = D_1\int_{\Gamma} x\,dy$$

In the second equality we have used Green's formula, which is legitimate since G is a domain whose orientation is coherent with that of Γ; in the fourth equality wes ave applied the mean value theorem for integrals (D_1 is the value of D vqsumed at some point belonging to $\bar{G}$). Hence, if $D > 0$ then the integrals $\int\limits_{\Gamma'} x'\,dy'$ and $\int\limits_{\Gamma} x\,dy$ are of one sign (see § 13.5 (4)) and Γ and Γ' are described in the same directions; if $D < 0$ these integrals have opposite signs and the contours Γ and Γ' are oppositely oriented.

It follows that for any function $f(x, y)$ defined and continuous on the closure $\bar{G}$ of an oriented measurable domain G there holds the equality

$$\iint\limits_{G} f\,dx\,dy = \iint\limits_{G'} f\frac{D(x, y)}{D(x', y')}\,dx'\,dy'$$

where G' is the oriented domain in the $x'y'$-plane corresponding to G. In this formula expressing the rule for change of variables in a double integral over an oriented domain the Jacobian of the transformation is taken without the sign of absolute value.

Let us also discuss a connection between the orientation of Γ and the sign of D. Let us take two noncolinear vectors $\boldsymbol{a}' = (a_1', a_2')$ and $\boldsymbol{a}'' = (a_1'', a_2'')$ in the xy-plane. If the determinant

$$\Delta = \begin{vmatrix} a_1' & a_1'' \\ a_2' & a_2'' \end{vmatrix}$$

is positive (see § 12.14) then the orientations of the pair $\boldsymbol{a}'$, $\boldsymbol{a}''$ and of the system Ox, Oy are the same (Fig. 13.14). If $\Delta < 0$ their orientations are opposite (Fig. 13.15).

Formulas (1) specify a mapping of the rectangular network of the coordinate lines in the xy-plane on a curvilinear one (see Figs. 13.16-13.18). There are two different characteristic classes of such mappings shown in Figs. 13.17 and 13.18. As is seen in the figures, the square $ABCD$ goes into the curvilinear "parallelogram" $A'B'C'D'$, the vector $\overrightarrow{AB}$ goes, to within infinitesimals of higher order, into the tangent to the arc $\overset{\frown}{A'B'}$ (at the point A') determined by the vector $\left(\frac{\partial x'}{\partial x}, \frac{\partial y'}{\partial x}\right)$ and the vector $\overrightarrow{AD}$ into the tangent to the arc $\overset{\frown}{A'D'}$ (at the point A') determined by the vector $\left(\frac{\partial x'}{\partial y}, \frac{\partial y'}{\partial y}\right)$. If the determinant $D' = \frac{D(x', y')}{D(x, y)}$ is positive the mutual disposition of these vectors is as shown in Fig. 13.17, and therefore the directions in which the variable points (x, y) and (x', y') corresponding to each other describe the contours $ABCD$ and $A'B'C'D'$ respectively are the same; hence the directions of traversing Γ and Γ' also coincide.

If $D' < 0$ the mutual orientation of the tangent vectors to $\overset{\frown}{A'B'}$ and to $\overset{\frown}{A'D'}$ changes to the opposite; therefore in this case Γ and Γ' are described in opposite directions (Fig. 13.18).

The definiton of the integrals over oriented domains G_+ and G lying in the other coordinate planes Oyz and Ozx are stated in like manner.

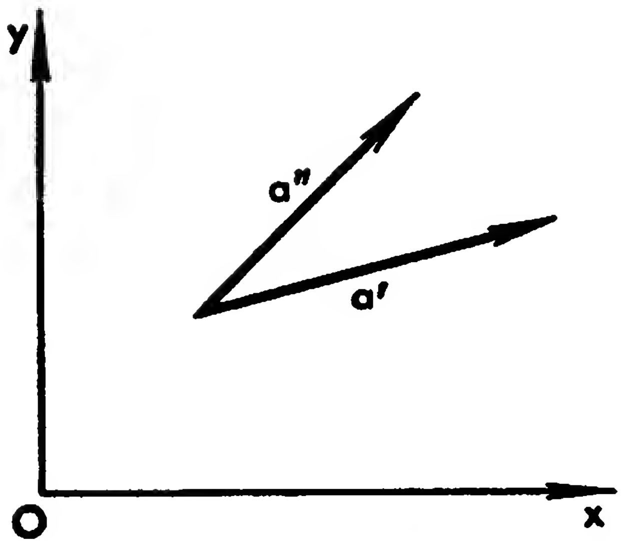

Fig. 13.14

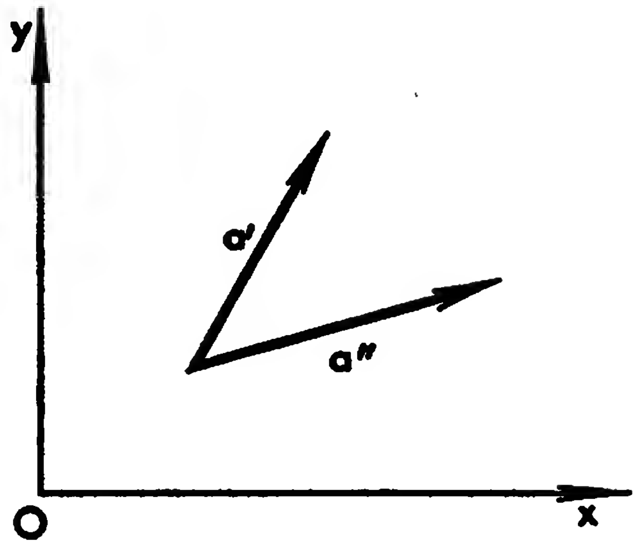

Fig. 13.15

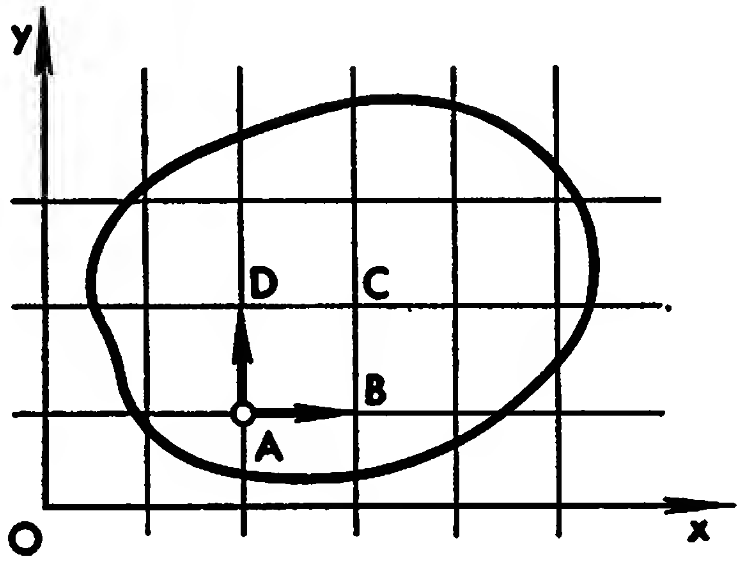

Fig. 13.16

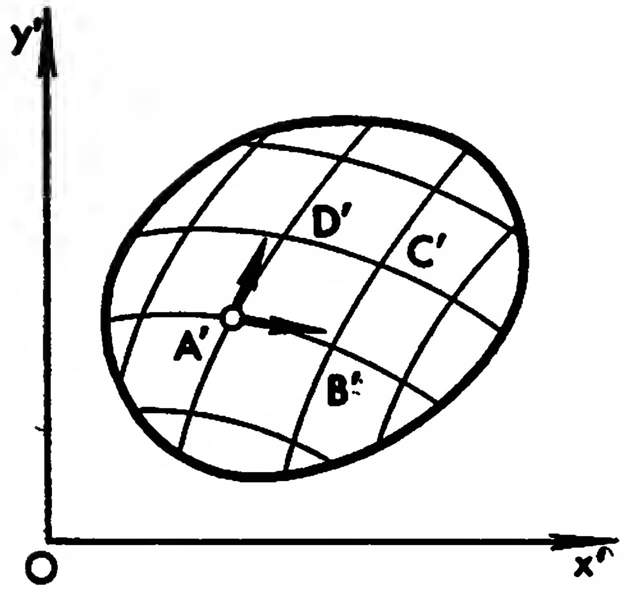

Fig. 13.17

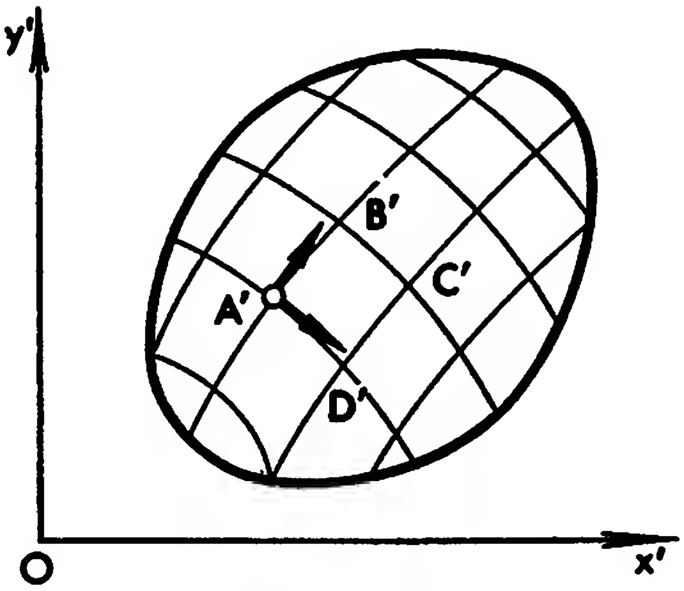

Fig. 13.18

§ 13.9. Flux of a Vector Through an Oriented Surface

Let

$$\boldsymbol{a}(x, y, z) = P\boldsymbol{i}+Q\boldsymbol{j}+R\boldsymbol{k}$$

be a continuous vector field defined in a domain H belonging to the three-dimensional space R with rectangular coordinates x, y, z and let S^* be an oriented smooth surface lying in H and determined by a vector parametric equation

$$\boldsymbol{r} = \boldsymbol{r}(u, v) = \varphi\boldsymbol{i}+\psi\boldsymbol{j}+\chi\boldsymbol{k} \qquad ((u, v) \in \Omega,\ |\dot{\boldsymbol{r}}_u \times \dot{\boldsymbol{r}}_v| > 0) \tag{1}$$

where Ω is a measurable domain in the uv-plane and φ, ψ and χ are continuously differentiable functions on $\bar{\Omega}$.

We shall choose the field of unit normals to S^* specified by the vector equation

$$\boldsymbol{n}(A) = \frac{\dot{\boldsymbol{r}}_u \times \dot{\boldsymbol{r}}_v}{|\dot{\boldsymbol{r}}_u \times \dot{\boldsymbol{r}}_v|}$$

(in this connection see § 13.7 (1) and (3)). Then the direction cosines of $\boldsymbol{n}(A)$ are expressed by the formulas

$$\cos(n, x) = \varkappa \frac{D(y, z)}{D(u, v)}, \quad \cos(n, y) = \varkappa \frac{D(z, x)}{D(u, v)}, \quad \cos(n, z) = \varkappa \frac{D(x, y)}{D(u, v)} \qquad (2)$$

where $\varkappa = \dfrac{1}{|\dot{\boldsymbol{r}}_u \times \dot{\boldsymbol{r}}_v|}$.

The same surface considered without orientation will be denoted as S.

By *the flux of the vector* $\boldsymbol{a}$ *through the oriented surface* S^* is meant the number

$$\int\limits_S (\boldsymbol{a}\, d\boldsymbol{s}^*) = \int\limits_S (\boldsymbol{an})\, dS \qquad (3)$$

that is the value of the integral (of the first type) over the surface S of the scalar product

$$(\boldsymbol{an}) = P\cos(n, x) + Q\cos(n, y) + R\cos(n, z)$$

of the vector $\boldsymbol{a}$ of the field by unit normal $\boldsymbol{n}$ specifying the orientation of S^*.

Since $(\boldsymbol{an})$ is a continuous function of the point $A \in S$ the surface integral of the first type (3) of $(\boldsymbol{an})$ over S exists (this was proved in § 13.6).

For instance, suppose that the vector field $\boldsymbol{a}$ in question defined in H is the velocity field of a stationary fluid flow, that is for each point $A \in H$ the vector $\boldsymbol{a}$ coincides with the velocity of the particle of fluid passing through that point, the velocity depending solely on A and being independent of time. Then the flux of the velocity vector $\boldsymbol{a}$ through an oriented surface S^* is equal to the volume of fluid flowing through S in unit time along the direction indicated by the normal $\boldsymbol{n}(A)$ specifying the orientation of S.

From (2) and from formula (3) in § 13.6 readily follows the equality

$$\int\limits_S (\boldsymbol{an})\, ds = \int\limits_S (P\cos(n, x) + Q\cos(n, y) + R\cos(n, z))\, ds =$$

$$= \iint\limits_\Omega \left(P\frac{D(y, z)}{D(u, v)} + Q\frac{D(z, x)}{D(u, v)} + R\frac{D(x, y)}{D(u, v)}\right) du\, dv \qquad (4)$$

whose right-hand member is an ordinary double integral over the domain Ω; the functions P, Q and R under the sign of double integration are taken for the values of x, y and z equal respectively to the functions φ, ψ and χ dependent on u and v.

It is often convenient to compute integral (4) in Cartesian coordinates. We shall demonstrate such calculations under the assumption that the smooth surface $\bar{S}$ in question can be projected in a one-to-one manner on each of the coordinate planes, the projections being measurable sets.

Thus, suppose that $\bar{S}$ can be described by each of the three functions

$$x = f_1(y, z) \qquad ((y, z) \in \bar{S}_x) \tag{4'}$$

$$y = f_2(z, x) \qquad ((z, x) \in \bar{S}_y) \tag{4''}$$

and

$$z = f_3(x, y) \qquad ((x, y) \in \bar{S}_z) \tag{4'''}$$

continuous on the projections of $\bar{S}$ on the planes $x = 0$, $y = 0$ and $z = 0$ respectively and possessing continuous partial derivatives which, generally speaking, exist only in the measurable interiors S_x, S_y and S_z of the projections (that is in the projections taken without their boundaries).

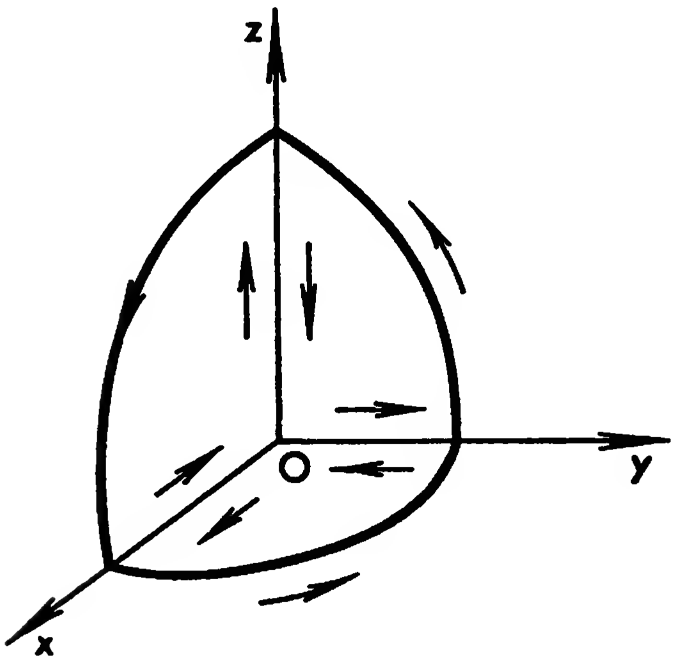

Fig. 13.19

Let us denote as S_x^*, S_y^* and S_z^* the corresponding oriented projections of the oriented surface S^* on the planes $x = 0$, $y = 0$ and $z = 0$ respectively. The specification of the positive direction of traversing the contour of S^* (which determines the orientation of S^*) induces the corresponding directions in which the boundaries of the domains S_x, S_y and S_z are described (see Fig. 13.19). The cosine of the angle between the normal n to S and the positive z-axis is given by the formula

$$\cos(n, z) = \pm\frac{1}{\sqrt{1+p^2+q^2}} \qquad \left(p = \frac{\partial f_3}{\partial x},\ q = \frac{\partial f_3}{\partial y}\right)$$

where the sign "+" or "−" is taken in accordance with the orientation of S^*. By formula (5) of § 13.6, we have

$$\int_S R\cos(n, z)\,ds = \int_{S_z} R(x, y, f_3(x, y))\frac{(\pm 1)}{\sqrt{1+p^2+q^2}}\sqrt{1+p^2+q^2}\,dx\,dy =$$

$$= \pm\int_{S_z} R(x, y, f_3(x, y))\,dx\,dy = \int_{S_z^*} R(x, y, f_3(x, y))\,dx\,dy =$$

$$= \int_{S^*} R(x, y, z)\,dx\,dy \tag{5}$$

where the fourth integral is taken over the oriented domain S_z^* (see § 13.8). As to the last integral, it should be considered as the notation of the preceding one. This is the so-called *surface integral of the second type*. In order to evaluate this integral we substitute $f_3(x, y)$ for z into the function $R(x, y, z)$ and take the integral of the resultant expression over the oriented projection S_z^*. As is known (see § 13.8), $\int\limits_{S_z^*} = \pm \int\limits_{S_z}$ where the sign "+" or "—" is taken depending on whether the domain S_z^* is oriented positively or negatively. Similar considerations apply to the other two integrals (see Fig. 13.19), and we obtain

$$\int\limits_{S} P\cos(n, x)\, ds = \int\limits_{S_x^*} P(f_1(x, y), y, z)\, dy\, dz = \int\limits_{S^*} P(x, y, z)\, dy\, dz$$

and

$$\int\limits_{S} Q\cos(n, y)\, ds = \int\limits_{S_y^*} Q(x, f_2(z, x), z)\, dz\, dx = \int\limits_{S^*} Q(x, y, z)\, dz\, dx$$

We have shown that the flux of the vector $\boldsymbol{a}$ through the oriented surface S^* specified by the normal $\boldsymbol{n}(A)$ can be computed by the formula

$$\int\limits_{S} (\boldsymbol{an})\, ds = \int\limits_{S^*} \big(P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy\big) \qquad (5)$$

If S^* is a surface which can be cut into a finite number N of parts S_j^* $\left(S^* = \sum\limits_{j=1}^{N} S_j^*\right)$ each of which can be projected in a one-to-one manner on each of the three coordinate planes then in order to compute the flux of $\boldsymbol{a}$ through S^* we can find the fluxes of $\boldsymbol{a}$ through each of these parts S_j^* as was shown above and then add up the results.

A spherical surface with centre at the origin is cut by the coordinate planes into 8 parts possessing the indicated property. The torus considered in Example 3 in § 7.20 is cut by the three coordinate planes and by the circular cylindrical surface of radius b having Oy as axis into sixteen such parts.

§ 13.10. Divergence. Gauss-Ostrogradsky* Theorem

Let E be the three-dimensional space with rectangular coordinates x, y, z and let $G \subset E$ be a domain (with a piecewise smooth boundary S) in which a vector field

$$\boldsymbol{a}(x, y, z) = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k} \qquad ((x, y, z) \in \bar{G}) \qquad (1)$$

is defined. We shall suppose that the functions P, Q and R and their derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial Q}{\partial y}$ and $\frac{\partial R}{\partial z}$ are continuous in $\bar{G}$ whence it follows that for the vector $\boldsymbol{a}$ there exists the continuous function

---

*K. F. Gauss (1777-1855), a German mathematician. For Ostrogradsky see § 8.7.

$$\operatorname{div} \boldsymbol{a} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \qquad ((x, y, z) \in \bar{G}) \tag{2}$$

which is called the *divergence of the vector* $\boldsymbol{a}$.

Let us assume that the surface S is oriented with the aid of the unit normal $\boldsymbol{n}$ directed to the exterior of G.

The aim of this section is to prove the formula

$$\int_G \operatorname{div} \boldsymbol{a}\, dG = \int_S (\boldsymbol{a}\boldsymbol{n})\, dS \tag{3}$$

which holds under some additional conditions imposed on G. This equality expresses the so-called *Gauss-Ostrogradsky theorem* (also known as the *divergence theorem*).

The Gauss-Ostrogradsky theorem reads: *the volume integral of the divergence of the vector* $\boldsymbol{a}$ *over the domain* G *is equal to the flux of that vector through the boundary* S *of the domain oriented with the aid of its outer normal.*

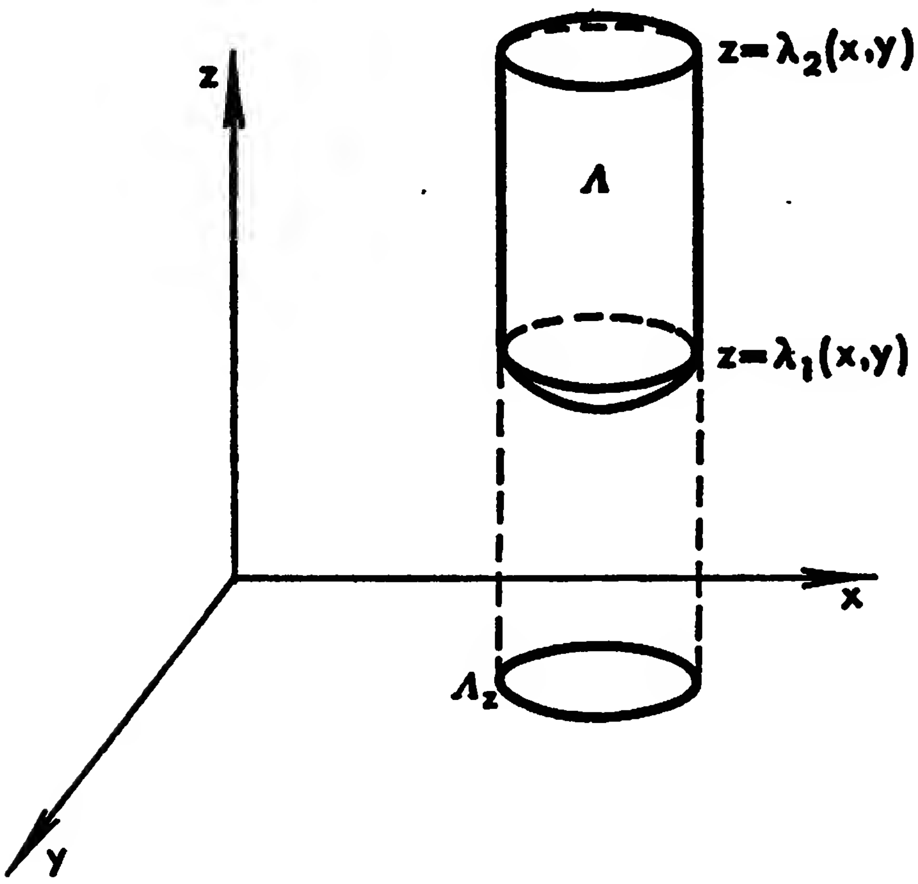

Fig. 13.20

To begin with, let us consider the domain Λ shown in Fig. 13.20. A domain of this type will be referred to as an *elementary* H_z-*domain*. The upper and lower boundaries of Λ are the surfaces σ_1 and σ_2 (with piecewise smooth boundary curves) determined by the equations $z = \lambda_1(x, y)$ and $z = \lambda_2(x, y)$, $\lambda_1(x, y) \leqslant \lambda_2(x, y)$; $(x, y) \in \bar{\Lambda}_z$, respectively, where Λ_z is a domain in the xy-plane with a (piecewise smooth) boundary γ, and the functions λ_1 and λ_2 are continuous on $\bar{\Lambda}_z$ and have continuous partial derivatives in the open set Λ_z. The lateral boundary of Λ is the cylindrical surface σ^* with the curve γ as directrix and generators parallel to the z-axis.

Let S^* be the boundary of Λ oriented with the aid of the outer normal (see the explanations below). This induces the corresponding orientations of the surface elements σ_1^* and σ_2^* and of the lateral surface σ^* of the domain Λ. For the domain Λ there hold the equalities (see the explanations below)

$$\int_\Lambda \frac{\partial R}{\partial z}\, d\Lambda = \iint_{\Lambda_z} dx\, dy \int_{\lambda_1(x,y)}^{\lambda_2(x,y)} \frac{\partial R}{\partial z}\, dz =$$

$$= \iint\limits_{A} \{R(x, y, \lambda_2(x, y)) - R(x, y, \lambda_1(x, y))\}\, dx\, dy =$$

$$= \iint\limits_{\sigma^*_{2,z}} R((x, y, \lambda_2(x, y))\, dx\, dy + \iint\limits_{\sigma^*_{1,z}} R(x, y, \lambda_1(x, y))\, dx\, dy =$$

$$= \iint\limits_{S^*} R(x, y, z)\, dx\, dy \quad (4)$$

The normal n to σ^*_1 forms an obtuse angle with the positive z-axis while the normal to σ^*_2 forms an acute angle with it; therefore the projection $\sigma^*_{1,z}$ of the element σ^*_1 on the plane $z = 0$ is oriented negatively while the projection $\sigma^*_{2,z}$ of σ^*_2 is oriented positively. This accounts for the passage from the third term in (4) to the fourth one. To the sum of the two integrals forming the fourth term we can formally add the integral

$$\iint\limits_{\sigma^*} R(x, y, z)\, dx\, dy = 0$$

which is equal to zero since $\cos(n, z) = 0$ for σ^*. Thus, the resultant sum of the three integrals is equal to the last integral in (4), that is to the flux of the vector $(0, 0, R)$ through S^*.

We have thus proved the Gauss-Ostrogradsky theorem for an elementary H_z-domain and for a vector of the form $(0, 0, R)$.

Further, let us call a domain G an *H_z-domain* if its closure $\bar{G}$ can be cut into a finite number of closures $\bar{G}_k$ of elementary H_z-domains so that the upper and lower parts of the boundaries of G_k's are some parts of the oriented boundary S^* of the domain G:

$$\bar{G} = \sum_{k=1}^{N} \bar{G}_k$$

We shall prove that for such a domain G and for a vector of the form $(0, 0, R)$ the Gauss-Ostrogradsky theorem also holds.

To this end, let us denote by $S_{1,k}$ and $S_{2,k}$ the lower and upper parts of the boundaries of $\bar{G}_k$'s respectively and by S_k the lateral parts of the boundaries of $\bar{G}_k$'s. Then we have (see the explanations below)

$$\int\limits_{G} \frac{\partial R}{\partial z}\, dG = \sum_{k=1}^{N} \int\limits_{G_k} \frac{\partial R}{\partial z}\, dG =$$

$$= \sum_{k=1}^{N} \left(\int\limits_{S^*_{1,k}} R(x, y, z)\, dx\, dy + \int\limits_{S^*_{2,k}} R(x, y, z)\, dx\, dy + \int\limits_{S^*_k} R(x, y, z)\, dx\, dy \right) =$$

$$= \int\limits_{S^*} R(x, y, z)\, dx\, dy$$

because the integrals over S^*_k's are obviously equal to zero while for the parts $S^*_{1,k}$ and $S^*_{2,k}$ we can assert that either their union coincides with the surface S^* or, if otherwise, the set

$$\sigma = S^* - \sum_1^N S^*_{1,k} - \sum_1^N S^*_{2,k}$$

is a part of S^* such that the normal at its every point is perpendicular to the z-axis and therefore the integral over σ is equal to zero.

By analogy with the above, we can define the notions of *H_x-domains* and *H_y-domains.* For instance, an H_x-domain possesses the property that its closure can be cut into a finite number of closures of *elementary H_x-domains*, the latter being understood in the same sense as the elementary H_z-domains with the only difference that the role of the z-axis is played by the x-axis. Similar considerations show that for an H_x-domain G there holds the equality

$$\int_G \frac{\partial P}{\partial x}\, dG = \iint_{S^*} P(x, y, z)\, dy\, dz$$

that is the Gauss-Ostrogradsky formula holds for the H_x-domains and for the vectors of the form $(P, 0, 0)$. Analogously, for an H_y-domain G there holds the formula

$$\int_G \frac{\partial Q}{\partial y}\, dG = \iint_{S^*} Q(x, y, z)\, dz\, dx$$

expressing the Gauss-Ostrogradsky theorem for a vector $(0, Q, 0)$.

Now let us suppose that a given domain G is simultaneously an H_x-, an H_y- and an H_z-domain; then obviously the Gauss-Ostrogradsky theorem holds for any continuously differentiable vector field $\boldsymbol{a} = (P, Q, R)$ defined on $\bar{G}$, that is we have the equality

$$\iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz =$$

$$= \iint_{S^*} (P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy) \qquad (5)$$

where the integral on the right-hand side is taken over the surface S^* oriented with the aid of the outer normal to G.

On putting $P = x$, $Q = y$ and $R = z$ in the Gauss-Ostrogradsky formula (5) (or (3)) we obtain the formula

$$|G| = \frac{1}{3} \iint_{S^*} (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

expressing the volume $|G|$ of the domain G in terms of the surface integral taken over its oriented (with the aid of the outer normal) boundary S^*.

Practically, we usually deal with domains which are simultaneously H_x-, H_y- and H_z-domains.

Example 1. The ball $x^2+y^2+z^2 \leq 1$ is an H_z-domain and even an elementary H_z-domain because its interior is bounded above and below by the smooth surfaces $z_1 = \sqrt{1-x^2-y^2}$ and $z_2 = -\sqrt{1-x^2-y^2}$ $(x^2+y^2 < 1)$ where the functions z_1 and z_2 are continuous in the closed circle $x^2+y^2 \leq 1$ whose boundary curve is smooth. It is evident that the ball in question is also an H_x- and an H_y-domain.

Example 2. Let us consider the torus T obtained by the rotation of the circle $(x-b)^2+y^2 = a^2$ $(0 < a < b)$ lying in the xy-plane about the y-axis. To show that T is an H_y-domain it suffices to cut the boundary surface of T into two parts with the aid of the xz-plane. Further, the planes $x = b-a$ and $x = a-b$ divide T into four elementary H_z-domains while the planes $z = b-a$ and $z = a-b$ divide it into four elementary H_x-domains.

The Gauss-Ostrogradsky formula transforms a volume integral over a domain into a surface integral taken over its boundary. In the next section we shall derive Stokes' formula with the aid of which a surface integral can be transformed, under certain conditions, into a line integral.

To elucidate the physical meaning of the notion of the divergence of a vector let us suppose that the given domain G is occupied by a stationary flow of a fluid so that the velocity of the particles of fluid passing through an arbitrary point $(x, y, z) \in G$ is equal to $\boldsymbol{a} = \boldsymbol{a}(x, y, z)$. Let us take an arbitrary but fixed point $A = (x, y, z) \in G$ and embed it in a ball $V_\varepsilon \subset G$ of radius $\varepsilon > 0$. Let S_ε^* be the boundary of the ball (the spherical surface) oriented with the aid of the outer normal. Then, by the Gauss-Ostrogradsky formula, we have

$$\iint\limits_{S_\varepsilon} (\boldsymbol{a}\, d\boldsymbol{s}) = \iiint\limits_{V_\varepsilon} \operatorname{div} \boldsymbol{a}\, dx\, dy\, dz$$

The left-hand member of this equality expresses the volume of the fluid flowing out of V_ε (to the exterior of S_ε) in unit time.

On applying the mean value theorem to the last equality we obtain

$$\iint\limits_{S_\varepsilon} (\boldsymbol{a}\, d\boldsymbol{s}) = |V_\varepsilon| \operatorname{div} \boldsymbol{a}_1 \tag{6}$$

where $|V_\varepsilon|$ is the volume of V_ε and $\boldsymbol{a}_1$ is the velocity of the fluid at some point belonging to V_ε. Let us divide both members of equality (6) by $|V_\varepsilon|$ and pass to the limit as $\varepsilon \to 0$; by the continuity of div $\boldsymbol{a}$ this limit exists and is equal to the divergence of $\boldsymbol{a}$ at the chosen point $A(x, y, z)$:

$$\operatorname{div} \boldsymbol{a} = \lim_{\varepsilon \to 0} \frac{1}{|V_\varepsilon|} \iint\limits_{S_\varepsilon} (\boldsymbol{a}\, d\boldsymbol{s}) \tag{7}$$

Thus, div $\boldsymbol{a}$ is equal to the value of the intensity of sources of fluid (continuously distributed over G) at the point $A(x, y, z)$. If div $\boldsymbol{a} = 0$ at the point A (or throughout G) then the intensity of sources is equal to zero at A (or throughout G). If div $\boldsymbol{a} < 0$ then at the corresponding point there is in fact not a source but a sink of fluid.

From the above physical considerations it appears evident that div $\boldsymbol{a}$ is an invariant with respect to all transformations of the rectangular coordinates. The same conclusion can be drawn on the basis of purely mathematical argument if we take into account that the flux of a vector field through the surface S_ε is invariant with respect to the transformations. It follows that if one and the same vector field $\boldsymbol{a}$ is considered relative to two different systems of rectangular coordinates (x, y, z) and (x', y', z') and is specified by two vector functions

$$\boldsymbol{a} = P(x, y, z)\boldsymbol{i}+Q(x, y, z)\boldsymbol{j}+R(x, y, z)\boldsymbol{k}$$

and

$$\boldsymbol{a} = P_1(x', y', z')\boldsymbol{i}_1+Q_1(x', y', z')\boldsymbol{j}_1+R_1(x', y', z')\boldsymbol{k}_1$$

respectively then

$$\operatorname{div} \boldsymbol{a} = \frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z} = \frac{\partial P_1}{\partial x'}+\frac{\partial Q_1}{\partial y'}+\frac{\partial R_1}{\partial z'}$$

at every point $A(x, y, z) = A'(x', y', z')$. This assertion can of course be proved directly without resorting to the Gauss-Ostrogradsky theorem.

The divergence of $\boldsymbol{a}$ can also be interpreted as the (symbolic) scalar product of the Hamiltonian operator ∇ and the vector $\boldsymbol{a}$:

$$\operatorname{div} \boldsymbol{a} = \nabla \boldsymbol{a} = \frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$$

From this point of view the above invariance can formally be proved as follows. Since ∇ and $\boldsymbol{a}$ are vectors and the scalar product of two vectors is invariant with respect to the transformations of rectangular coordinates so is the divergence $\nabla \boldsymbol{a}$.

The Gauss-Ostrogradsky theorem can also be stated for the case when G is a domain in the xy-plane and $\boldsymbol{a}(x, y) = P(x, y)\boldsymbol{i}+Q(x, y)\boldsymbol{j}$ is a vector field defined in it. Denoting by $\boldsymbol{n}(A)$ the outer normal to the boundary Γ of the domain $G\,(A \in \Gamma)$ which is supposed to be a piecewise smooth contour we can write the equality

$$\iint\limits_G \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) dx\, dy = \int\limits_\Gamma (\boldsymbol{a}\boldsymbol{n})\, ds$$

where ds is the differential of arc length of the curve Γ.

If we agree that the direction of the tangent to the contour Γ coincides at every point of Γ with the positive direction of traversing Γ in which the arc length s is reckoned along Γ then

$$\cos(n, x) = \cos(T, y) = \frac{dy}{ds} \quad \text{and} \quad \cos(n, y) = -\cos(T, x) = -\frac{dx}{ds}$$

Therefore $(\boldsymbol{a}\boldsymbol{n})\, ds = P\, dy-Q\, dx$, and we obtain

$$\iint\limits_G \left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right) dx\, dy = \int\limits_\Gamma (P\, dy-Q\, dx)$$

On replacing P by Q and Q by $-P$ in the last formula we arrive at Green's formula which was already derived in § 13.5.

Let G be a bounded domain with doubly continuously differentiable boundary S and let G_λ be the part of G bounded by the surface S_λ whose points are at a distance of $\lambda > 0$, reckoned along the normal to S, from the boundary S of G (see § 7.25). Let us also consider a vector field $\boldsymbol{a}$ defined and continuous on $\bar{G}$ and possessing continuous partial derivatives on G (the derivatives may be unbounded in the vicinity of the boundary of G). We shall suppose that for sufficiently small λ the domain G_λ satisfies the conditions under which the Gauss-Ostrogradsky theorem is applicable.

Let us show that in that case the Gauss-Ostrogradsky formula

$$\int_G \operatorname{div} \boldsymbol{a}\, dx = \int_S (\boldsymbol{an})\, ds \tag{8}$$

where $x = (x_1, x_2, x_3)$ continues to hold if its left-hand member is understood in the following improper sense:

$$\int_G \operatorname{div} \boldsymbol{a}\, dx = \lim_{\lambda \to 0} \int_{G_\lambda} \operatorname{div} \boldsymbol{a}\, dx \tag{9}$$

As to the right-hand member of (8), it is an ordinary surface integral of the second type over the smooth surface S oriented with the aid of its outer normal because the vector $\boldsymbol{a}$ is continuous on S.

Indeed, according to the Gauss-Ostrogradsky theorem which was already proved for domains of the type of G_λ, we have

$$\int_{G_\lambda} \operatorname{div} \boldsymbol{a}\, dx = \int_{S_\lambda} (\boldsymbol{an})\, ds_\lambda \qquad (0 < \lambda \leq \mu) \tag{10}$$

for a sufficiently small μ because for sufficiently small values of λ the boundary S_λ of G_λ is a smooth surface (see § 7.25) and the vector $\boldsymbol{a}$ is not only continuous on the set $\bar{G}_\lambda$ but also has continuous partial derivatives in it.

Let us show that

$$\lim_{\lambda \to 0} \int_{S_\lambda} (\boldsymbol{an})\, ds_\lambda = \int_S (\boldsymbol{an})\, ds \tag{11}$$

under the assumption that S can be cut into a finite number of smooth surface elements S^l:

$$S = \sum_{l=1}^{N} S^l \tag{12}$$

By virtue of (10) this will imply the validity of (8) and (9).

To prove (11) let us consider the parts S_λ^l $(l = 1, \ldots, N)$ of S_λ consisting of the points of G lying on the normals to S^l at a distance λ from S^l. Decomposition (12) of S into the parts S_λ induces the corresponding decomposition of S_λ:

$$S_\lambda = \sum_{l=1}^{N} S_\lambda^l \tag{13}$$

For definiteness, let us suppose that S^l is determined by an equation $z = f(x, y)$ $((x, y) \in \Omega)$. Then the Cartesian coordinates (ξ, η, ζ) of the points of S_λ^l are determined by equations

$$\xi = \varphi(x, y, \lambda), \quad \eta = \psi(x, y, \lambda), \quad \zeta = \chi(x, y, \lambda) \qquad (x, y) \in \Omega$$

where Ω is the closure of a bounded domain in the xy-plane having a piecewise smooth boundary curve and φ, ψ and χ are continuously differentiable functions of (x, y, λ) describing the smooth surface S_λ^l for $0 \leqslant \lambda \leqslant \mu$ where μ is sufficiently small. For expressions of these functions see § 7.25 (5) ($n = 3$ and $t = \lambda$).

Therefore (see the explanations below) we have

$$\int_{S_\lambda^l} (\boldsymbol{a}\boldsymbol{n})\, dS_\lambda^l = \pm \iint_{\Omega} \frac{\boldsymbol{a}(\dot{\boldsymbol{r}}_x \times \dot{\boldsymbol{r}}_y)}{|\dot{\boldsymbol{r}}_x \times \dot{\boldsymbol{r}}_y|}\, dx\, dy \to \int_{S^l} (\boldsymbol{a}\boldsymbol{n})\, ds \qquad (\lambda \to 0) \tag{14}$$

where the vector product of $\dot{\boldsymbol{r}}_x$ and $\dot{\boldsymbol{r}}_y$ is given by the formula

$$\dot{\boldsymbol{r}}_x \times \dot{\boldsymbol{r}}_y = \frac{D(\psi, \chi)}{D(x, y)}\boldsymbol{i} + \frac{D(\chi, \varphi)}{D(x, y)}\boldsymbol{j} + \frac{D(\varphi, \psi)}{D(x, y)}\boldsymbol{k}$$

and the values of ξ, η and ζ entering into $\boldsymbol{a} = \boldsymbol{a}(\xi, \eta, \zeta)$ are replaced by the functions φ, ψ and χ respectively. Indeed, the integrand in the middle integral in (14) is a continuous function of (x, y, λ) on the closed set $(x, y) \in \Omega$, $0 \leqslant \lambda \leqslant \mu$, and therefore it is allowable to pass to the limit, as $\lambda \to 0$, under the integral sign (see § 12.13, Theorem 1). An analogous situation takes place when S^k $(k = 1, \ldots, N)$ can be projected in a one-to-one manner on the plane $x = 0$ or $y = 0$. From (12), (13) and (14) follows (11) which implies (8) and (9).

§ 13.11. Rotation of a Vector. Stokes'* Theorem

Let us consider a continuously differentiable vector field

$$\boldsymbol{a} = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$$

defined in a domain in the three-dimensional space R.

The *rotation of the vector* $\boldsymbol{a}$ is the vector

$$\operatorname{rot} \boldsymbol{a} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

It can be regarded as the (formal) vector product of the Hamiltonian operator ∇ by the vector $\boldsymbol{a}$:

$$\operatorname{rot} \boldsymbol{a} = \nabla \times \boldsymbol{a}$$

---

* G. G. Stokes (1819-1903), a British physicist and mathematician.

As is known from vector algebra, the vector product of two vectors is an axial vector, that is it is invariant with respect to all the transformations of orthogonal coordinates preserving their orientation (in the sense that right-handed systems go into right-handed ones while left-handed systems go into left-handed ones). It is known (see § 7.6) that the symbol ∇ can be regarded as a vector since when we pass from one rectangular coordinates system (x, y, z) to another system (x', y', z') the transformation rule for the "components" $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ of ∇ is the same as that for the components of an ordinary vector. Therefore rot $\boldsymbol{a} = \nabla \times \boldsymbol{a}$ is an axial vector, that is it is invariant with respect to the transformations of rectangular coordinates preserving their orientations. Consequently, we can assert without calculations that if the vector $\boldsymbol{a} = P(x, y, z)\boldsymbol{i} + Q(x, y, z)\boldsymbol{j} + R(x, y, z)\boldsymbol{k}$ has the components $P_1(x', y', z')$, $Q_1(x', y', z')$ and $R_1(x', y', z')$ in a new coordinate system (x', y', z') (having the same orientation as the old system (x, y, z)), that is if

$$\boldsymbol{a} = P_1(x', y', z')\boldsymbol{i}' + Q_1(x', y', z')\boldsymbol{j}' + R_1(x', y', z')\boldsymbol{k}'$$

then there holds the identity

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k} =$$

$$= \left(\frac{\partial R_1}{\partial y'} - \frac{\partial Q_1}{\partial z'}\right)\boldsymbol{i}' + \left(\frac{\partial P_1}{\partial z'} - \frac{\partial R_1}{\partial x'}\right)\boldsymbol{j}' + \left(\frac{\partial Q_1}{\partial x'} - \frac{\partial P_1}{\partial y'}\right)\boldsymbol{k}'$$

where $\boldsymbol{i}', \boldsymbol{j}'$ and $\boldsymbol{k}'$ are the unit vectors of the coordinate axes of the system (x', y', z').

We shall prove *Stokes' formula*

$$\iint\limits_S (\boldsymbol{n} \text{ rot } \boldsymbol{a})\, ds = \int\limits_\Gamma (\boldsymbol{a}\, d\boldsymbol{l}) \tag{1}$$

expressing *Stokes' theorem* which states that *the flux of the vector rot* $\boldsymbol{a}$ *through an oriented surface* S^* *is equal to the circulation of the vector* $\boldsymbol{a}$ *over the coherently oriented contour* Γ *bounding the surface* S^*.*

We shall begin with the proof of Stokes' theorem for a smooth surface element which can be projected in a one-to-one manner on each of the three coordinate planes.

Let us consider an oriented smooth surface element S^* with a piecewise smooth edge Γ representable by equations of each of the following forms:

$$z = f(x, y) \qquad (x, y) \in S_z, \qquad x = \varphi(y, z) \qquad (y, z) \in S_x$$

and

$$y = \psi(z, x) \qquad (z, x) \in S_y$$

---

* Here S^* denotes the surface S oriented with the aid of the normal $\boldsymbol{n}$. The left-hand member of (1) is a surface integral of the first type.

We thus suppose that each of these equations can be uniquely resolved with respect to any of the variables x, y and z and that the functions f, φ and ψ are continuously differentiable on the corresponding projections of S on the coordinate planes.

We have

$$\iint_S (\boldsymbol{n} \operatorname{rot} \boldsymbol{a})\, ds = \iint_S \left\{\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) \cos(n, x) + \right.$$

$$\left. + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) \cos(n, y) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cos(n, z)\right\} ds \qquad (2)$$

Let us consider the sum of those terms entering into the right-hand side of (2) which contain P. It can be expressed (see the explanations below) as

$$-\iint_S \left\{\frac{\partial P}{\partial y} \cos(n, z) - \frac{\partial P}{\partial z} \cos(n, y)\right\} ds =$$

$$= -\iint_S \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\,\frac{\partial f}{\partial y}\right) \cos(n, z)\, ds = -\iint_{S_z^*} \frac{\partial}{\partial y} P(x, y, f(x, y))\, dx\, dy =$$

$$= \int_{\Gamma_z} P[x, y, f(x, y)]\, dx = \int_0^{s_1^0} P[\varphi_1(s_1),\ \psi_1(s_1),\ f(\varphi_1(s_1),\ \psi_1(s_1))]\, \varphi_1'(s_1)\, ds_1 =$$

$$= \int_0^{s_1^0} P(\varphi_1(s_1), \psi_1(s_1), \chi_1(s_1))\, \varphi_1'(s_1)\, ds_1 = \int_\Gamma P(x, y, z)\, dx \qquad (3)$$

The proportion

$$\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : (-1) = \cos(n, x) : \cos(n, y) : \cos(n, z)$$

shows that

$$\cos(n, y) = -\frac{\partial f}{\partial y} \cos(n, z)$$

which implies the first equality in (3). The second equality is written on the basis of § 13.9 (5). The third equality follows from Green's formula. Finally, the last three equalities in (3) are valid if we suppose that the oriented contour Γ is representable by piecewise smooth functions

$$x = \varphi_1(s_1), \quad y = \psi_1(s_1) \quad \text{and} \quad z = \chi_1(s_1), \qquad 0 \leqslant s_1 \leqslant s_1^0$$

The first two of these functions specify the projection Γ_z of Γ on the plane $z = 0$ oriented in the coherent way. It should also be taken into account that Γ is the edge of the surface S determined by the equality $z = f(x, y)$ and therefore $\chi_1(s_1) = f(\varphi_1(s_1), \psi_1(s_1))$, $0 \leqslant s_1 \leqslant s_1^0$.

The equalities

$$\iint_S \left(\frac{\partial Q}{\partial x} \cos(n, z) - \frac{\partial Q}{\partial z} \cos(n, x)\right) ds = \int_\Gamma Q(x, y, z)\, dy \qquad (4)$$

and

$$\iint_S \left(\frac{\partial R}{\partial y}\cos(n, x) - \frac{\partial R}{\partial x}\cos(n, y)\right) ds = \int_\Gamma R(x, y, z)\, dz \tag{5}$$

are proved analogously.

From (3), (4) and (5) follows Stokes' formula (1).

We have proved Stokes' theorem for an oriented surface element projectable simultaneously on each of the three coordinate planes. There is another important simple case when Stokes' theorem holds to which the above argument is inapplicable. Namely, we mean the case when S^* is a part of a plane parallel to one of the coordinate axes. The validity of Stokes' theorem for such S^* can be verified with the aid of direct calculations similar to (3). However, it is also possible to use the following argument. The integrals entering into Stokes' formula are invariant with respect to the transformations of the rectangular coordinates preserving their orientation. A transformation of this type can always be chosen so that S^* can be projected in a one-to-one manner on each of the coordinate planes of the new coordinate system, which proves the desired assertion.

Stokes' formula continues to hold for any oriented surface S^* (with a piecewise smooth edge Γ) which can be decomposed with the aid of piecewise smooth curves into a finite number of smooth surface elements projectable in a one-to-one manner on each of the three coordinate planes.

Indeed, let $S^* = \sigma_1^* + \sigma_2^* + \ldots + \sigma_N^*$ be such a decomposition and let $\Gamma_1, \ldots, \Gamma_N$ be the coherently oriented contours bounding $\sigma_1^*, \ldots, \sigma_N^*$ respectively. Then, according to what was proved above,

$$\iint_S (\operatorname{rot} \boldsymbol{a}\, d\boldsymbol{\sigma}) = \sum_{j=1}^{N} \iint_{\sigma_j} (\operatorname{rot} \boldsymbol{a}\, d\boldsymbol{\sigma}) = \sum_{j=1}^{N} \int_{\Gamma_j} (\boldsymbol{a}\, d\boldsymbol{s}) = \int_\Gamma (\boldsymbol{a}\, d\boldsymbol{s})$$

because the parts of the integrals $\int_{\Gamma_j}$ $(j = 1, \ldots, N)$ corresponding to the arcs of Γ_j not belonging to Γ mutually cancel out since each of these arcs is traversed twice in the opposite directions.

An oriented surface which can be split into a finite number of (plane) triangles is called a *polyhedral* surface and is an example of a simple surface to which Stokes' formula is applicable.

We shall also note the following. Let σ_ε^* denote a circular plane area with centre at a point $A(x, y, z)$ and radius ε oriented with the aid of its definitely chosed unit normal vector $\boldsymbol{n}$; let γ_ε be the oriented contour bounding σ_ε^*. According to Stokes' formula, we have

$$\int_{\gamma_\varepsilon} (\boldsymbol{a}\, d\boldsymbol{s}) = \int_{\sigma_\varepsilon} (\boldsymbol{n} \operatorname{rot} \boldsymbol{a})\, d\sigma_\varepsilon = \int_{\sigma_\varepsilon} \operatorname{rot}_n \boldsymbol{a}\, d\sigma_\varepsilon = |\sigma_\varepsilon| \operatorname{rot}_n \boldsymbol{a}_1$$

where $\operatorname{rot}_n \boldsymbol{a}$ is a scalar function equal to the projection of $\operatorname{rot} \boldsymbol{a}$ on $\boldsymbol{n}$ and $\operatorname{rot}_n \boldsymbol{a}_1$ is the value of this function assumed at some point belonging to σ_ε.

It follows that the value of the function $\mathrm{rot}_n\, \boldsymbol{a}$ at the point A is expressed by the formula

$$\mathrm{rot}_n\, \boldsymbol{a} = \lim_{\varepsilon\to\infty} \frac{1}{|\sigma_\varepsilon|} \int\limits_{\gamma_\varepsilon} (\boldsymbol{a}\, d\boldsymbol{s}) \tag{6}$$

where in the passage to the limit as $\varepsilon \to 0$ it is supposed that the vector $\boldsymbol{n}$ remains invariable. The right-hand side of (6) is a number which is the same in any right-handed (left-handed) coordinate system. However, if a right-handed system is replaced by a left-handed one while $\boldsymbol{n}$ is invariable the direction in which the contour γ_ε of σ_ε is described changes to the opposite and therefore the sign of the right-hand member of (6) also changes. This again shows, in another way, that rot $\boldsymbol{a}$ is invariant with respect to the orientation preserving transformations of rectangular coordinates.

Remark. Let Ω be a domain in which a continuously differentiable field of a vector $\boldsymbol{a}$ is defined such that rot $\boldsymbol{a} = 0$. If a closed contour $\Gamma \subset \Omega$ is such that it is possible to pull over it an oriented smooth surface with Γ as its (oriented) edge then, according to Stokes' theorem, the line integral of $\boldsymbol{a}$ taken round the contour Γ is equal to zero:

$$\int\limits_{\Gamma} (\boldsymbol{a}\, d\boldsymbol{l}) = \iint\limits_{S} (\boldsymbol{n}\, \mathrm{rot}\, \boldsymbol{a}) = 0$$

This property can be used for proving Theorem 3 in § 13.3 but such a proof is connected with some sophisticated considerations because it should be taken into account that in any domain there are closed (knotted) contours over which it is impossible to stretch smooth surfaces.

§ 13.12. Differentiation of Integral with Respect to Parameter

We shall begin with the proof of the equality

$$\frac{\partial}{\partial x} \int\limits_{\Omega} f(x, y)\, dy = \int\limits_{\Omega} \frac{\partial}{\partial x} f(x, y)\, dy \qquad (a \leqslant x \leqslant b) \tag{1}$$

on condition that Ω is a closed measurable set in the space of the points $y = (y_1, \ldots, y_m)$ and the function f and its derivative $\frac{\partial f}{\partial x}$ are continuous on the set $H = [a, b]\times\Omega$ $(x \in [a, b], y \in \Omega)$.

In particular, if Ω is a closed interval $[c, d]$ formula (1) takes the form

$$\frac{\partial}{\partial x} \int\limits_{c}^{d} f(x, y)\, dy = \int\limits_{c}^{d} \frac{\partial}{\partial x} f(x, y)\, dy \tag{1'}$$

under the assumption that f and $\frac{\partial f}{\partial x}$ are continuous on $[a, b]\times[c, d]$.

Indeed, let us put

$$F(x) = \int_{\Omega} f(x, y)\, dy$$

then

$$\frac{F(x+h)-F(x)}{h} = \int_{\Omega} \frac{1}{h}[f(x+h, y)-f(x, y)]\, dy =$$

$$= \int_{\Omega} \frac{\partial f}{\partial x}(x+\theta h, y)\, dy \to \int_{\Omega} \frac{\partial f}{\partial x}(x, y)\, dy \qquad (h \to 0, \; 0 < \theta < 1)$$

because

$$\left| \int_{\Omega} \left[\frac{\partial f}{\partial x}(x+\theta h, y) - \frac{\partial f}{\partial x}(x, y)\right] dy \right| \leqslant \int_{\Omega} \omega\left(|h|, \frac{\partial f}{\partial x}\right) dy =$$

$$= |\Omega|\, \omega\left(|h|, \frac{\partial f}{\partial x}\right) \to 0 \qquad (h \to 0)$$

where $\omega\left(\delta, \frac{\partial f}{\partial x}\right)$ is the modulus of continuity of $\frac{\partial f}{\partial x}$ on the (bounded and closed) set H.

Now we shall suppose that f is a function $f = f(x, y)$ of two variables x and y and prove a generalization of formula (1) to an integral of the form

$$F(x) = \int_{\varphi(x)}^{\psi(x)} f(x, y)\, dy \qquad (a \leqslant x \leqslant b) \qquad (2)$$

where the functions φ and ψ satisfy the inequality $\varphi(x) \leqslant \psi(x)$ $(a \leqslant x \leqslant b)$ and are continuously differentiable while the function $f(x, y)$ of the scalar variables x and y and its partial derivative $\frac{\partial f}{\partial x}$ are continuous on the set of points (x, y) (see also § 7.11) whose coordinates satisfy the inequalities $\varphi(x) \leqslant y \leqslant \psi(x)$ and $a \leqslant x \leqslant b$.

Let us show that the function $F(x)$ possesses the derivative on $[a, b]$ determined by the formula

$$F'(x) = f(x, \psi(x))\psi'(x) - f(x, \varphi(x))\varphi'(x) + \int_{\varphi(x)}^{\psi(x)} \frac{\partial}{\partial x} f(x, y)\, dy \qquad (3)$$

To this end we construct the auxiliary function

$$\Phi(x, u, v) = \int_{u}^{v} f(x, y)\, dy \qquad (4)$$

defined on the set H of the points (x, u, v) specified by the inequalities $\varphi(x) \leqslant u \leqslant v \leqslant \psi(x)$ and $a \leqslant x \leqslant b$.

The function $z = F(x)$ can be regarded as a function of a function of the form $z = \Phi(x, u, v)$, $u = \varphi(x)$, $v = \psi(x)$ $(a \leqslant x \leqslant b)$ and its derivative can be computed by the well-known formula

$$F'(x) = \frac{dz}{dx} = \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial u}\varphi'(x) + \frac{\partial \Phi}{\partial v}\psi'(x) \tag{5}$$

where the right-hand side is taken for the values $u = \varphi(x)$ and $v = \psi(x)$ of the variables u and v; however, we have to show that the partial derivatives of Φ are continuous functions of (x, u, v) and to express them in terms of f, φ and ψ. By the continuity of $f(x, y)$ with respect to y and by virtue of the theorem on the differentiation of an integral with respect to the upper and lower limits of integration, it follows from (4) that

$$\frac{\partial \Phi}{\partial v} = f(x, v) \quad \text{and} \quad \frac{\partial \Phi}{\partial u} = -f(x, u) \tag{6}$$

and, since f is continuous, the right-hand members of (6) are continuous with respect to (x, u, v) and therefore so are the left-hand members.

The derivative $\frac{\partial f}{\partial x}$ is continuous by the hypothesis, and therefore, by virtue of (1), we have

$$\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\int_u^v f(x, y)\,dy = \int_u^v \frac{\partial}{\partial x} f(x, y)\,dy \tag{7}$$

(in this connection see the remark below). Further, we can formally regard $\frac{\partial f}{\partial x}$ as a function of the variables (x, u, v, y): $\frac{\partial}{\partial x} f(x, y) = \gamma(x, u, v, y)$. It is obviously continuous with respect to these variables, and u and v can be considered as continuous functions of (x, u, v):

$$u = \lambda(x, u, v) \qquad v = \mu(x, u, v) \qquad (x, u, v) \in H$$

Hence, using this notation we can write

$$\frac{\partial \Phi}{\partial x} = \int_{\lambda(x, u, v)}^{\mu(x, u, v)} \gamma(x, u, v, y)\,dy$$

Consequently $\frac{\partial \Phi}{\partial x}$ is a continuous function of (x, u, v) (see § 12.13, Theorem 2).

We have justified equality (5).

The substitution of (6) and (7) into (5) and the change of variables $u = \varphi(x)$, $v = \psi(x)$ now lead to formula (4).

Remark. Strictly speaking, equality (7) was proved only for the points (x_0, u_0, v_0) for which

$$\varphi(x_0) < u_0 < v_0 < \psi(x_0) \tag{8}$$

In this case it is possible to choose a number $\delta > 0$ so small that the rectangle $\{|x-x_0| \leqslant \delta, u_0 \leqslant y \leqslant v_0\}$ will belong to the domain of definition of $f(x, y)$. This enables us to apply formula (1) (with $c = u_0$ and $d = v_0$). If $u_0 = \varphi(x_0)$ or $v_0 = \psi(x_0)$ (or both) there may be no such rectangle for any $\delta > 0$. However, the right-hand member of (7) makes sense and is continuous for all the points (x, u, v) of the closed set H, and therefore the partial derivative $\frac{\partial \Phi}{\partial x}$ defined originally only for the points (x_0, u_0, v_0) of form (8) can be continuously extended to the whole set H if we put it equal to the right member of (7). This defines the (generalized) derivative $\frac{\partial \Phi}{\partial x}$ on H. The theorem on the derivative of a composite function continues to hold for the derivatives understood in this generalized sense (see § 7.11).

§ 13.13. Improper Integrals

Let G be a measurable open set in the n-dimensional space and let x^0 be a point belonging to G.

Let us take the (closed) ball $\omega(\varepsilon) = \{|x-x^0| \leqslant \varepsilon\}$ of radius $\varepsilon > 0$ with centre at x^0 and construct the (open) set $G_\varepsilon = G - \omega(\varepsilon)$.

We shall say that *an integral of the form*

$$\int_G f\,dx \tag{1}$$

has only one singularity, at the point x^0, if the function f is defined and unbounded on G, but is bounded and integrable on G_ε for an arbitrarily small $\varepsilon > 0$.

It should be stressed that if integral (1) has (only one) singularity at the point x^0 the integrand function $f(x)$ is not Riemann integrable because it is unbounded on the measurable open set G (see Theorem 1 in § 12.10 and what precedes it).

If integral (1) *has only one singularity at the point x^0 we say that it exists in the improper sense if the limit*

$$\lim_{\varepsilon \to 0} \int_{G_\varepsilon} f(x)\,dx = \int_G f(x)\,dx \tag{2}$$

exists. In this case we assign to expression (1) the numerical value equal to limit (2) which is the *improper integral of f over G.*

Since for $0 < \delta_1 < \delta_2$ we have

$$\int_{G_{\delta_1}} - \int_{G_{\delta_2}} = \int_{G_{\delta_1} - G_{\delta_2}}$$

Cauchy's necessary and sufficient criterion for the existence of a limit implies that, under the conditions stated, improper integral (2) exists if and only if, given an arbitrary $\varepsilon > 0$, there is $\delta > 0$ such that for any δ_1 and δ_2

satisfying the inequalities $0 < \delta_1, \delta_2 < \delta$ there holds the inequality

$$\left| \int\limits_{G_{\delta_1}-G_{\delta_2}} f\,dx \right| < \varepsilon$$

It follows that if Ω is an arbitrary measurable open set containing the point x^0 and $G_1 = \Omega G$ then the integrals $\int\limits_G f\,dx$ and $\int\limits_{G_1} f\,dx$ exist or do not exist simultaneously (see Figs. 13.21 and 13.22) because the application of Cauchy's criterion to them leads to one and the same conclusion.

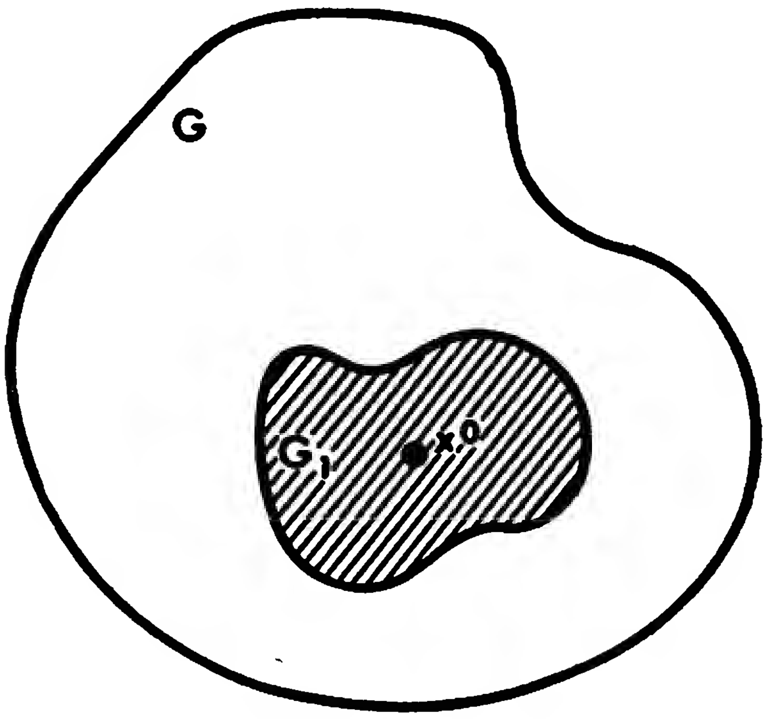

Fig. 13.21

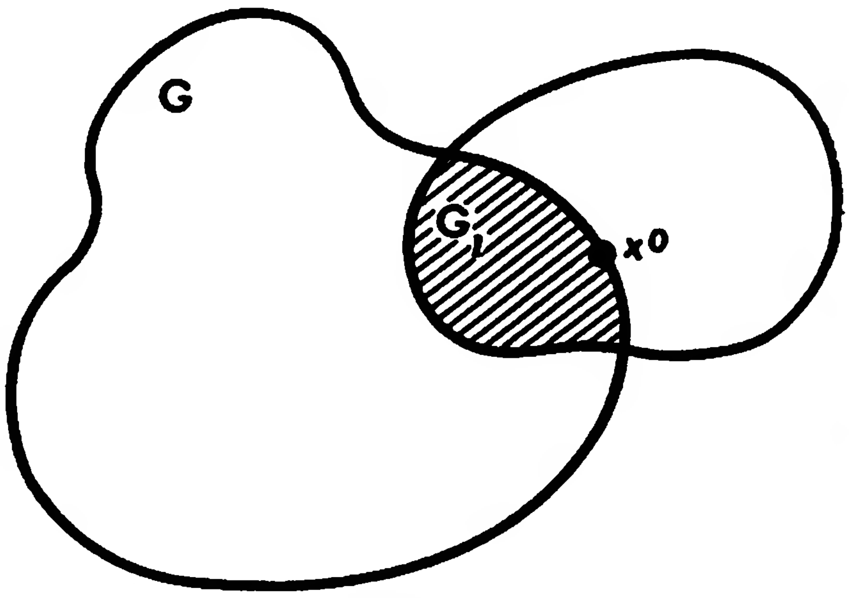

Fig. 13.22

Let us consider the special case of dimension 1. Let $f(x)$ be a function defined on a (one-dimensional) interval (a, b) such that the integral $\int\limits_a^b f(x)\,dx$ has an only singularity at a point $x^0 \in [a, b]$.

If $x^0 = a$ or $x^0 = b$ the above definition of the improper integral reduces to the ordinary definition which was stated in § 9.12. In the case when $x^0 \in (a, b)$, that is when x^0 is an interior point of the closed interval $[a, b]$, the improper integral of $f(x)$ is regarded as existent in accordance with the definition stated in this section if the limit

$$\lim_{\varepsilon\to 0}\left(\int\limits_a^{x^0-\varepsilon} f\,dx + \int\limits_{x^0+\varepsilon}^{b} f\,dx\right) = \int\limits_a^b f\,dx \qquad (3)$$

exists. But this is nothing but the definition of Cauchy's principal value for a one-dimensional improper integral stated at the end of § 9.16 (but not of the ordinary Riemann improper integral because the definition of the latter requires that the limit (as $\varepsilon \to 0$) of each of the integrals on the left-hand side of (3) should exist).

Given any constants A and B, we have the obvious equality

$$\int_G (Af(x)+B\varphi(x))\,dx = A\int_G f(x)\,dx + B\int_G \varphi(x)\,dx \tag{4}$$

for improper integrals which, as usual, reads: *if the improper integrals on the right-hand side of equality* (4) *exist then the integral on its left-hand side also exists and is equal to the right-hand side of* (4).

If f is a nonnegative function on G then (finite or infinite) limit (2) always exists because the expression under the sign of the limit does not decrease when ε monotonically tends to zero.

In case this limit is finite we write

$$\int_G f\,dx < \infty$$

and if it is infinite we write

$$\int_G f\,dx = \infty$$

It is clear that for a nonnegative function of one variable the definition of the integral in the sense of Cauchy's principal value and that of the Riemann improper integral coincide in the case when $x^0 \in (a, b)$ because in this case the existence of the limit of the sum on the left-hand side of (3) implies the existence of the limit of each of the two summands entering into that sum.

Obviously, we can also say that *the improper integral of the above-described nonnegative function defined on G exists if and only if the expression under the sign of the limit in* (2) *is bounded above by a constant M independent of $\varepsilon > 0$.*

Let us state one more equivalent definition: *the improper integral over G of a nonnegative function f with only one singularity at $x^0 \in \bar{G}$ is the limit*

$$\int_G f\,dx = \lim_{n\to\infty} \int_{G-\lambda_n} f\,dx \tag{5}$$

where λ_n are domains possessing the following properties:

(i) $\lambda_n \ni x^0$,
(ii) $\lambda_n \supset \lambda_{n+1}$,
(iii) *the diameter $d(\lambda_n)$ tends to zero as $n \to \infty$.*

If there are two nonnegative functions f and φ defined on G whose integrals have only one singularity at x^0 and if $f(x) \leqslant \varphi(x)$ $(x \in G)$ then

$$\int_{G_\varepsilon} f\,dx \leqslant \int_{G_\varepsilon} \varphi\,dx$$

for any $\varepsilon > 0$. Both members of this inequality are monotone increasing as ε monotonically tends to zero, and therefore, on passing to the limit for

$\varepsilon \to 0$, we obtain the inequality

$$\int_G f\,dx \leqslant \int_G \varphi\,dx \tag{6}$$

whose members can be finite or infinite. If the integral on the right-hand side of (6) is finite then so is the one on the left-hand side and if the integral on the left-hand side is infinite then the integral on the right-hand side is also infinite.

Integral (1) with one singularity at a finite point x^0 is said to be *absolutely convergent* if the integral of the absolute value of $f(x)$ is convergent:

$$\int_G |f|\,dx < \infty \tag{7}$$

An absolutely convergent integral (1) is convergent since (7) implies that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\varepsilon > \int_{G_{\delta_1}-G_{\delta_2}} |f|\,dx \geqslant \left| \int_{G_{\delta_1}-G_{\delta_2}} f\,dx \right|, \qquad 0 < \delta_1 < \delta_2 < \delta$$

Example 1. Let us consider the integral

$$\int_\Omega \frac{dx}{r^\alpha} \tag{8}$$

where $r = \sqrt{x_1^2 + \ldots + x_n^2}$ and Ω is the unit ball in the n-dimensional space with centre at the origin.

For $\alpha > 0$ expression (8) is obviously an improper integral with only one singularity at the origin. Its value is defined as the limit

$$\int_\Omega \frac{dx}{r^\alpha} = \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \frac{dx}{r^\alpha}$$

On passing to spherical (spacial polar) coordinates with the aid of the transformation

$$\begin{aligned} x_1 &= r \cos \varphi_1 \\ x_2 &= r \sin \varphi_1 \cos \varphi_2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{n-1} &= r \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{n-2} \cos \varphi_{n-1} \\ x_n &= r \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{n-2} \sin \varphi_{n-1} \end{aligned}$$

and taking into account that the Jacobian of this transformation is

$$I = r^{n-1} \sin \varphi_1^{n-2} \sin \varphi_2^{n-3} \ldots \sin \varphi_{n-1}$$

we obtain the equality

$$I_\alpha = \sigma_n \lim_{\varepsilon \to 0} \int_\varepsilon^1 r^{n-\alpha-1}\, dr \tag{9}$$

where $\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the area of the spherical surface bounding the unit ball in the n-dimensional space.

It follows from (9) that

$$I_\alpha < \infty \quad \text{if} \quad \alpha < n$$

and

$$I_\alpha = \infty \quad \text{if} \quad \alpha \geqslant n$$

As a generalization of this example we can consider the integral

$$\int_\Omega \frac{|\varphi(x)|\, dx}{r^\alpha} \tag{10}$$

where φ is a continuous function on Ω. Let us put

$$M = \max_{x \in \Omega} |\varphi(x)|$$

For $\alpha < n$ we have

$$\int_\Omega \frac{|\varphi(x)|\, dx}{r^\alpha} \leqslant M \int_\Omega \frac{dx}{r^\alpha} < \infty$$

that is integral (10) is absolutely convergent.

Now let $\alpha \geqslant n$ and $|\varphi(0)| > 0$. Then there is a ball ω with centre at the origin and radius so small that $|\varphi(x)| > \frac{|\varphi(0)|}{2} = m > 0$ on that ball. Therefore

$$\left| \int_\omega \frac{\varphi(x)}{r^\alpha}\, dx \right| = \int_\omega \frac{|\varphi(x)|}{r^\alpha}\, dx \geqslant m \int_\omega \frac{1}{r^\alpha}\, dx = \infty$$

and consequently integral (10) is divergent for $\alpha \geqslant n$.

The notion of a Riemann multiple integral is defined for a measurable and, consequently, bounded domain. If the domain is unbounded then, under certain conditions, it is possible to state the definition of the notion of the improper integral over such domain.

Let us consider an unbounded domain G lying in the n-dimensional space and possessing the property that, given any ball $\omega(R)$ with centre at the origin and radius R, the intersection $G(R) = G\omega(R)$ of G with that ball is a measurable set. Further, let there be given a function $f(x)$ defined on G and integrable on $G(R)$ for any R. In this case we say that *f has only one singularity at the point at infinity* and define the *improper integral of f over G* as the limit

$$\lim_{R \to \infty} \int_{G(R)} f\, dx = \int_G f\, dx \tag{11}$$

All that was said about an improper integral with only one singularity at a finite point x^0 can be repeated with some modifications for an improper integral having one singularity at the point at infinity, and we shall not go into particulars here.

Example 2. The transformation to polar coordinates brings the integral

$$I_\alpha = \int\limits_{r>1} \frac{dx}{r^\alpha} = \lim_{R\to\infty} \int\limits_{1<r<R} \frac{dx}{r^\alpha}$$

to the expression

$$I_\alpha = \sigma_n \lim_{R\to\infty} \int_1^R r^{n-\alpha-1}\,dr = \begin{cases} \dfrac{\sigma_n}{n-\alpha} & \text{for} \quad a > n \\ \infty & \text{for} \quad a \leqslant n \end{cases}$$

Finally, let us discuss a more general case when the closure of a domain G where a function f is defined can be broken up into a finite number of closures of open sets so that these closures can only intersect along some parts of their boundaries:

$$\bar{G} = \bar{G}_1 + \ldots + \bar{G}_N \tag{12}$$

If each of the integrals $\int\limits_{G_j} f\,dx$ $(j = 1, \ldots, N)$ has only one singularity $x^j \in G_j$ and if the domain G is unbounded then only one of these integrals has the point at infinity as its singular point. Besides, the points x^j are pairwise distinct.

In this case the *improper integral of f on G* is defined as the sum

$$\int\limits_G f\,dx = \sum_{j=1}^{N} \int\limits_{G_j} f\,dG \tag{13}$$

If at least one of the integrals entering into this sum is divergent, the integral on the left-hand side of (13) is also considered divergent. Similarly, integral (13) is said to be *absolutely convergent* if and only if all the integrals entering into sum (13) are absolutely convergent.

Here we shall not carry out the proof (which is quite simple) of the fact that the definition stated leads to the result (number) independent of all the possible ways in which the domain G can be split into parts with the indicated properties.

Example 3. It is evident that the integral $\int\limits_{R_n} \frac{dx}{r^\alpha}$ where R_n is the whole n-dimensional space is divergent for any real α.

Example 4. The integral

$$I = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dx\,dy$$

can be defined as the limit

$$I = \lim_{N\to\infty} \int_0^N \int_0^N e^{-x^2-y^2}\, dx\, dy = \lim_{N\to\infty} \int_0^N e^{-x^2}\, dx \int_0^N e^{-y^2}\, dy = \left(\int_0^\infty e^{-x^2}\, dx\right)^2$$

where the improper integral with respect to one variable standing on the right-hand side is convergent. On the other hand, denoting by Ω (N) the quarter circle of radius N and centre at the origin lying in the first quadrant we can write

$$I = \lim_{N\to\infty} \iint_{\Omega(N)} e^{-\varrho^2} \varrho\, d\varrho\, d\theta = \lim_{N\to\infty} \int_0^{\pi/2} d\theta\, \frac{1}{2} \int_0^N e^{-\varrho^2}\, d(\varrho^2) = \frac{\pi}{2} \cdot \frac{1}{2} = \frac{\pi}{4}$$

whence follows

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

Example 5. The function $\psi(x)$ defined as being equal to $e^{-|x|}$ at the points $x = (x_1, \ldots, x_n)$ with irrational coordinates and to $-e^{-|x|}$ at all the other points of the n-dimensional space R_n is not absolutely integrable although the integral of $|\psi|$ over R_n is convergent:

$$\int_{R_n} |\psi|\, dx < \infty$$

Indeed, this is so since the function $\psi(x)$ is discontinuous throughout R_n.

Exercise. Check that for the fundamental solution v of the heat conductivity equation (see the exercise in § 7.7) we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} v\, dx\, dy\, dz = 1 \qquad (t_0 > t)$$

Hint. Pass to polar (spherical) coordinates.

Remark. Let Ω be a bounded domain with a doubly continuously differentiable boundary S. Let us denote by S_λ the surface lying within Ω whose points are at a distance of $\lambda > 0$ from S, reckoned along the inner normal to S (see § 7.25), and let $\Omega_\lambda \subset \Omega$ be the domain bounded by S_λ.

If $f(x)$ is an unbounded function defined on Ω and integrable on Ω_λ for any sufficiently small $\lambda > 0$ we can define the *improper integral of f over Ω* as the limit

$$\int_\Omega f(x)\, dx = \lim_{\lambda\to 0} \int_{\Omega_\lambda} f(x)\, dx$$

provided it exists.

On the basis of this definition it is also possible to introduce the notion of an absolutely convergent integral.

The idea on which this definition of an improper integral of a function having singularities on the boundary of the domain of integration is based is similar to that of definition (2) considered in this section. A narrower definition of this type was stated in § 12.22. Of course, both definitions coincide in the case of a nonnegative function defined in a domain of the form indicated here. However, the definition stated here is more general because it can be shown that if an integral is convergent in the sense of § 12.22 then it is absolutely convergent. The definition stated here was in fact already used in the derivation of the generalized Gauss-Ostrogradsky theorem at the end of § 13.5 and in §§ 13.6 and 13.10.

§ 13.14. Uniform Convergence of Improper Integrals

Let $\Omega \subset R_m$ and $D \subset R_n$ be some sets in the m-dimensional and n-dimensional spaces R_m and R_n respectively. We shall also suppose that the set D is measurable. By ω_δ we shall denote the open ball of radius δ with centre at y^0 where the point y^0 belongs to $\bar{D}$.

Let us consider the integral

$$F(x) = \int_D f(x, y)\, dy \qquad (x \in \Omega) \tag{1}$$

having only one singularity at the point y^0. This means that $f(x, y)$ as function of y is unbounded on D but bounded and integrable on $D - \omega_\delta$ for any $\delta > 0$.

Integral (1) *is said to be uniformly convergent with respect to* $x \in \Omega$ *if it is convergent for any* $x \in \Omega$ *and if, given any* $\varepsilon > 0$, *there is* $\delta_0 > 0$ *such that for any* δ *satisfying the condition* $0 < \delta < \delta_0$ *the inequality*

$$\left| \int_{D\omega_\delta} f(x, y)\, dy \right| < \varepsilon \qquad (D\omega_\delta = D \cap \omega_\delta) \tag{2}$$

holds for every $x \in \Omega$.

It is important in this definition that δ_0 and δ are independent of $x \in \Omega$.

Taking a positive δ we form the integral

$$F_\delta(x) = \int_{D-\omega_\delta} f(x, y)\, dy$$

which obviously has no singularities. Then inequality (2) can be rewritten in the form

$$|F(x) - F_\delta(x)| = \left| \int_{D\omega_\delta} f(x, y)\, dy \right| < \varepsilon$$

and thus we see that for any $\varepsilon > 0$ there is δ_0 such that for all δ satisfying the inequalities $0 < \delta < \delta_0$ and for any $x \in \Omega$ there holds the inequality

$$|F(x) - F_\delta(x)| < \varepsilon$$

As is known, the last inequality expresses the fact that

$$\lim_{\delta \to 0} F_\delta(x) = F(x) \text{ uniformly on } \Omega \tag{3}$$

Conversely, it is obvious that (3) implies (2).

Hence, relation (3) can be regarded as an equivalent definition of the uniform convergence of integral (1) on Ω.

There holds the following theorem.

Theorem 1. *If the function $f(x, y)$ is continuous on $\bar{\Omega} \times \bar{D}$ everywhere except at the points of the form (x, y^0) and if integral (1) is uniformly convergent with respect to $x \in \bar{\Omega}$ then this integral is a continuous function of $x \in \bar{\Omega}$.*

Proof. It follows from the conditions of the theorem that the function $f(x, y)$ is continuous at the points (x, y) belonging to the bounded closed set $\bar{\Omega} \times (\bar{D} \backslash \omega_\delta)$ and, besides, the set $\bar{D}$ is measurable. Therefore the function $F_\delta(x)$ is continuous on $\bar{\Omega}$ (see Theorem 1 in § 12.13). Further, $F_\delta(x)$ tends to $F(x)$ as $\delta \to 0$ uniformly on $\bar{\Omega}$. Hence, by virtue of Theorem 2 in § 11.7, the function $F(x)$ is continuous on $\bar{\Omega}$ because, although this theorem was proved for sequences of functions $\{f_n\}$ dependent on the natural parameter n, it obviously continues to hold (and the proof remains quite analogous) when n is a parameter tending continuously to a number n_0. In the case under consideration we have $n = \delta \to 0$.

Theorem 2. *If the conditions of Theorem 1 hold and Ω is measurable (in R_m) then the function $F(x)$ specified by the equality*

$$F(x) = \int_D f(x, y)\, dy \qquad (x \in \bar{G}) \tag{4}$$

can be integrated (with respect to x) on Ω under the integral sign, that is

$$\int_\Omega F(x)\, dx = \int_D dy \int_\Omega f(x, y)\, dx \tag{5}$$

Proof. From the proof of Theorem 1 we know that under the conditions of this theorem the functions $F_\delta(x)$ and $F(x)$ are continuous on Ω and $F_\delta(x)$ uniformly tends to $F(x)$ as $\delta \to 0$. This means that

$$\eta_\delta = \max_{x \in \Omega} |F_\delta(x) - F(x)| \to 0, \qquad \delta \to 0$$

It follows that

$$\int_\Omega F_\delta(x)\, dx \to \int_\Omega F(x)\, dx, \qquad \delta \to 0 \tag{7}$$

because

$$\left|\int\limits_{\Omega} F_\delta(x)\,dx-\int\limits_{\Omega} F(x)\,dx\right| \leqslant \int\limits_{\Omega} |F_\delta(x)-F(x)|\,dx \leqslant \eta_\delta|\Omega| \to 0, \qquad \delta\to 0$$

From (7) follows

$$\int\limits_{D\setminus\omega_\delta} dy \int\limits_{\Omega} f(x,y)\,dx = \int\limits_{\Omega} dx \int\limits_{D\setminus\omega_\delta} f(x,y)\,dy = \int\limits_{\Omega} F_\delta(x)\,dx \to \int\limits_{\Omega} F(x)\,dx =$$

$$= \int\limits_{\Omega} dx \int\limits_{D} f(x,y)\,dy, \qquad \delta\to 0 \quad (8)$$

which proves equality (6).

The first equality in (8) is obtained by an ordinary change of the order of integration with respect to x and y, which is legitimate, for any $\delta > 0$, since the function $f(x, y)$ is continuous on the measurable closed set $\Omega\times(D\setminus\omega_\delta)$ (see § 12.2).

It should be noted that the integral with respect to y on the right-hand side of (5) is an improper integral with one singularity at the point $y^0 \in \bar{\Omega}$. Its existence has been proved.

Theorem 3. *Let G be a measurable open set in the space R_n, y^0 be a point belonging to $\bar{G}$ and $[a, b]$ be the range of a scalar variable x. Further, let a function $f(x, y)$ be continuous and have a continuous partial derivative $\frac{\partial f}{\partial x}$ everywhere on the set $[a, b]\times\bar{G}$ ($x \in [a, b]$, $y \in \bar{G}$) except possibly at points of the form (x, y^0) in whose vicinity $f(x, y)$ may not be bounded.*

Then if the integral

$$F(x) = \int\limits_{G} f(x,y)\,dy \qquad (a \leqslant x \leqslant b) \tag{9}$$

is convergent and if the integral

$$F_1(x) = \int\limits_{G} \frac{\partial}{\partial x} f(x,y)\,dy \tag{10}$$

is uniformly convergent with respect to $x \in [a, b]$, there holds the equality

$$F_1(x) = F'(x) \tag{11}$$

that is it is allowable to differentiate F under the integral sign:

$$\frac{\partial}{\partial x}\int\limits_{G} f(x,y)\,dy = \int\limits_{G} \frac{\partial}{\partial x} f(x,y)\,dy \tag{12}$$

Proof. Let us put

$$F_\delta(x) = \int\limits_{G\setminus\omega_\delta} f(x,y)\,dy$$

where $\delta > 0$.

Then we can write (see § 13.12)

$$\frac{\partial}{\partial x} F_\delta(x) = \int\limits_{G\backslash\omega_\delta} \frac{\partial}{\partial x} f(x, y)\, dy = F_{1\delta}(x)$$

since f and $\frac{\partial f}{\partial x}$ are continuous on $[a, b]\times[G\backslash\omega_\delta]$.

By the hypothesis,

$$F_\delta(x) \to F(x), \qquad \delta \to 0, \; x \in [a, b] \tag{13}$$

Furthermore, by the uniform convergence of integral (10), we have

$$\lim_{\delta\to 0} \frac{\partial}{\partial x} F_\delta(x) = \lim_{\delta\to 0} F_{1\delta}(x) = F_1(x) \tag{13'}$$

uniformly on $[a, b]$.

On the basis of Theorem 3′ in § 11.8 and by virtue of the remark to that theorem, it follows from (13) and (13′) that $F(x)$ possesses a derivative on $[a, b]$ equal to $F_1(x)$.

In all the theorems proved here it is very important that the improper integrals in question are uniformly convergent with respect to the parameters. If a convergent integral is not uniformly convergent it is said to be *nonuniformly convergent*. Generally speaking, for integrals of this kind the above three theorems do not hold.

An important criterion for uniform convergence is the so-called *Weierstrass' test* which can be stated as the following theorem.

Theorem 4. *Let integral* (1) *have only one singularity at a point* $y^0 \in \bar{D}$ *for all* $x \in \Omega$ *and let there exist a nonnegative function* $\varphi(y)$ *such that*

$$|f(x, y)| \leq \varphi(y) \quad \text{for all} \quad (x, y) \in \bar{G}\times\bar{D} \tag{14}$$

and the improper integral

$$\int\limits_D \varphi(y)\, dy < \infty \tag{15}$$

having singularity at y^0 *exists. Then integral* (1) *is uniformly convergent with respect to* $x \in \bar{G}$.

Proof. It follows from (15) that, given any $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\int\limits_{D\omega_\delta} \varphi(y)\, dy < \varepsilon \qquad (0 < \delta < \delta_0)$$

where ω_δ is the ball with centre at y^0 and radius δ. Therefore, by virtue of (14), we have

$$\left|\int\limits_{D\omega_\delta} f(x, y)\, dy\right| \leq \int\limits_{D\omega_\delta} \varphi(y)\, dy < \varepsilon$$

for all $x \in \bar{G}$. The theorem has been proved.

Example 1. The integral

$$\psi(a) = \int_0^1 x^{a-1}\, dx \qquad (a > 0) \tag{16}$$

exists for all $a > 0$. For $a > 0$ its singularity, provided there is such, can only be at one point $x = 0$. More precisely, for $0 < a < 1$ the point $x = 0$ is singular while for $a \geqslant 1$ the integrand is a continuous function on the closed interval $[0, 1]$ and the integral has no singularities. However, for testing the integral for uniform convergence it is inessential whether or not there is a singularity at the point $x = 0$; it is quite sufficient to know that if the integral has a singularity it must be at the point $x = 0$.

To investigate this integral with a possible singularity at the point $x = 0$ we must consider the integral

$$\left| \int_0^\delta x^{a-1}\, dx \right| = \frac{\delta^a}{a}$$

For an arbitrary $\varepsilon > 0$ it is impossible to choose $\delta > 0$ so that this integral is less than ε in its absolute value for all $a > 0$ because for any fixed δ we have $\lim\limits_{a \to 0} (\delta^a/a) = \infty$ $(a > 0)$. Therefore integral (16) is not uniformly convergent with respect to $a > 0$. It is also obvious that it is nonuniformly convergent for $a \in (0, a_0)$ where a_0 is an arbitrary fixed positive number.

At the same time, integral (16) is uniformly convergent on any interval $a_0 \leqslant a < \infty$ $(a_0 > 0)$ and, moreover, on every finite closed interval $[a_0, a_1]$, which can readily be shown by applying Weierstrass' test. Indeed, if $a_0 \leqslant a$ then we have $x^{a-1} \leqslant x^{a_0-1}$ on the interval $[0, \delta]$ with $0 < \delta < 1$, and the integral

$$\int_0^\delta x^{a_0-1}\, dx < \infty$$

is convergent.

The function $\psi(a)$ is continuous for all $a > 0$. Indeed, let us take an arbitrary $a_0 > 0$ and let

$$0 < a_1 < a_0 < a_2$$

The integrand $x^{a-1} = \varphi(x, a)$ is continuous on the rectangle $\Delta = \{0 \leqslant x \leqslant 1, a_1 \leqslant a \leqslant a_2\}$ except possibly at points (x, a) with $x = 0$, and integral (16) is uniformly convergent for $a \in [a_1, a_2]$. Hence, by Theorem 1, the function $\psi(a)$ is continuous on $[a_1, a_2]$ and, in particular, at the point $a = a_0$.

For $a > 0$ there holds the formula

$$\psi'(a) = \int_0^1 \frac{\partial}{\partial a} x^{a-1}\, dx = \int_0^1 x^{a-1} \ln x\, dx \tag{17}$$

Again, to check the validity of the formula for $a = a_0 > 0$ we choose numbers a_1 and a_2 such that $0 < a_1 < a_0 < a_2$ and, in order to apply Theorem 3, show that integral (17) is uniformly convergent with respect to $a \in [a_1, a_2]$. Here it is convenient to resort to Weierstrass' test.

Since $\lim\limits_{x \to 0} x^\lambda \ln x = 0$ $(\lambda > 0)$ and the function $x^\lambda \ln x$ is continuous on (0, 1) there is a positive constant C such that $|x^\lambda \ln x| \leqslant C$ $(0 < x \leqslant 1)$. Consequently, for $0 < \lambda < a_1$ we have

$$|x^{a-1} \ln x| = |x^{a-\lambda-1} x^\lambda \ln x| \leqslant C x^{a-\lambda-1} \leqslant C x^{a_1-\lambda-1}, \qquad a \in [a_1, a_2]$$

on the closed interval [0, 1], and the integral on the right-hand side of (17) is uniformly convergent because the integral

$$\int_0^1 C x^{a_1-\lambda-1}\, dx < \infty$$

is convergent. Finally, taking into account that the function $\frac{\partial}{\partial a} x^{a-1} = x^{a-1} \ln x$ is continuous on the rectangle $[0, 1] \times [a_1, a_2]$ except possibly at points of form $(0, a)$ we see that, by virtue of Theorem 3, equality (17) is fulfilled.

Example 2. *Beta function.* The function

$$B(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}\, dx \tag{18}$$

is called the *beta function* or *Euler's integral of the first kind.* The singularities of integral (18), provided there are such, can only be at the points $x = 0$ and $x = 1$. Therefore to test the integral for uniform convergence it is convenient to split it into the two integrals

$$B_1(a, b) = \int_0^{1/2} x^{a-1}(1-x)^{b-1}\, dx$$

and

$$B_2(a, b) = \int_{1/2}^1 x^{a-1}(1-x)^{b-1}\, dx$$

The singularity of the integral B_1, provided there is such, can only be at the point $x = 0$. This integral is convergent for $a > 0$ and any b because

$$\int_0^{1/2} x^{a-1}(1-x)^{b-1}\, dx \leqslant M_b \int_0^{1/2} x^{a-1}\, dx < \infty, \quad M_b = \max_{0 \leqslant x \leqslant 1/2} (1-x)^{b-1}$$

and is divergent for $a \leq 0$ because

$$\int_0^{1/2} x^{a-1}(1-x)^{b-1}\,dx \geq m_b \int_0^{1/2} x^{a-1}\,dx = \infty, \qquad m_b = \min_{1\leq x\leq 1/2} (1-x)^{b-1} > 0$$

Similarly, the integral $B_2(a, b)$ is convergent for $b > 0$ and divergent for $b \leq 0$. Therefore the beta function makes sense only for $a > 0$ and $b > 0$.

In order to prove that $B_1(a, b)$ is a continuous function of the variables a and b at an arbitrary point (a_0, b_0) with $a_0 > 0$ and $b_0 > 0$ let us take a rectangle $\Delta = \{a_1 \leq a \leq a_2;\ b_1 \leq b \leq b_2\}$ $(a_1, b_1 > 0)$ such that the point (a_0, b_0) is strictly inside it. The integral $\int_0^{\delta} x^{a-1}(1-x)^{b-1}\,dx$ can now be estimated as follows:

$$\int_0^{\delta} x^{a-1}(1-x)^{b-1}\,dx \leq M_{b_1} \int_0^{\delta} x^{a-1}\,dx = M_{b_1}\frac{\delta^{a}}{a} \leq M_{b_1}\frac{\delta^{a_1}}{a_1}$$

For any $\varepsilon > 0$ there is $\delta_0 > 0$ such that

$$M_{b_1}\frac{\delta^{a_1}}{a_1} \leq \frac{M_{b_1}\delta_0^{a_1}}{a_1} < \varepsilon$$

for $0 < \delta < \delta_0$, whence it follows that the integral determining $B_1(a, b)$ is uniformly convergent with respect to $(a, b) \in \Delta$ because

$$\int_0^{\delta} x^{a-1}(1-x)^{b-1}\,dx < \varepsilon$$

for any $(a, b) \in \Delta$ and $0 < \delta < \delta_0$. Since the integrand is a continuous function of (x, a, b) $\big(x > 0, (a, b) \in \Delta\big)$ the function B_1 is continuous at the point (a_0, b_0). The continuity of $B_2(a, b)$ is proved in like manner.

There holds the relation

$$\frac{\partial}{\partial a}\int_0^{1} x^{a-1}(1-x)^{b-1}\,dx = \int_0^{1} x^{a-1}\ln x\,(1-x)^{b-1}\,dx \qquad (0 < a, b)$$

Indeed, both integrals entering into this relation are convergent, and it can readily be shown that the second integral is uniformly convergent in a sufficiently small neighbourhood of the point a; besides, the integrand in the right-hand member of the relation is continuous with respect to (x, a) except possibly at points of form $(0, a)$. A similar argument readily shows that for any $k, l = 0, 1, \ldots$ there exists the partial derivative $\dfrac{\partial^{k+l}B}{\partial a^k\,\partial b^l}$ continuous on

$[a, b]$ $(a, b > 0)$ which is expressed by the formula

$$\frac{\partial^{k+l}B}{\partial a^k \partial b^l} = \int_0^1 x^{a-1} \ln^k x (1-x)^{b-1} \ln^{l} (1-x)\, dx$$

§ 13.15. Uniformly Convergent Integral over Unbounded Domain

We shall consider an integral of the form

$$F(x) = \int_D f(x, y)\, dy \qquad (x \in G) \tag{1}$$

taken over an unbounded domain D such that it has the point at infinity as its only singularity for any $x \in G$.

Integral (1) *is said to be uniformly convergent with respect to* $x \in G$ if it is convergent for all $x \in G$ and if, given any $\varepsilon > 0$, there is a sufficiently large number R_0 independent of x and such that for any R satisfying the inequality $R > R_0$ there holds the inequality

$$\left| \int_{D-\omega_R} f(x, y)\, dy \right| < \varepsilon$$

where ω_R is the ball of radius R with centre at the origin.

If a function $f(x, y)$ *is continuous on* $\bar{G} \times \bar{D}$ *and integral* (1) *is uniformly convergent with respect to* $x \in \bar{G}$ *then the function* $F(x)$ *is continuous on* $\bar{G}$. *If, in addition,* G *is a measurable set then there holds the equality*

$$\int_G F(x)\, dx = \int_G dx \int_D f(x,y)\, dy = \int_D dy \int_G f(x, y)\, dx \tag{2}$$

If x *is a scalar variable running through a closed interval* $[a, b]$ *and the function* $f(x,y)$ *is continuous together with its partial derivative* $\frac{\partial f}{\partial x}$ *on* $[a, b] \times \times \bar{D}$ *and if integral* (1) *is convergent while the integral*

$$F_1(x) = \int_D \frac{\partial}{\partial x} f(x, y)\, dy$$

is uniformly convergent with respect to $x \in [a, b]$ *then* $F'(x) = F_1(x)$, *that is*

$$\frac{\partial}{\partial x} \int_D f(x, y)\, dy = \int_D \frac{\partial}{\partial x} f(x, y)\, dy \tag{3}$$

Finally, *if for the function* $f(x, y)$ *in question there holds the inequality* $|f(x, y)| \leqslant \varphi(y)$ $(x \in G)$ *and the improper integral* $\int_D \varphi(y)\, dy$ *is convergent then integral* (1) *is convergent for any* $x \in G$; *moreover, by Weierstrass' test, it is uniformly convergent.*

The assertions stated here are analogous to the corresponding Theorems 1-4 in the foregoing section. Their proofs are also quite similar. It should also be noted that these theorems are analogous to the corresponding theorems on continuity, integrability and differentiability of uniformly convergent series of functions. It is possible to elaborate unified proofs of all these theorems by using the notion of a more general improper (Stieltjes) integral which, on the one hand, includes as special cases the integrals considered here and, on the other hand, also includes infinite series.

Example 1. *Gamma function.* The integral

$$\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}\,dx \tag{4}$$

is called the *gamma function* or *Euler's integral of the second kind.* It has the point $x = \infty$ as its singularity and for $0 < a < 1$ it has one more singularity at the point $x = 0$. Therefore in order to investigate this integral it is advisable to split it into two integrals:

$$\Gamma(a) = \int_0^1 x^{a-1}e^{-x}\,dx + \int_1^\infty x^{a-1}e^{-x}\,dx = \varphi_1(a)+\varphi_2(a)$$

The first integral is uniformly convergent for all $a \geqslant a_0 > 0$ where a_0 is an arbitrary fixed positive number. Indeed, we have

$$x^{a-1}e^{-x} \leqslant x^{a_0-1}\cdot 1 = x^{a_0-1} \qquad (0 \leqslant x \leqslant 1)$$

and, since

$$\int_0^1 x^{a_0-1}\,dx < \infty$$

our assertion follows from Weierstrass' test. However, it is nonuniformly convergent on the whole interval $0 < a < \infty$ because, for any $\delta < 1$, we have

$$\int_0^\delta x^{a-1}e^{-x}\,dx \geqslant m\int_0^\delta x^{a-1}\,dx = \frac{m\delta^a}{a} \to \infty \qquad (a \to 0),\ m = \min_{0\leqslant x\leqslant 1} e^{-x}$$

and hence for an arbitrary $\varepsilon > 0$ there is no $\delta > 0$ such that the integral $\int_0^\delta x^{a-1}e^{-x}\,dx$ becomes less than ε for all $a > 0$.

It is evident that the second integral is convergent for any real a. If a_0 is an arbitrary number then for $a \leqslant a_0$ we have

$$x^{a-1}e^{-x} \leqslant x^{a_0-1}e^{-x} \qquad (1 \leqslant x < \infty)$$

and, since

$$\int_{1}^{\infty} x^{a_0-1}e^{-x}\,dx < \infty$$

we see that, by Weierstrass' test, the second integral is uniformly convergent for all $a \leqslant a_0$. At the same time, it is not uniformly convergent on the whole real axis $-\infty < a < \infty$ because we have

$$\int_{N}^{\infty} x^{a-1}e^{-x}\,dx \geqslant N^{a-1}\int_{N}^{\infty} e^{-x}\,dx = N^{a-1}e^{-N} \to \infty \qquad (a \to \infty)$$

for $a > 1$ and any fixed $N > 1$.

We have shown that if $a_0 > 0$ then both integrals $\varphi_1(a)$ and $\varphi_2(a)$ converge uniformly for a ranging over a closed interval $[a_1, a_2]$ $(0 < a_1 < a_0 < a_2)$, whence, by the obvious continuity properties of the integrand, it follows that the gamma function is continuous at a_0 (for any $a_0 > 0$).

It can easily be shown that the integral

$$\int_{0}^{\infty} \frac{\partial}{\partial a} x^{a-1}e^{-x}\,dx = \int_{0}^{\infty} x^{a-1}\ln x e^{-x}\,dx \tag{5}$$

can be split into two integrals (taken from 0 to 1 and from 1 to ∞ respectively) which are uniformly convergent for a ranging over any interval (a_1, a_2) where $a_1 > 0$, whence, by the continuity of the integrand in (5) for $x > 0$, it follows that

$$\Gamma'(a) = \int_{0}^{\infty} x^{a-1}\ln x e^{-x}\,dx$$

In a similar way we can show that

$$\Gamma^{(k)}(a) = \int_{0}^{\infty} x^{a-1}(\ln x)^{k} e^{-x}\,dx$$

and thus the function $\Gamma(a)$ is infinitely differentiable for $a > 0$. In the theory of functions of a complex variable it is proved that $\Gamma(a)$ is in fact an analytic function.

Now we note that for $a > 1$ we have

$$\Gamma(a) = \int_{0}^{\infty} x^{a-1}e^{-x}\,dx = \lim_{N\to\infty}\left\{-x^{a-1}e^{-x}\Big|_{0}^{N} + (a-1)\int_{0}^{N} x^{a-2}e^{-x}\,dx\right\} =$$

$$= (a-1)\Gamma(a-1) \tag{6}$$

Therefore for natural values $a = n$ of the argument there hold the relations

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \ldots = n!\,\Gamma(1) = n!\int\limits_0^\infty e^{-x}\,dx = n! \tag{7}$$

whence it is seen that the gamma function is a natural generalization of the factorial.

Example 2. The integral

$$A(\lambda) = \int\limits_0^\infty \frac{\sin \lambda x}{x}\,dx \qquad (-\infty < \lambda < \infty) \tag{8}$$

has singularities at the points at infinity $x = \pm\infty$ and is convergent for any indicated value of λ. However, its convergence is uniform on the set $\lambda_0 \leqslant |\lambda| < \infty$ and nonuniform on the closed interval $[0, \lambda_0]$ for any positive λ_0. Indeed, for $\lambda > 0$ we have

$$R_N(\lambda) = \int\limits_N^\infty \frac{\sin \lambda x}{x}\,dx = \int\limits_N^\infty \frac{\sin \lambda x}{\lambda x}\,d(\lambda x) = \int\limits_{N\lambda}^\infty \frac{\sin z}{z}\,dz =$$

$$= \lim_{M\to\infty}\left\{-\frac{\cos z}{z}\Big|_{N\lambda}^{M} - \int\limits_{N\lambda}^{M} \frac{\cos z}{z^2}\,dz\right\} = \frac{\cos N\lambda}{N\lambda} - \int\limits_{N\lambda}^\infty \frac{\cos z\,dz}{z^2}.$$

whence

$$|R_N(\lambda)| \leqslant \frac{1}{N\lambda} + \int\limits_{N\lambda}^\infty \frac{dz}{z^2} = \frac{1}{N\lambda} + \frac{1}{N\lambda} = \frac{2}{N\lambda}$$

and hence, for $\lambda_0 \leqslant \lambda < \infty$, we obtain

$$|R_N(\lambda)| \leqslant \frac{2}{N\lambda_0} \to 0, \qquad \lambda_0 \to \infty$$

Thus, for any $\varepsilon > 0$ there is N_0 such that $|R_N(\lambda)| < \varepsilon$ for all $N > N_0$ and for any $\lambda \in [\lambda_0, \infty)$. On the other hand, the inequality

$$\left|\int\limits_{N\lambda}^\infty \frac{\sin z}{z}\,dz\right| < \varepsilon$$

cannot hold for a fixed N, however large, and for all $\lambda \in [0, \lambda_0]$ where $\varepsilon > 0$ is a given arbitrary number. For, when N is fixed and $\lambda \to 0$ the left-hand member of this inequality tends (see § 9.14 (8)) to the integral

$$A = \int\limits_0^\infty \frac{\sin z}{z}\,dz > 0$$

Integral (8) is uniformly convergent for $\lambda \in [N, N']$ where $0 < N < N'$; therefore, taking into account that the integrand is a continuous function of $(\lambda, x) \in [N, N'] \times [0, \infty)$, we obtain

$$\int_N^{N'} d\lambda \int_0^\infty \frac{\sin \lambda x}{x}\,dx = \int_0^\infty dx \int_N^{N'} \frac{\sin \lambda x}{x}\,d\lambda = \int_0^\infty \frac{\cos Nx - \cos N'x}{x^2}\,dx \tag{9}$$

The value of the function $\frac{\sin \lambda u}{u}$ at the point $u = 0$ is taken to be equal to λ here whence it becomes clear that the function $\frac{\sin \lambda x}{x}$ dependent on (λ, x) is continuous at any point $(\lambda, 0)$ $(\lambda \in [0, \infty))$.

Equality (9) continues to hold for $N = 0$:

$$\int_0^{N'} d\lambda \int_0^\infty \frac{\sin \lambda x}{x}\,dx = \int_0^\infty \frac{1-\cos N'x}{x^2}\,dx \tag{10}$$

However, the above considerations do not apply to this case because integral (8) is not uniformly convergent on $[0, N']$. But equality (10) can be derived directly from (9) by passing to the limit for $N \to 0$ because the difference between the right-hand members of (10) and (9) tends to zero as $N \to 0$:

$$\int_0^\infty \frac{1-\cos Nx}{x^2}\,dx = 2\int_0^\infty \frac{\sin^2 (N/2)x}{x^2}\,dx = N\int_0^\infty \frac{\sin^2 u}{u^2}\,du \to 0 \qquad (N \to 0) \tag{11}$$

The inner integral on the left-hand side of (10) is equal to $A(\lambda) = A$ for $\lambda > 0$, and therefore the left-hand member of (10) is equal to $N'A$ while, by virtue of (11), its right-hand member is equal to

$$N' \int_0^\infty \frac{\sin^2 u}{u^2}\,du$$

On cancelling by N' we obtain

$$A = \int_0^\infty \frac{\sin x}{x}\,dx = \int_0^\infty \frac{\sin^2 x}{x^2}\,dx \tag{12}$$

(as is shown in § 15.9 (9), $A = \pi/2$).

It should be noted that if $\lambda < 0$ then $A(\lambda) = -A$.

Exercises

1. Consider the integral

$$U(x, y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi(t) \frac{y\,dt}{(x-t)^2+y^2}$$

where $\varphi(t)$ is a bounded function integrable over every finite closed interval and the point (x, y) belongs to the upper half-plane (this is the *Poisson integral for the upper half-plane*). Show that this integral and all its partial derivatives are uniformly convergent on any bounded closed set of points (x, y) in the upper half-plane (i.e. for $y > 0$). Prove that $U(x, y)$ is a harmonic function in the upper half-plane. Hint: take into account that $\frac{y}{x^2+y^2}$ is a harmonic function for $x^2+y^2 > 0$.

2. Consider the integral

$$U(\varrho, \theta) = \frac{1}{\pi}\int_0^{2\pi} P_\varrho(t-\theta)\varphi(t)\,dt$$

(this is the *Poisson integral for the circle;* see Example 3 in § 11.8) where $\varphi(t)$ is a periodic function with period 2π integrable over the period. Prove that this integral and all its partial derivatives (with respect to ϱ and θ) are uniformly convergent on any circle $\varrho < \varrho_0$ $(\varrho_0 < 1)$.

§ 13.16. Uniformly Convergent Improper Integral with Variable Singularity

In §§ 13.14 and 13.15 we considered improper integrals of the form

$$F(x) = \int_D f(x, y)\,dy \qquad (x \in \Omega) \tag{1}$$

with a singularity at a fixed point y^0 independent of x. In the general case the location of the singularity may depend on x, that is $y^0 = \alpha(x)$. For instance, this is the case for the *volume potential*

$$F(x) = \int_D \frac{\mu(y)\,dy}{r^\lambda} \qquad (x \in R_n) \tag{2}$$

$$0 < \lambda < n, \quad r = |x-y| = \left(\sum_1^n (x_j-y_j)^2\right)^{1/2}$$

where the function μ is continuous on the closure $\bar{D}$ of the measurable domain D. Here we have $\alpha(x) = x$ for $x \in \bar{D}$; if $x \notin \bar{D}$ the integral has no singularity.

Another example is the *logarithmic potential* which is a line integral of the form

$$\Phi(x) = \int_C \lambda(\xi) \ln\frac{1}{r}\,ds, \qquad x = (x_1, x_2) \tag{3}$$

where C is a smooth contour in the $\xi_1\xi_2$-plane not intersecting itself, $\lambda(\xi)$ is a continuous function on C, s is the arc length reckoned along C and $r = \sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}$.

The so-called *single-layer potential*

$$\psi(x) = \iint\limits_{S} \frac{\lambda}{r}\, ds \tag{4}$$

can also serve as an example of this kind.

Integral (4) is taken over a smooth surface S lying in the three-dimensional space of the points $x = (x_1, x_2, x_3)$. Usually S is a boundary of a bounded domain and λ is a continuous function defined on S; r is the distance from the point x to the point $u \in S$ with respect to which the integration is carried out.

We shall investigate improper integrals with variable singularities of the form

$$F(x) = \int\limits_{D} K(x-y(u))\alpha(x, u)\, du, \qquad x = (x_1, \ldots, x_n) \in \Omega \tag{5}$$

where Ω is a domain in the space R_n, D is a measurable domain belonging to another space R_m whose dimension m, in the general case, can be equal to or less than that of R_n $(1 \leq m \leq n,\ D \subset R_m)$, $\alpha(x, u)$ is a continuous function of $x \in \Omega$ and $u \in D$ and the function $y = y(u)$ specifies a continuously differentiable mapping $D \ni u \to y \in R_n$ of D into R_n and describes part S of an m-dimensional differentiable manifold in R_n (see § 17.1). Finally, $K(z) = K(z_1, \ldots, z_n)$ is a function of $z \in R_n$ continuous everywhere except at the point $z = 0$ in whose neighbourhood K is unbounded.

Integrals (2), (3) and (4) are special cases of integral (5).

We shall say that integral (5) is *uniformly convergent on* Ω if, given any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\int\limits_{|x-y(u)|<\delta} |K(x-y(u))\alpha(x, u)|\, du < \varepsilon \qquad \text{for all} \quad x \in \Omega \tag{6}$$

Here the integration extends over all $u \in \bar{D}$ for which the inequality written under the integral sign is fulfilled. Of course it may happen that the distance from the point x to the manifold S is so large that the set of the indicated points u turns out to be empty; in such a case integral (6) is equal to zero. (Integral (6) is taken over an open set and therefore it exists at least in the sense of the Lebesgue integral; see Chapter 19.)

For example, integral (2) is uniformly convergent because

$$\int\limits_{|x-y|<\delta} \frac{|\mu(y)|}{|x-y|^{\lambda}}\, dy \leq M \int\limits_{|x-y|<\delta} \frac{dy}{|x-y|^{\lambda}} = M \int\limits_{|y|<\delta} \frac{dy}{|y|^{\lambda}} < \varepsilon \tag{7}$$

provided that, for the given $\varepsilon > 0$, the number $\delta > 0$ is sufficiently small. Here we have $|\mu(y)| \leq M$, $y \in D$, $y = u$, $\mu(y) = \alpha(x, y)$, $K(z) = |z|^{-\lambda}$. The uniform convergence of integrals (3) and (4) will be discussed later.

Theorem 1. *If integral* (5) *satisfying the conditions indicated above is uniformly convergent on* Ω *then it is also absolutely convergent and* F *is a continuous function of* x *on* Ω.

Proof. Let us fix an arbitrary $\varepsilon > 0$ and then choose $\delta > 0$ so that the inequality

$$\int\limits_{|x-y(u)|<3\delta} |K(x-y(u))\,\alpha|\,du < \varepsilon \qquad (x \in \Omega)$$

is fulfilled. Further, to prove the continuity of F at x^0 let us put $F(x) = F_1(x)+F_2(x)$ where

$$\left.\begin{aligned} F_1(x) &= \int\limits_{|x^0-y(u)|<2\delta} K(x-y(u))\,\alpha\,du \\ F_2(x) &= \int\limits_{|x^0-y(u)|\geqslant 2\delta} K(x-y(u))\,\alpha\,du \end{aligned}\right\} \tag{9}$$

(these integrals extend over the values $u \in \bar{D}$ over which the inequalities under the integral signs hold).

Taking the ball V determined by the inequality $|x-x^0| \leqslant \delta$ we see that the function F_2 is continuous on it since the integral representing F_2 is taken over the measurable closed set ε of points u satisfying the inequality $|x^0-y(u)| \geqslant 2$ while the integrand is a function of $(x, u) \in V\times\varepsilon$ which is continuous since the function $K(x-y(u))$ is continuous because

$$|x-y(u)| \geqslant |x^0-y(u)|-|x-x^0| \geqslant 2\delta-\delta = \delta > 0$$

Further, for sufficiently small δ, the function F_1 satisfies the inequality $|F_1(x)| < \varepsilon$ because if $|x-x^0| < \delta$ and $|x^0-y(u)| < 2\delta$ then $|x^0-y(u)| < 3\delta$. Consequently the function F is continuous at x^0. The absolute convergence of integral (5) follows from that of integral (6).

Theorem 1 and what was said about the volume potential imply that integral (2) is a continuous function of $x \in R_n$.

Theorem 2. *There holds the equality*

$$\frac{\partial}{\partial x_j}\int\limits_D K(x-y(u))\,\mu(u)\,du = \int\limits_D \frac{\partial}{\partial x_j} K(x-y(u))\,\mu(u)\,du \tag{10}$$

$$(j = 1, \ldots, n)$$

where the function $y(u)$, *the kernel* $K(x)$ *and the differentiated kernel* $K_1(x) = \frac{\partial}{\partial x_j} K(x)$ *satisfy the conditions indicated in connection with* (5) *and* $\mu(u)$ *is a continuous function on* $\bar{D}$.

The proof of this theorem is based on the following lemma.

Lemma 1. *Let* $\psi(x)$, $F(x)$ *and* $F_\delta(x)$ $(0 < \delta < \delta_0)$ *be functions defined on an interval* (a, b) *and let* $x_0 \in (a, b)$. *Also let the function* $F_\delta(x)$ *be continuously differentiable and the function* $\psi(x)$ *be continuous. Furthermore, let the condition*

$$F_\delta(x) \to F(x), \quad \delta \to 0, \qquad x \in (a, b) \tag{11}$$

be fulfilled and let for any $\varepsilon > 0$ *there exist* $\delta_0 > 0$ *such that*

$$|F'_\delta(x)-\psi(x)| < \varepsilon \tag{12}$$

for all x *and* δ *satisfying the inequalities*

$$|x-x_0| < \delta < \delta_0 \tag{13}$$

Then

$$F'(x_0) = \psi(x_0) \tag{14}$$

In other words,

$$\left(\lim_{\delta\to 0} F_\delta(x)\right)'_{x=x_0} = \lim_{\delta\to 0} F'_\delta(x_0) \tag{14'}$$

Proof of Lemma 1. Since $\psi(x)$ is continuous it follows from (12) that for any $\varepsilon > 0$ there is δ_0 such that

$$|F'_\delta(x)-\psi(x_0)| \leqslant |F'_\delta(x)-\psi(x)|+|\psi(x)-\psi(x_0)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

for δ and x satisfying (13).

This shows that

$$\lim_{x\to x_0,\ \delta\to 0} F'_\delta(x) = \psi(x_0) \tag{15}$$

Further, by the continuous differentiability of $F_\delta(x)$, for the indicated values of δ we have

$$F_\delta(x) = F_\delta(x_0)+\int\limits_{x_0}^{x} F'_\delta(u)\,du = F_\delta(x_0)+(x-x_0)F'_\delta(x_0+\theta(x-x_0)) \quad (0<\theta<1)$$

and, by virtue of (15),

$$\frac{F_\delta(x)-F_\delta(x_0)}{x-x_0} = F'_\delta(x_0+\theta(x-x_0)) \xrightarrow[x\to x_0,\ \delta\to 0]{} \psi(x_0)$$

Thus, given any $\varepsilon > 0$, there is $\delta_0 > 0$ such that

$$\left|\frac{F_\delta(x)-F_\delta(x_0)}{x-x_0}-\psi(x_0)\right| < \varepsilon$$

for x and δ satisfying (13).

Now let us pass to the limit as $\delta \to 0$ for the indicated values of x; this results in

$$\left|\frac{F(x)-F(x_0)}{x-x_0}-\psi(x_0)\right| \leqslant \varepsilon, \qquad |x-x_0| < \delta$$

which proves (14).

Note. It is natural to say that (12) expresses the property of *local uniform convergence* of $F'_\delta(x)$ to $\psi(x)$ ($\delta \to 0$) at the point x_0. If we replace this property, entering into Lemma 1, by the stronger requirement that the convergence of $F'_\delta(x)$ to $\psi(x)$ on Ω should be uniform in the ordinary sense we arrive at Theorem 3′ in § 11.8 which was already proved.

Proof of Theorem 2. Let us prove equality (10) for a point $x = x^0 \in D$. To this end we write

$$F_\delta(x) = \int\limits_{|x^0-y(u)| \geq 2\delta} K(x-y(u))\,\mu(u)\,du \tag{16}$$

and

$$\frac{\partial}{\partial x_j} F_\delta(x) = \int\limits_{|x^0-y(u)| \geq 2\delta} \frac{\partial}{\partial x_j} K(x-y(u))\,\mu(u)\,du \tag{17}$$

As is already known, for $|x-x^0| < \delta$ and $|x^0-y(u)| \geq 2\delta$ we have the inequality $|x^0-y(u)| \geq \delta$ showing that the integrand functions in (16) and (17) are continuous in x and u. Indeed, the kernels K and $\frac{\partial}{\partial x_j} K = K_1$ have singularities only at the origin in the space R_n. Thus, the functions $F_\delta(x)$ and $\frac{\partial}{\partial x_j} F_\delta(x)$ are continuous for $|x-x^0| < \delta$, and the differentiation under the integral sign in (17) is legitimate.

For any $x \in \Omega$ there also exist the limit functions

$$\lim_{\delta\to 0} F_\delta(x) = \int\limits_D K(x-y(u))\,\mu(u)\,du = F(x)$$

and

$$\lim_{\delta\to 0} \frac{\partial}{\partial x_j} F_\delta(x) = \int\limits_D \frac{\partial}{\partial x_j} K(x-y(u))\,\mu(u)\,du = \psi(x)$$

By Theorem 1, they are continuous. The conditions imposed on K_1 guarantee that for any $\varepsilon > 0$ there is $\delta_0 > 0$ such that

$$\left|\psi(x) - \frac{\partial}{\partial x_j} F_\delta(x)\right| = \left|\int\limits_{|x^0-y(u)| \leq 2\delta} \frac{\partial}{\partial x_j} K(x-y(u))\,\mu(u)\,du\right| < \varepsilon$$

for x and δ satisfying (13) (see what was said after formula (9)).

We see that $F_\delta(x)$ regarded as a function of x_j for fixed $x_k = x_k^0$; $k = 1, \ldots, j-1, j+1, \ldots, n$, satisfies the conditions of Lemma 1 and therefore

$$\frac{\partial}{\partial x_j} F(x^0) = \psi(x^0)$$

We have thus proved (10) for any $x^0 \in D$.

The integral on the right-hand side of (10) is a continuous function of x on $\bar{\Omega}$.

Example 1. For $\lambda < n-1$ it is legitimate to differentiate the volume potential (2) under the integral sign:

$$F'_{x_j}(x) = \int\limits_D \mu(y) \frac{\partial}{\partial x_j} \frac{1}{r^\lambda}\,dy = -\lambda \int\limits_D \mu(y) \frac{x_j-y_j}{r^{\lambda+2}}\,dy \tag{18}$$

What was established above implies that $F(x)$ is a continuous function, and we also take into account here that the integrand in (18) is of form (5) ($K(x-y) = |x-y|^{-\lambda-2}$, $m = n$, $y(u) = y$) and possesses all the properties established for (5) and that integral (18) is uniformly convergent because

$$\int\limits_{|x-y|<\delta} \left|\mu \frac{\partial}{\partial x_j} r^{-\lambda}\right| dy \leqslant M \int\limits_{|x-y|<\delta} \frac{dy}{r^{\lambda+1}} < \varepsilon$$

for sufficiently small δ.

Example 2. Logarithmic potential (3) is a continuous function of $x = (x_1, x_2)$. For $x \notin C$ this follows from the well-known theorem on the continuity of an integral with respect to parameter. Therefore the most interesting part of the proof concerns the continuity of $\Phi(x)$ at the points $x \in C$. Let $x^0 \in C$; without loss of generality we can assume that x^0 lies at the origin ($x^0 = (0, 0)$) and that the tangent to C at the origin coincides with the x_1-axis. Let us represent integral (3) as the sum

$$F(x) = \int\limits_{C_1} \mu(x) \ln \frac{1}{r} ds + \int\limits_{C_2} \mu(x) \ln \frac{1}{r} ds$$

where C_1 is a small arc of C containing the origin inside it and $C_2 = C - C_1$. It is clear that the integral along C_2 is continuous at the origin because the integrand has no singularity in its vicinity. We shall assume that the arc C_1 is so small that its equation can be written in an explicit form $x_2 = f(x_1)$ ($|x| \leqslant a$). The integral

$$F_1(x) = \int\limits_{C_1} \mu(x) \ln \frac{1}{r} ds = \int\limits_{-a}^{a} \ln \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-f(\xi))^2}} \lambda(\xi_1)\, d\xi_1 \qquad (19)$$

$$\lambda(\xi_1) = \mu(\xi_1, f(\xi_1))\sqrt{1+f'^2(\xi_1)}$$

apparently belongs to type (5) ($u = x_1$, $m = 1$ and $n = 2$). Its uniform convergence with respect to the points (x_1, x_2) belonging to a ball V with centre at the origin follows from the inequalities below in which $M \geqslant |\lambda(\xi_1)|$ and $|\xi_1| < a$:

$$\left| \int\limits_{\sqrt{(x_1-\xi_1)^2+(x_2-f(\xi_1))^2} < \delta} \lambda \ln \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-f(\xi_1))^2}} d\xi_1 \right| \leqslant$$

$$\leqslant M \left| \int\limits_{|x_1-\xi_1|<\delta} \ln \frac{1}{|x_1-\xi_1|} d\xi_1 \right| = M \left| \int\limits_{|\xi_1|<\delta} \ln \frac{1}{|\xi_1|} d\xi_1 \right| < \varepsilon$$

where δ is sufficiently small.

Example 3. The continuity of single-layer potential (4) at an arbitrary point $x^0 \in S$ can be established as follows. We can assume, without losing generality, that x^0 lies at the origin, i.e. $x^0 = (0, 0, 0)$, and that the plane $x_3 = 0$ is tangent to S at x^0. By analogy with the above, we consider the integral

$$\psi(x) = \iint\limits_{C_1} \frac{\lambda}{r}\, ds = \iint\limits_{\sigma} \frac{\mu(\xi_1, \xi_2)\, d\xi_1\, d\xi_2}{\sqrt{(x-\xi_1)^2+(x_2-\xi_2)^2+(x_3-f(\xi_1, \xi_2))^2}}$$

$$\mu = \lambda\sqrt{1+\left(\frac{\partial f}{\partial \xi_1}\right)^2+\left(\frac{\partial f}{\partial \xi_2}\right)^2}$$

where $\xi_3 = f(\xi_1, \xi_2)$ is a function describing C_1 which is continuously differentiable in a domain σ of the $\xi_1\xi_2$-plane. This is an integral of type (5). Its uniform convergence follows from the fact that the absolute value of the part of this integral corresponding to the set of points (ξ_1, ξ_2) satisfying the inequality

$$\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-f)^2} < \delta$$

does not exceed

$$M \iint\limits_{\sqrt{(x-\xi_1)^2+(x_2-\xi_2)^2} < \delta} \frac{d\xi_1\, d\xi_2}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2}} = M \iint\limits_{\sqrt{\xi_1^2+\xi_2^2} < \delta} \frac{d\xi_1\, d\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} < \varepsilon$$

CHAPTER 14

Normed Linear Spaces. Orthogonal Systems

§ 14.1. Space C of Continuous Functions

The reader is advised to recall the material of §§ 6.1, 6.2 and 6.3 before studying the present section.

We begin this section with adding to the material of §§ 6.1 and 6.3 the definition of the *completeness* of a space.

Let E be a normed linear space and let a sequence of elements $x_n \in E$ converge to an element $x \in E$. Then for any $\varepsilon > 0$ there is $N > 0$ such that the inequality

$$\|x_n - x\| < \frac{\varepsilon}{2} \qquad (n > N)$$

holds.

Consequently, for n, $m > N$ we can write

$$\|x_n - x_m\| = \|x_n - x + x - x_m\| \leqslant \|x_n - x\| + \|x - x_m\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We have proved that if a sequence of elements $x_n \in E$ is convergent to an element $x \in E$, then the condition of Cauchy's criterion is satisfied: given any $\varepsilon > 0$, there is $N > 0$ such that the inequalities n, $m > N$ imply

$$\|x_n - x_m\| < \varepsilon$$

Generally speaking, the converse of this assertion does not hold: there are normed linear spaces E containing sequences of elements which satisfy Cauchy's condition but are not convergent to any element of E. Later on (§ 19.7) we shall deal with some examples of such spaces. Now we proceed to the definition of the completeness.

A normed linear space E is said to be *complete* if any sequence of elements $x_n \in E$ ($n = 1, 2, \ldots$) satisfying Cauchy's condition converges to an element $x \in E$.

A complete linear space is also called a *Banach space*.*

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* S. Banach (1892-1945), a distinguished Polish mathematician who contributed many outstanding results to functional analysis and, particularly, to the theory of normed linear spaces.

We have already dealt with a special case of a Banach space, namely with the space R_1 of real numbers.

The Space C Let Ω be a bounded closed set in the space R_n. The collection of all real (or complex*) functions $f = f(x)$ $(x \in \Omega)$ continuous on Ω is denoted as $C = C(\Omega)$, and with each function $f \in C$ is associated the number

$$||f||_C = \max_{x \in \Omega} |f(x)| \tag{1}$$

which is the *norm of f (corresponding to the metric of the space C).*

The space C of continuous functions (on Ω) is a real (or complex) normed linear space with zero element $\theta = \theta(x) \equiv 0$.

Indeed, it is evident that C is a real (or complex) linear space in the sense of the definition stated in § 6.1. Furthermore (see § 6.3), we have

(i) $||f||_C \geqslant 0$, and the equality $||f||_C = 0$ implies $f = \theta$;

(ii) $||\alpha f||_C = \max\limits_{\Omega} |\alpha f(x)| = |\alpha| \max\limits_{\Omega} |f(x)| = |\alpha|\, ||f||_C$;

(iii) $||f+\varphi||_C = \max\limits_{\Omega} |f(x)+\varphi(x)| \leqslant \max\limits_{\Omega} |f(x)| + \max\limits_{\Omega} |\varphi(x)| =$

$= ||f||_C + ||\varphi||_C$.

According to definition (1), for any functions $f, f_1, f_2, \ldots$ belonging to C we have the equalities

$$||f-f_k||_C = \max_{x \in \Omega} |f(x)-f_k(x)| \qquad (k = 1, 2, \ldots) \tag{2}$$

If the right-hand side of (2) tends to zero as $k \to \infty$ then (see § 11.7) the sequence of functions $\{f_k\}$ is uniformly convergent to f on Ω. Thus, *the convergence of a sequence of functions in the space C (with respect to the metric of C) is equivalent to its uniform convergence on Ω.*

Now let us suppose that a sequence of functions $f_k \in C$ satisfies Cauchy's condition (with respect to the metric of C): given any $\varepsilon > 0$, there is $N > 0$ such that

$$\varepsilon > ||f_k - f_l||_C = \max_{\Omega} |f_k(x) - f_l(x)|$$

for all $k, l > N$. Then, as was shown in § 11.7, the sequence $\{f_k\}$ is uniformly convergent to a function $f = f(x) \in C$ on Ω and, consequently, it is convergent in the norm of the space C to the element $f \in C$:

$$||f_k - f||_C = \max_{\Omega} |f_k(x) - f(x)| \to 0, \qquad k \to \infty$$

---

* A continuous complex function is defined by the equality $f(x) = \varphi(x) + i\psi(x)$ where φ and ψ are continuous real functions. Thus, in this case $\max\limits_{x \in \Omega} |f(x)| = \max\limits_{x \in \Omega} (\varphi^2(x) + \psi^2(x))^{1/2}$.

Thus, *if a sequence of functions $f_k \in C$ satisfies the condition of Cauchy's criterion, there is a function $f \in C$ to which the given sequence converges with respect to the metric of C, that is*

$$\lim_{k\to\infty} f_k = f \qquad \text{(in the metric of } C\text{)}$$

We have proved that *C is a complete normed linear space, that is a Banach space.*

§ 14.2. Spaces L' L'_p and l_p

Let Ω be an open set in R_n. We shall denote by $L' = L'(\Omega)$ the collection (*space*) of the (real or complex) functions f absolutely integrable on Ω (generally speaking, in the sense of the improper Riemann integral*). The norm $||f||_L$ of an element $f \in L'$ is defined as the integral $\int_\Omega |f(x)|dx$:

$$||f||_L = \int_\Omega |f(x)|\, dx \tag{1}$$

If $f = \varphi + i\psi$ is a complex function then

$$\int_\Omega |f|\, dx = \int_\Omega \sqrt{\varphi^2+\psi^2}\, dx$$

We thus automatically assume that if the set Ω is bounded it is (Jordan) measurable and if it is unbounded then it is *locally measurable* in the sense that every set $\omega\Omega$ (the intersection of ω and Ω) is measurable where $\omega \subset R_n$ is an arbitrary ball.

Functional analysis also deals with *the space $L = L(\Omega)$ of functions f which are Lebesgue measurable on Ω and have finite norm* (1) *where the integral is understood in Lebesgue's sense* (see Chapter 19**).

Most of the facts that will be established for the functions $f \in L'$ can be extended (sometimes with slight modifications) to the functions $f \in L$. In this connection, in some cases we shall make brief remarks with references to Chapter 19.

Since we have called integral (1) the norm of L', by the zero element of L' (or of L) should be meant any function $\theta = \theta(x)$ satisfying the condition

$$\int_\Omega |\theta(x)|\, dx = 0 \tag{2}$$

---

[*] If $f \in L'$ then the integral $\int f\,dx$ has a finite number of singularities (see § 13.13) and $\int_\Omega |f|\, dx < \infty$.

[**] In this case Ω is supposed to be measurable or locally measurable in Lebesgue's sense.

The function $\theta(x) \equiv 0$ is an example of such a function but it is not a unique function of the kind (see Theorem 1 below). However, two functions $f_1(x)$ and $f_2(x)$ differing by $\theta(x)$ are not considered as different elements of $L'(\Omega)$ (or of $L(\Omega)$); they are identified in the sense that they represent one and the same element ($f_1 = f_2$) of the space $L'(\Omega)$ ($L(\Omega)$).

Let us show that $L'(\Omega)$ ($L(\Omega)$) is a linear space and that integral (1) possesses all the properties of a norm. Indeed,

(i) $\|f\|_L \geqslant 0$, and the equality $\|f\|_L = 0$ implies $f = \theta$;

(ii) if $f \in L'$ (L) and α is a number then αf belongs to L' (L) and the equality

$$\|\alpha f\|_L = \int_\Omega |\alpha f(x)|\, dx = |\alpha| \int_\Omega |f(x)|\, dx = |\alpha|\, \|f\|_L$$

holds;

(iii) if $f, \varphi \in L'$ (L) then also $f+\varphi \in L'$ (L) and

$$\|f+\varphi\|_L = \int_\Omega |f(x)+\varphi(x)|\, dx \leqslant \int_\Omega |f(x)|\, dx + \int_\Omega |\varphi(x)|\, dx = \|f\|_L + \|\varphi\|_L$$

If $f, f_1, f_2, \ldots \in L'$ (L) then

$$\|f-f_k\|_L = \int_\Omega |f(x)-f_k(x)|\, dx \qquad (k = 1, 2, \ldots) \tag{3}$$

Hence, *the convergence $f_k \to f$ with respect to the metric of L is equivalent to the fact that the integral on the right-hand side of* (3) *tends to zero.* In such a case we also say that *f_k tends to f in the mean* (*in the mean of order one*; see § 6.3).

However, *the space L′ is not complete while the space L = L(Ω) of the functions Lebesgue integrable (summable) on Ω* (see § 19.3, Property 20) *is complete.*

Theorem 1. *Let $\theta = \theta(x) \in L'(\Omega)$. For the equality*

$$\int_\Omega |\theta(x)|\, dx = 0 \tag{4}$$

to hold it is necessary and sufficient that $\theta(x) = 0$ on the set $\Omega' \subset \Omega$ of points at which $\theta(x)$ is continuous.

Proof. Let us suppose that equality (4) holds and that there is a point $x^0 \in \Omega'$ at which θ is continuous such that $|\theta(x^0)| > 0$. Then there exists a ball $\omega \subset \Omega$ with centre at x^0 on which $|\theta(x)| > \eta > 0$. This conclusion contradicts (4):

$$\int_\Omega |\theta(x)|\, dx \geqslant \int_\omega |\theta(x)|\, dx \geqslant \eta\, |\omega| > 0$$

Now let $\theta \in \mathring{L}'(\Omega)$ and $\theta(x) = 0$ for all $x \in \Omega'$.

Let us denote by Ω_ε the set obtained from Ω by deleting the balls of radii $\varepsilon > 0$ with centres at the singular points of the integral $\int\limits_{\Omega} \theta(x)\,dx$ (there is a finite number of these balls) and by deleting the exterior of the ball of radius $1/\varepsilon$ with centre at the origin.

By Lebesgue's theorem, $\Omega - \Omega'$ is of Lebesgue measure zero, and therefore the set Ω' is dense in Ω and the lower integral of $|\theta(x)|$ over Ω_ε is equal to zero; consequently the integral of $|\theta(x)|$ on Ω_ε is itself equal to zero, and therefore

$$\int\limits_{\Omega} |\theta(x)|\,dx = \lim_{\varepsilon \to 0} \int\limits_{\Omega_\varepsilon} |\theta(x)|\,dx = 0$$

In the case of the Lebesgue integral we can state the following proposition which is an analogue of Theorem 1 (see § 19.3, properties 1 and 16).

Let $\theta = \theta(x) \in L(\Omega)$. *Then for equality* (4) *(with the integral understood in Lebesgue's sense) to hold it is necessary and sufficient that the function* $\theta(x)$ *be equal to zero almost everywhere on* Ω.

The Spaces l_p *and* L_p. Let p be a number satisfying the inequalities $1 \leqslant p < < \infty$. A sequence $a = \{a_k\}$ of (real or complex) numbers a_k is said to belong to the space l_p if the norm $\|a\|_{l_p}$ defined as the number $\left(\sum\limits_1^\infty |a_k|^p\right)^{1/p}$ is finite:

$$\|a\|_{l_p} = \left(\sum_1^\infty |a_k|^p\right)^{1/p} < \infty \tag{5}$$

A (real or complex) function $f(x)$ defined in a domain $\Omega \subset R_n$ is said to belong to the space $L_p'(\Omega)$ if the integral of f over Ω has a finite number of singularities and if the norm $\|f\|_{L_p}(\Omega)$ defined as the number $\left(\int\limits_{\Omega} |f(x)|^p\,dx\right)^{1/p}$ is finite:

$$\|f\|_{L_p(\Omega)} = \left(\int\limits_{\Omega} |f(x)|^p\,dx\right)^{1/p} < \infty \tag{6}$$

(as is seen, $L_1'(\Omega) = L'(\Omega)$). Functional analysis also deals with the space $L_p(\Omega)$ of the Lebesgue measurable functions defined on Ω with finite Lebesgue integrals (6) (see Chapter 19).

Let $\varphi(u)$ and $\psi(v)$ $(u, v \geqslant 0)$ be two strictly increasing and mutually inverse continuous functions assuming zero values at the points $u = 0$ and $v = 0$ respectively.

On plotting the graphs of these functions in the uv-plane we readily see that there holds the inequality

$$ab \leqslant \Phi(a) + \Psi(b) \qquad (a, b \geqslant 0)$$

where

$$\Phi(x) = \int_0^x \varphi(t)\,dt \quad \text{and} \quad \Psi(x) = \int_0^x \psi(t)\,dt$$

and that the equality is satisfied here if and only if $\varphi(a) = b$.

For the case when $\varphi(u) = u^{\alpha}$ and $\psi(v) = v^{1/\alpha}$ $\left(\alpha > 0,\ 1+\alpha = p \text{ and } \frac{1}{p}+\frac{1}{q} = 1\right)$ we obtain

$$ab \leqslant \frac{a^p}{p}+\frac{b^q}{q} \qquad (a, b \geqslant 0)$$

whence it follows that if

$$\boldsymbol{a} = \{a_k\} \in l_p \quad \text{and} \quad \boldsymbol{b} = \{b_k\} \in l_q$$

then

$$\sum_1^{\infty} |a_k b_k| \leqslant \frac{1}{p}\sum_1^{\infty} |a_k|^p+\frac{1}{q}\sum_1^{\infty} |b_k|^q \tag{7}$$

Similarly, if $f(x) \in L_p'(\Omega)$ $(L_p(\Omega))$ and $\varphi(x) \in L_q'(\Omega)$ $(L_q(\Omega))$ then

$$\int_{\Omega} |f(x)\,\varphi(x)|\,dx \leqslant \frac{1}{p}\int_{\Omega} |f(x)|^p\,dx+\frac{1}{q}\int_{\Omega} |\varphi(x)|^q\,dx \qquad [(8)$$

On applying (7) to the sequences

$$\frac{\boldsymbol{a}}{\|\boldsymbol{a}\|_{l_p}} = \left\{\frac{a_k}{\|\boldsymbol{a}\|_{l_p}}\right\} \quad \text{and} \quad \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|_{l_q}} = \left\{\frac{b_k}{\|\boldsymbol{b}\|_{l_q}}\right\}$$

(where $\|\boldsymbol{a}_{l_p}\|$, $\|\boldsymbol{b}\|_{l_q} > 0$) we obtain

$$\frac{1}{\|\boldsymbol{a}\|_{l_p}\|\boldsymbol{b}\|_{l_p}}\sum_1^{\infty} |a_k b_k| \leqslant \frac{1}{p}+\frac{1}{q} = 1$$

whence

$$\sum_1^{\infty} |a_k b_k| \leqslant \left(\sum_1^{\infty} |a_k|^p\right)^{1/p}\left(\sum_1^{\infty} |b_k|^q\right)^{1/q} \tag{9}$$

It is also obvious that (9) continues to hold when one or both factors on the right-hand side are equal to zero.

Similarly, the application of (8) to the functions

$$\frac{f(x)}{\|f\|_{L_p(\Omega)}} \quad \text{and} \quad \frac{\varphi(x)}{\|\varphi\|_{L_q(\Omega)}}$$

where $\|f\|_{L_p(\Omega)}$, $\|\varphi\|_{L_q(\Omega)} > 0$ leads to the inequality

$$\frac{1}{\|f\|_{L_q(\Omega)}\|\varphi\|_{L_p(\Omega)}}\int_{\Omega} |f\varphi|\,dx \leqslant 1$$

whence

$$\int_{\Omega} |f\varphi|\, dx \leqslant \left(\int_{\Omega} |f|^p\, dx\right)^{1/p} \left(\int_{\Omega} |\varphi|^q\, dx\right)^{1/q} \tag{10}$$

The relation obtained continues to hold when one or both factors on the right-hand side are equal to zero.

Inequalities (9) and (10) are referred to as *Hölder's* inequalities*. In the case $p = 2$ the first of them is called *Cauchy's inequality* and the second *Bunyakovsky's** inequality* (see § 6.2 (8) and (10)).

If $a, b \in l_p$ then

$$\sum_1^{\infty} |a_k+b_k|^p \leqslant \sum_1^{\infty} |a_k+b_k|^{p-1} |a_k| + \sum_1^{\infty} |a_k+b_k|^{p-1} |b_k| \leqslant$$
$$\leqslant \left(\sum_1^{\infty} |a_k+b_k|^p\right)^{(p-1)/p} \left(\sum_1^{\infty} |a_k|^p\right)^{1/p} + \left(\sum_1^{\infty} |a_k+b_k|^p\right)^{(p-1)/p} \left(\sum_1^{\infty} |b_k|^p\right)^{1/p}$$

where we have applied Hölder's inequality to each of the summands in the middle member. It follows that

$$\left(\sum_1^{\infty} |a_k+b_k|^p\right)^{1/p} \leqslant \left(\sum_1^{\infty} |a_k|^p\right)^{1/p} + \left(\sum_1^{\infty} |b_k|^p\right)^{1/p} \tag{11}$$

Similarly if $f, \varphi \in L_p'$ then

$$\int_{\Omega} |f+\varphi|^p\, dx \leqslant \int_{\Omega} |f+\varphi|^{p-1} |f|\, dx + \int |f+\varphi|^{p-1} |\varphi|\, dx \leqslant$$
$$\leqslant \left(\int_{\Omega} |f+\varphi|^p\, dx\right)^{(p-1)/p} \left[\left(\int_{\Omega} |f|^p\, dx\right)^{1/p} + \left(\int_{\Omega} |\varphi|^p\, dx\right)^{1/p}\right]$$

whence

$$\left(\int_{\Omega} |f+\varphi|^p\, dx\right)^{1/p} \leqslant \left(\int_{\Omega} |f|^p\, dx\right)^{1/p} + \left(\int_{\Omega} |\varphi|^p\, dx\right)^{1/p} \tag{12}$$

Inequalities (11) and (12) are known as *Minkowski's*** inequalities*.

The meaning of relations (7)-(12) is that if their right-hand members are finite then so are the left-hand ones. They imply that l_p and L_p' (L_p) *are (real or complex) normed linear spaces*.

Expressions (5) and (6) are (Banach) norms (see § 6.3) because, together with relations (11) and (12), the properties

$$\|\lambda a\|_{l_p} = |\lambda|\, \|a\|_{l_p} \quad \text{and} \quad \|\lambda f\|_{L_p} = |\lambda|\, \|f\|_{L_p}$$

hold for them where λ is an arbitrary number, and equalities $\|a\|_{l_p} = \|f\|_{L_p} = 0$ imply that $a = 0$ (i.e. $a_k = 0;\ k = 1, 2, \ldots$) and that $f(x) = 0$

---

* O. Hölder (1859-1937), a German mathematician.

** Academician V. Ya. Bunyakovsky (1804-1889), a Russian mathematician.

*** H. Minkowski (1864-1909), a German mathematician and physicist.

at the points of continuity (while in Lebesgue's theory the latter property holds almost everywhere).

Thus, l_p and L'_p (L_p) are normed spaces. The zero element of l_p is the sequence whose all members are equal to zero ($a_k = 0$; $k = 1, 2, \ldots$) and the zero element of L'_p is represented by any function $\theta(x) \in L'_p$ which is equal to zero at its points of continuity while the zero element of L_p is a function equal to zero almost everywhere on Ω in the sense of Lebesgue's measure.

It should be noted that *if* $f \in L'_p(\Omega)$ ($L_p(\Omega)$) *where* Ω *is a bounded (measurable) set then* $f \in L'(\Omega)$ ($L(\Omega)$) and there holds (see (10) for $\varphi \equiv 1$) the inequality

$$\int_\Omega |f(x)|\,dx = \int_\Omega |f(x)\cdot 1|\,dx \leqslant |\Omega|^{1/q}\left(\int_\Omega |f(x)|^p\,dx\right)^{1/p} \tag{13}$$

$$(1 < p < \infty,\ 1/p + 1/q = 1)$$

In the case $p = 2$ the spaces l_p and $L'_p(L_p)$ possess some special properties; for instance it is possible to define scalar products of their elements. In this connection they will be studied in more detail in § 14.3 and also in § 14.6 (Example 1) where, in particular, it will be proved that l_2 is a complete space. It can be shown quite similarly that for any p the space l_p is also complete.

§ 14.3. Spaces L'_2 and L_2

Let G be a (bounded) measurable set in R_n and let $L_{2,r}(G)$ be the collection of all the possible (complex or real) Riemann integrable functions on G. It is obvious that $L_{2,r}(G)$ is a (complex or real) linear space. With any two functions φ and ψ belonging to $L_{2,r}(G)$ we can associate the number

$$(\varphi, \psi) = (\varphi, \psi)_G = \int_G \varphi(x)\overline{\psi}(x)\,dx \tag{1}$$

where the integral is taken in the ordinary (proper) Riemann sense. Expression (1) possesses the three properties (see § 6.2) of a scalar product. Indeed, for $\varphi, \psi, \chi \in L_{2,r}(G)$ we have

(i) $\overline{(\varphi, \psi)_G} = \int_G \overline{\varphi}\psi\,dx = (\psi, \varphi)_G$,

(ii) $(\alpha\varphi+\beta\psi, \chi) = \alpha(\varphi, \chi)_G+\beta(\psi, \chi)_G$,

(iii) $(\varphi, \varphi)_G \geqslant 0$, and the equality $(\varphi, \varphi)_G = 0$ implies $\varphi(x) = \theta(x)$ (see § 14.2 (2)) where θ is a Riemann integrable function for which the integral of the square of its modulus is equal to zero. Therefore, as was shown in § 6.2, for any two functions $\varphi, \psi \in L_{2,r}(G)$ we have the inequality

$$\int_G |\varphi(x)\overline{\psi}(x)|\,dx \leqslant \left(\int_G |\varphi(x)|^2\,dx\right)^{1/2}\left(\int_G |\psi(x)|^2\,dx\right)^{1/2} \tag{2}$$

Now let us assume that Ω is an open set, possibly unbounded, such that its intersection with any ball is a measurable set. We shall denote by $L'_2 =$

$= L'_2(\Omega)$ the collection of all (complex or real) functions f defined on Ω, whose integrals $\int\limits_\Omega f\,dx$ have finite numbers of singularities (provided there are such), which satisfy the condition $|f(x)|^2 \in L'(\Omega) = L'$ (see § 14.2).

Let us take two arbitrary functions $\varphi, \psi \in L'_2(\Omega)$ and construct the set Ω_ε ($\varepsilon > 0$) by deleting from the set Ω the balls of radii $\varepsilon > 0$ with centres at the points where the integrals of φ and ψ over Ω have singularities (the number of these balls is finite) and, if Ω is unbounded, by deleting the ball $|x| \geqslant \varepsilon^{-1}$ too. If the integrals of φ and ψ have no singularities we simply put $\Omega_\varepsilon = \Omega$.

Thus, $\psi \in L_{2,r}(\Omega_\varepsilon)$ for any $\varepsilon > 0$ and

$$\int\limits_{\Omega_\varepsilon} |\varphi(x)\,\overline{\psi}(x)|\,dx \leqslant \left(\int\limits_{\Omega_\varepsilon} |\varphi(x)|^2\,dx\right)^{1/2} \left(\int\limits_{\Omega_\varepsilon} |\psi(x)|^2\,dx\right)^{1/2} \leqslant$$

$$\leqslant \left(\int\limits_{\Omega} |\varphi(x)|^2\,dx\right)^{1/2} \left(\int\limits_{\Omega} |\psi(x)|^2\,dx\right)^{1/2}$$

whence it follows that, since $\varepsilon > 0$ is an arbitrary number, the integral $\int\limits_\Omega |\varphi(x)\,\overline{\psi}(x)|\,dx$ exists and the inequality

$$\int\limits_{\Omega} |\varphi(x)\,\overline{\psi}(x)|\,dx \leqslant \left(\int\limits_{\Omega} |\varphi|^2\,dx\right)^{1/2} \left(\int\limits_{\Omega} |\psi|^2\,dx\right)^{1/2} \tag{3}$$

is fulfilled*. We have proved that if $\varphi, \psi \in L'_2(\Omega)$ then $\varphi\overline{\psi} \in L'(\Omega)$ and inequality (3) holds. Thus, for any $\varphi, \psi \in L'_2(\Omega)$, makes sense the (absolutely convergent) integral

$$(\varphi, \psi) = \int\limits_\Omega \varphi(x)\,\overline{\psi}(x)\,dx \tag{4}$$

understood as a Riemann integral (generally, in the improper sense). It can readily be verified that integral (4) possesses the three properties of a scalar product (§ 6.2) on condition that the zero element is understood as a function $\theta = \theta(x) \in L'_2$ for which

$$\int\limits_\Omega |\theta(x)|^2\,dx = 0$$

(see the foregoing section).

Now, following §§ 6.2 and 6.3, we can define a norm for the functions $f \in L'_2$ by putting

$$\|f\|_{L_2} = \left(\int\limits_\Omega |f|^2\,dx\right)^{1/2}$$

---

* Inequality (3) also follows from § 14.2 (10) for $p = 2$ but here we have proved it in a completely different way.

The space L_2' equipped with this norm becomes a normed space. If a sequence of functions $f_k \in L_2'$ is convergent in norm to a function $f \in L_2'$ this means that

$$||f-f_k||_{L_2} = \left(\int_\Omega |f(x)-f_k(x)|^2\, dx\right)^{1/2} \to 0, \qquad k \to \infty$$

In such a case *the sequence* $\{f_k\}$ *is said to be convergent to* f *on* Ω *in the mean square.*

The space L_2' *and also the space* L' *are incomplete. A complete space of this kind is the space* $L_2 = L_2(\Omega)$ *of functions Lebesgue measurable on* Ω *for which the squares of their moduli are Lebesgue integrable. The space* L_2 *is called a Hilbert space* after D. Hilbert (1862-1943), the great German mathematician.

The space L_2 is more perfect than L_2' in this sense but its theory involves the Lebesgue integral. On the other hand, the space L_2' includes a sufficiently wide class of functions which are often quite sufficient for practical purposes.

It should be noted that if a function f belongs to $L_2'(\Omega)$ and the set Ω is bounded then $f \in L'(\Omega)$ (see § 14.2 (13)). For instance, we have $x^{-\alpha} \in L_2'(0, 1) \subset L'(0, 1)$ when $\alpha < 1/2$. If the inequalities $1/2 \leq \alpha < 1$ take place, the function $x^{-\alpha}$ does not belong to $L_2(0, 1)$ but belongs to $L'(0, 1)$. Further, we see that $x^{-\alpha} \in L_2'(1, \infty)$ if $\alpha > 1/2$ while $x^{-\alpha} \in L'(1, \infty)$ only if $\alpha > 1$.

§ 14.4. Approximation with Finite Functions

The *support* (or *carrier*) of a function $\varphi(x)$ is the closure of the set of the points x at which $\varphi(x) \neq 0$.

A function $\varphi(x)$ is said to be *finite** *in the open set* $\Omega \subset R_n$ if it is defined in R_n and has a bounded support F belonging to Ω. A bounded support F is often called a *compact support* to stress that each sequence of points $x^k \in F \subset R_n$ contains a subsequence convergent to a point $x^0 \in F$.

A function $\varphi(x)$ will be called *finitely-valued*** if there exists a finite system of pairwise nonintersecting rectangles (rectangular parallelepipeds with edges parallel to the coordinate axes) Δ on each of which the function φ is constant and, besides, $\varphi = 0$ outside these rectangles. These rectangles can be closed or open (or half-open in the sense that some of the faces of Δ belong and some do not belong to it).

A finitely-valued function is obviously finite in R_n.

There holds the following theorem.

---

* The notion of a *finite* function (which vanishes outside some closed subset of Ω) must not be confused with the notion of a *bounded* function (whose range is contained in some finite interval) although in the latter case the word "finite" is also sometimes used. To avoid the ambiguity it is advisable to use in the former case the term "a function of finite (or compact) support". — *Tr.*

** A *finitely-valued* (i.e. piecewise constant) function is also called a *step function* or a *simple function* although these terms are sometimes used in a slightly different sense (e.g. see §§ 16.2 and 19.3). — *Tr.*

Theorem 1. *For every function $f \in L'_p(\Omega)$ $(L_p(\Omega))$ and any $\varepsilon > 0$ there exists a function φ finite in Ω which is finitely-valued or continuous* and such that*

$$\int_{\Omega} |f(x)-\varphi(x)|^p \, dx < \varepsilon \qquad (1 \leq p < \infty) \tag{1}$$

Let us begin with the demonstration of the idea of Theorem 1 by means of graphs for the case $p = 1$ when $f(x)$ is a function defined on the x-axis.

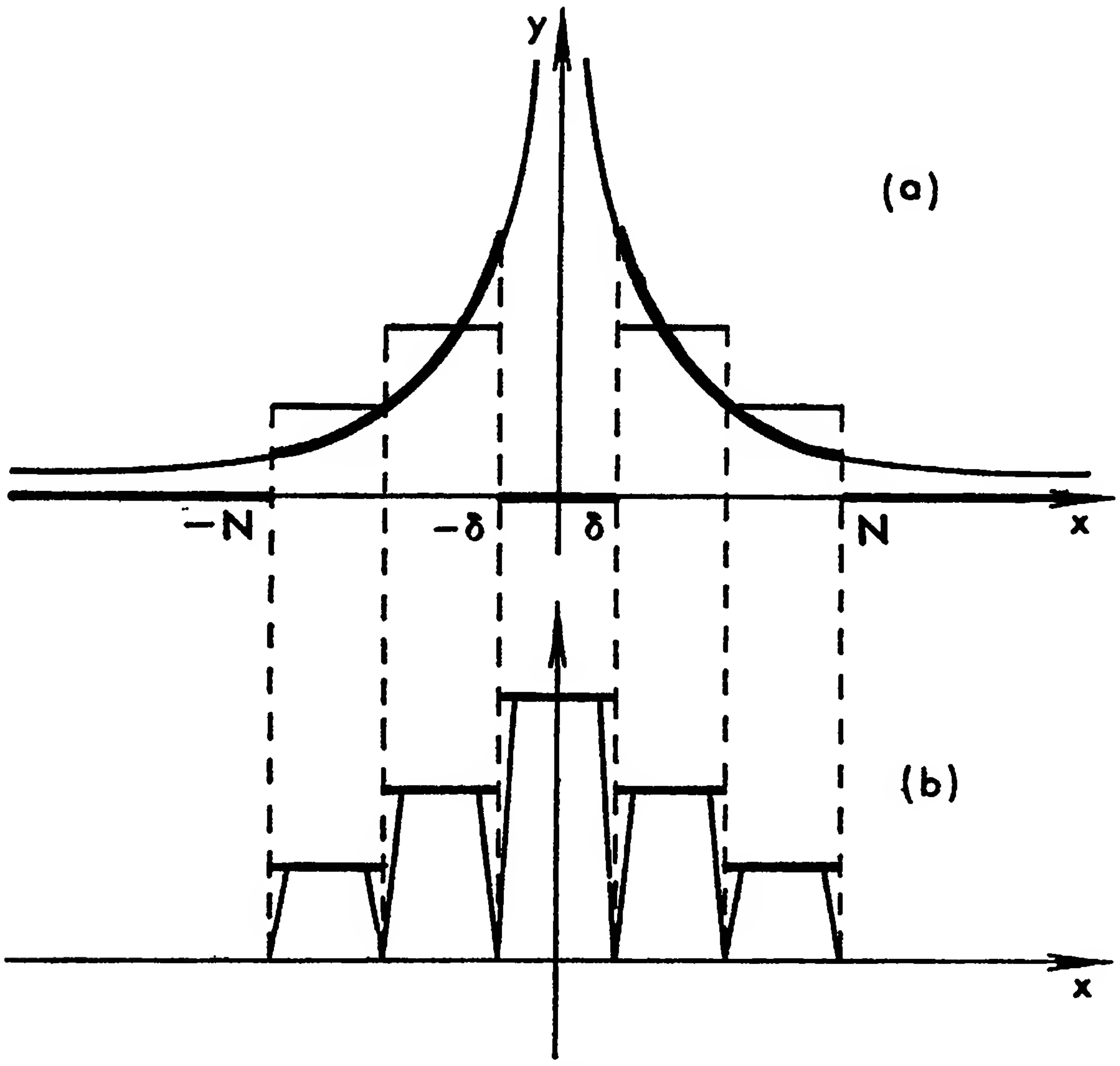

Fig. 14.1

In Fig. 14.1*a* we see the graph of a function f having singularities at the points $-\infty$, 0 and $+\infty$ which will be regarded as belonging to $L'(-\infty, \infty)$. For a sufficiently small $\delta > 0$ and a sufficiently large N the finitely-valued function $\psi(x)$ whose graph is shown in heavy line in Fig. 14.1*a* satisfies the inequality

$$\int_{-\infty}^{\infty} |f(x)-\psi(x)| \, dx < \frac{\varepsilon}{3}$$

($\psi(x) = f(x)$ for $\delta \leq |x| \leq N$ and $\psi(x) = 0$ for the other values of x). In the figure the function $f(x) = \psi(x)$ is shown as being continuous on the intervals $(-N, -\delta)$ and (δ, N) but it is also allowable for it to be discontinuous on condition that it is integrable; it is possible to construct a finitely-valued function $\chi(x)$ for it such that

---

* Here "continuous" can be replaced by "infinitely differentiable" (see property (iv) in § 18.2).

$$\int_{-\infty}^{\infty} |\psi(x)-\chi(x)|\,dx < \frac{\varepsilon}{3}$$

(see the heavy "step-like" line in Fig. 14.1*b*). Now it only remains to approximate $\chi(x)$ by a continuous finite function $\varphi(x)$ as is shown in Fig. 14.1*b* (where each horizontal line segment serving as a part of the graph of $\chi(x)$ (given in heavy line) is replaced by a polygonal line representing the continuous function $\psi(x)$).

A rigorous proof of Theorem 1 follows from Theorems 2, 3 and 4 proved below.*

Theorem 2. *A function $f \in L_p'(\Omega)$, $1 \leqslant p < \infty$* (see §§ 14.2 and 14.3) *can be approximated, to an arbitrary accuracy relative to the metric of $L_p'(\Omega)$, with a bounded function $\psi \in L_p'(\Omega)$ having a compact support.*

Proof. If Ω is a bounded set and f is bounded on it we can simply put $\psi = f$. If otherwise, the integral $\int_\Omega f\,dx$ has singularities. Let Ω_η $(\eta > 0)$ be the (bounded) set obtained from Ω by deleting from it the (closed) balls of radii η and centres at the singular points of the integral $\int_\Omega f\,dx$ (the number of these balls is finite) and, if Ω is unbounded, by deleting the ball $|x| \geqslant \eta^{-1}$ too. Let us put

$$\psi(x) = \begin{cases} f(x) & \text{on} \quad \Omega_\eta \\ 0 & \text{on} \quad \Omega-\Omega_\eta \end{cases}$$

Obviously ψ is a bounded function with compact support on Ω belonging to $L_p'(\Omega)$, and, besides, for sufficiently small η we have

$$\int_\Omega |f(x)-\psi(x)|^p\,dx = \int_{\Omega-\Omega_\eta} |f|^p\,dx < \varepsilon \tag{2}$$

The theorem has been proved.

Theorem 3. *A function f defined and Riemann integrable on an open set Ω can be approximated, with an arbitrary accuracy relative to the metric of $L_p(\Omega)$, by a finitely-valued function φ which is finite in Ω and satisfies the inequality*

$$|\varphi(x)| \leqslant K = \sup_{x\in\Omega} |f(x)| \tag{3}$$

when it is real and the inequality

$$|\varphi(x)| \leqslant 2K \tag{4}$$

when it is complex.

---

* For the space L (and consequently for the space L' as well) Theorem 1 follows from § 19.3 (property 18); in the case of the space L_p it is implied by § 18.2 (property (iv)) and by Theorem 4 proved below.

Proof. We shall begin with the case of a real function f. Since it is integrable on Ω, given any $\varepsilon > 0$, it is possible to construct a rectangular network with mesh-size h such that the lower integral sum of the function f extending over those cubes Δ_j ($j = 1, \ldots, N$) of the network which are completely inside Ω differs from the integral $\int_\Omega f\,dx$ by less than ε:

$$\int_\Omega f\,dx - \sum_{j=1}^{N} m_j |\Delta_j| < \frac{\varepsilon}{2}, \qquad m_j = \inf_{x \in \Delta_j} f(x)$$

Let us take the finitely-valued function

$$\varphi(x) = \begin{cases} m_j \text{ on } \Delta_j \\ 0 \text{ outside } \sum_1^N \Delta_j = G \end{cases}$$

which obviously satisfies (3) (for the function φ to be one-valued, we can define it on the open or half-open cubes Δ). It is obvious that the support of $\varphi(x)$ belongs to G and that inequality (3) holds. Taking into account the inequality $\varphi(x) \leqslant f(x)$ which is fulfilled for the points $x \in G$ we obtain

$$\begin{aligned} \int_\Omega |f(x)-\varphi(x)|\,dx &= \int_G |f(x)-\varphi(x)|\,dx + \int_{\Omega-G} |f|\,dx = \\ &= \left(\int_\Omega f\,dx - \sum m_j |\Delta_j|\right) + \int_{\Omega-G} (|f|-f)\,dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

provided that h is sufficiently small for the inequality

$$\int_{\Omega-G} (|f|-f)\,dx \leqslant 2 \int_{\Omega-G} |f|\,dx < \frac{\varepsilon}{2}$$

to hold. Such a choice of $h > 0$ is possible because f is bounded on Ω and the measure of $|\Omega - G|$ can be made arbitrarily small for sufficiently small h.

We have proved the assertion of the theorem for a real function f and for the metric of the space $L(\Omega)$. Now we shall prove it for the case of a real function f and for the metric of the space $L_p(\Omega)$ ($1 < p < \infty$). To this end, using what has already been proved for the given $\varepsilon > 0$, we construct a finitely-valued function φ with support belonging to Ω which satisfies inequality (3) so that

$$\left(\int_\Omega |f(x)-\varphi(x)|\,dx\right)^{1/p} < \frac{\varepsilon}{|\Omega|^{(p-1)/p} (2K)^{(p^2-1)/p}}$$

Then (see § 14.2 (10)) we obtain

$$\int_{\Omega} |f-\varphi|^p\,dx = \int_{\Omega} |f-\varphi|^{1/p}\,|f-\varphi|^{(p^2-1)/p}\,dx \leqslant$$

$$\leqslant \left(\int_{\Omega} |f-\varphi|\,dx\right)^{1/p} \left(\int_{\Omega} |f-\varphi|^{p+1}\,dx\right)^{(p-1)/p} \leqslant$$

$$\leqslant \frac{\varepsilon}{|\Omega|^{(p-1)/p}\,(2K)^{(p^2-1)/p}} \left(\int_{\Omega} (2K)^{p+1}\,dx\right)^{(p-1)/p} = \varepsilon$$

which completes the proof of the theorem for a real function $f \in L_p'(\Omega)$.

If $f = f_1 + if_2$ is a complex function satisfying the conditions of the theorem, its real and imaginary parts f_1 and f_2 also satisfy these conditions. As has already been proved, for these real functions f_1 and f_2 there exist finitely-valued real functions φ_1 and φ_2 which provide the corresponding approximations for them (in the metric of $L_p(\Omega)$). Therefore the complex function $\varphi = \varphi_1 + i\varphi_2$ which is obviously finitely-valued and finite in Ω satisfies inequality (4) and serves as the desired approximation of f.

Theorem 4. *A finitely-valued function $f(x)$ with support Ω can be approximated (to an arbitrary accuracy) in the metric of $L_p(\Omega)$ with a continuous function $\varphi(x)$ finite in the interior of Ω.*

Note that, according to the definition of a finitely-valued function, the support Ω of $f(x)$ is the union of a finite number of rectangles with faces parallel to the coordinate axes.

Proof. We shall begin the proof with the case of the simplest finitely-valued function $f(x)$ equal to a number m on a rectangle $\varDelta$ and to zero outside it. Let $\varDelta' \subset \varDelta''$ be two rectangles belonging to $\varDelta$ and having the same centre as $\varDelta$ and let us consider the continuous function which is defined by the relations

$$\varphi_{\varDelta}(x) = \begin{cases} 1 & \text{on } \varDelta' \\ 0 & \text{outside } \varDelta'' \end{cases} \tag{5}$$

at the points belonging to $\varDelta'$ and $\varDelta''$ and is linear on $\varDelta'' - \varDelta'$ along the rays issued from the centre of $\varDelta$.

It is clear that the function $\varphi_{\varDelta}(x)$ and, together with it, the function $\varphi(x) = m\varphi_{\varDelta}(x)$ are continuous and finite in the interior of $\varDelta$. It is also evident that for any $\varepsilon > 0$ there is a rectangle $\varDelta'$ and, together with it, a rectangle $\varDelta''$ which are so close to $\varDelta$ that

$$\left(\int_{R_n} |f(x)-\varphi(x)|^p\,dx\right)^{1/p} < \varepsilon$$

In the general case a finitely-valued function $f(x)$ equal to numbers $m_1, \ldots, m_N$ on some (open or half-open) nonintersecting rectangles $\varDelta_1, \ldots$

$\ldots, \Delta_N$ respectively can be represented as the sum

$$f(x) = \sum_1^N f_k(x)$$

of the functions $f_k(x) = m_k \chi_{\Delta_k}(x)$ $(k = 1, \ldots, N)$ where

$$\chi_\Delta = (x) = \begin{cases} 1 & \text{for} \quad x \in \Delta \\ 0 & \text{for} \quad x \notin \Delta \end{cases}$$

By what has been proved, given any $\varepsilon > 0$, there are functions $\varphi_k(x)$ continuous and finite in the interiors of Δ_k such that

$$\int_{R_n} |f_k - \varphi_k|^p \, dx < \frac{\varepsilon}{N}$$

Now, since the function

$$\varphi(x) = \sum_1^N \varphi_k(x)$$

is continuous and finite in the interior of $\sum\limits_1^N \Delta_k$ and satisfies the inequality

$$\int_{\Sigma\Delta_k} |f - \varphi|^p \, dx = \int_{R_n} |f - \varphi|^p \, dx = \sum_{k=1}^N \int_{R_n} |f_k - \varphi_k|^p \, dx < N \frac{\varepsilon}{N} = \varepsilon$$

we obtain what we had to prove.

The above argument applies both to real and to complex functions.

In particular, we have shown that any function $f \in L_p'(\Omega)$ can be approximated in the corresponding metric with a finitely-valued function of the form

$$\varphi(x) = \begin{cases} m_k & \text{on } \Delta_k \qquad (k = 1, \ldots, N) \\ 0 & \text{outside} \quad \sum\limits_1^N \Delta_k \end{cases} \tag{6}$$

where Δ_k are some pairwise disjoint cubes.

Every finitely-valued function can be regarded as a member of the family of the functions of the form

$$\varphi(x) = f(x; x', \ldots, x^N, \eta_1, \ldots, \eta_N, m_1, \ldots, m_N)$$

dependent on the N vector parameters $x^1, \ldots, x^N$ and on $2N$ scalar parameters $\eta_1, \ldots, \eta_N$ and $m_1, \ldots, m_N$ where x^k are the centres of the corresponding cubes, η_k are the lengths of their edges and m_k are the numbers entering into equality (6). It is readily seen that if an approximation to the function f with the aid of a function φ_1 belonging to that family has already be found, that is if

$$\left(\int_\Omega |f(x) - \varphi_1(x)^p \, dx\right)^{1/p} = \varepsilon_1 < \varepsilon$$

then the function φ in the last inequality can be replaced by another function φ_1 (belonging to the indicated family) determined by some rational values of the parameters which are so close to the values of the parameters specifying the original function φ that

$$\left(\int_\Omega |\varphi_1(x)-\varphi(x)|^p dx\right)^{1/p} < \varepsilon-\varepsilon_1$$

We thus see that for an arbitrary function $f \in L_p'(\Omega)$ and for any $\varepsilon > 0$ there is a finitely-valued function φ determined by rational values of the parameters which approximates f to an accuracy of ε with respect to the metric of $L'(\Omega)$:

$$\left(\int_\Omega |f(x)-\varphi(x)|^p\, dx\right)^{1/p} < \varepsilon$$

Since all such functions (determined by rational values of the parameters) can be numbered in a consecutive order, they form a countable set.

We have thus proved the following important theorem.

Theorem 5. *In the space $L_p'(\Omega)$ $(L_p(\Omega))$ there exists a countable sequence of (finitely-valued) functions $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$, ... such that for any function $f(x) \in L_p'(\Omega)$ and any given number $\varepsilon > 0$ there is an element $\varphi_k(x)$ of that sequence such that*

$$\left(\int_\Omega |f-\varphi_k|^p\, dx\right)^{1/p} < \varepsilon \tag{7}$$

The property of the space $L_p'(\Omega)$ $(L_p(\Omega))$ indicated in Theorem 5 is completely analogous to the well-known property of the set of all real (complex) numbers: *the set of all real (complex) numbers contains a countable subset which is everywhere dense* (see § 14.5) *in that set* (this is in fact the subset of all real rational numbers (complex numbers with rational real and imaginary parts)).

Theorem 6. *Let f be a function belonging to the space $L_p'(\Omega)$ $(L_p(\Omega))$. If Ω is a part of $R_n = R$ we shall extend f to the whole space R by putting $f = 0$ outside Ω. Then the extended function f* (we retain the same notation for it) *possesses the limiting property*

$$\varphi(t) \equiv \left(\int_R |f(x+t)-f(x)|^p\, dx\right)^{1/p} \to 0,\, t \to 0 \qquad (1 \leq p < \infty) \tag{8}$$

Proof. The extension f of the function $f \in L_p'(\Omega)$ $(L_p(\Omega))$ constructed as is indicated in the theorem obviously belongs to $L_p' = L_p'(R)$ $(L_p = L_p(R))$, and therefore Theorem 1 applies to it. Let us choose an arbitrary $\varepsilon > 0$ and find a continuous function φ finite in R_n and such that

$$\left(\int_R |f(x)-\varphi(x)|^p\, dx\right)^{1/p} < \frac{\varepsilon}{3}$$

The support of φ is a bounded set $F \subset g' \subset g$ where g and g' are concentric balls with radii ϱ' and ϱ, $\varrho' < \varrho$, respectively. The function φ is continuous on the bounded and closed set g and therefore is uniformly continuous on it. Let us denote as $\omega(\delta)$ the modulus of continuity of φ on g:

$$\omega(\delta) = \sup_{|x'-x''|<\delta} |\varphi(x')-\varphi(x'')| \qquad (x', x'' \in g)$$

Then, denoting by $|g'|$ the measure of g' we obtain, for sufficiently small δ, the inequalities

$$\left(\int_R |f(x+t)-f(x)|^p\,dx\right)^{1/p} \leq \left(\int_R |f(x+t)-\varphi(x+t)|^p\,dx\right)^{1/p} +$$

$$+\left(\int_{g'} |\varphi(x+t)-\varphi(x)|^p\,dx\right)^{1/p} + \left(\int_R |\varphi(x)-f(x)|^p\,dx\right)^{1/p} <$$

$$< \frac{\varepsilon}{3}+\omega(\delta)|g'|^{1/p}+\frac{\varepsilon}{3} < \varepsilon \qquad (|t| < \delta < \varrho-\varrho')$$

The theorem has been proved.

§ 14.5. Linear Spaces. Fundamentals of the Theory of Normed Linear Spaces

Let E be a real (complex) linear space (see § 6.1). It is evident that if a subset E_1 of E is such that $x \in E_1$ implies $\alpha x \in E_1$ where α is an arbitrary real (complex) number and $x, y \in E_1$ implies $x+y \in E_1$ then E_1 is itself a linear space. In this case E_1 is spoken of as a (linear) *subspace* of E.

A finite system of elements $x_1, \ldots, x_n \in E$ is said to be *linearly independent* if the equality

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = \theta$$

implies $\alpha_j = 0$; $j = 1, \ldots, n$. If otherwise, the system is said to be *linearly dependent*.

A linear space E' is called *finite-dimensional* or, more precisely, *n-dimensional* if it contains a system of n linearly independent elements $x_1, \ldots, x_n$ and if every system of $n+1$ elements belonging to it is linearly dependent. It can readily be seen that in this case any element $x \in E'$ can be expressed in a unique manner as a sum of the form

$$x = \sum_{k=1}^{n} \alpha_k x_k \tag{1}$$

where α_k ($k = 1, \ldots, n$) are some numbers (real or complex depending on the nature of the space in question).

It can also be shown that, conversely, *if $x_1, \ldots, x_n$ is a linearly independent system of elements then the linear space E' consisting of all the elements of form* (1) *is n-dimensional.* To this end it is sufficient to prove that any $n+1$ elements of E' form a linearly dependent system.

The system of the functions

$$1,\ x,\ x^2,\ \ldots,\ x^{n-1} \qquad (a \leq x \leq b) \tag{2}$$

is linearly independent in the space $C(a, b)$ because the zero element of $C(a, b)$ is the function identically equal to zero on $[a, b]$ and the equalities

$$\sum_0^{n-1} \alpha_k x^k \equiv 0 \qquad (a \leq x \leq b) \tag{3}$$

imply that $\alpha_0 = \alpha_1 = \ldots = \alpha_{n-1} = 0$.

System (2) regarded as belonging to the space $L_p'(a, b)$ is also linearly independent because for a sum $\sum\limits_0^{n-1} \alpha_k x^k$ to be the zero element of this space it is necessary and sufficient that the equality

$$\int_a^b \left| \sum_0^{n-1} \alpha_k x^k \right|^p dx = 0$$

hold, and since the integrand is continuous this equality implies (see Theorem 1 in § 14.2) (3), i. e. a_j $(j = 0, 1, \ldots, n-1)$ are equal to zero.

Therefore the collection of all polynomials $P_{n-1}(x)$ $(a \leq x \leq b)$ of a given degree $n-1$* is an n-dimensional linear subspace of $C(a, b)$ and of $L_p'(a, b)$.

A linear space E is called *infinite-dimensional* if for any arbitrarily large n there is a linearly independent system of elements $x_1, \ldots, x_n$ belonging to E.

A sequence of elements

$$x_1,\ x_2,\ x_3,\ \ldots \tag{4}$$

is said to be *linearly independent* if its any subsystem consisting of a finite number of elements is linearly independent. Such an infinite sequence will be referred to as a *countable linearly independent system of elements.*

In every infinite-dimensional linear space E there is a countable linearly independent system of elements since any element $x_1 \neq \theta$ can be regarded as a linearly independent system (consisting of that single element): we shall choose it as the first element of a sequence of type (4) which will be constructed by induction. Suppose that we have already found a linearly independent system

$$x_1,\ \ldots,\ x_n \tag{5}$$

belonging to E. Since E is infinite-dimensional there must exist an element $x_{n+1} \in E$ such that the system $x_1, x_2, \ldots, x_n, x_{n+1}$ is linearly independent because, if otherwise, any element $x \in E$ would be representable in the form $x = \sum\limits_{k=1}^{n} \alpha_k x_k$ where α_k's are some numbers, and the set of the elements of the form $\sum_1^n \alpha_k x_k$ would coincide with E and contain only n linearly independent elements.

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* More precisely, of degree not higher than $n-1$.

On continuing this process indefinitely we obtain a sequence of type (4) which is obviously linearly independent. It should be taken into account that any finite subsystem of n elements $x_{k_1}, \ldots, x_{k_n}$ $(k_1 < k_2 < \ldots < k_n)$ arbitrarily chosen from such a sequence (4) forms a linearly independent system, which follows from the fact that the wider system $x_1, x_2, \ldots, x_{k_n}$ is linearly independent.

The sequence of functions $1, x, x^2, \ldots$ serves as an example of a countable linearly independent system in $C(a, b)$ and in $L'_p(a, b)$.

Now let us suppose that E is a normed linear space. A set $G \subset E$ is said to be *dense* in E if for any element $x \in E$ and any positive number ε there is an element $y \in G$ such that

$$||x-y|| < \varepsilon$$

By virtue of this definition, we conclude, on the basis of Theorem 1 in § 14.4, that the set of all continuous (and even infinitely differentiable) functions finite in Ω is dense in $L'_p(\Omega)$ (and also in $L_p(\Omega)$); the set of all finitely-valued functions with support belonging to Ω is also dense in these spaces.

Here is one more example. A function φ defined on a closed interval $[a, b]$ is called *polygonal* if it is continuous on that interval $(\varphi \in C(a, b))$ and if there exists a partition of the interval such that φ is a linear function on each of the subintervals of the partition. For any function $f \in C(a, b)$ and for any $\varepsilon > 0$, by virtue of the continuity of the former, there exists a polygonal function $\varphi(x)$ such that

$$||f-\varphi||_{C(a, b)} = \max_{x \in [a, b]} |f(x)-\varphi(x)| < \varepsilon$$

Consequently, the polygonal functions defined on $[a, b]$ form a dense set in $C(a, b)$.

A space E is called *separable* if it is infinite-dimensional and contains a countable set dense in E. An instance of a separable space is the space $C(a, b)$ (see the exercises below).

By Theorem 5 in § 14.4, the (infinite-dimensional) space $L'_p(\Omega)$ $(L_p(\Omega))$ is separable because it contains the set of finitely-valued functions which is countable and dense in $L'_p(\Omega)$ $(L_p(\Omega))$. See Exercise 5 below.

A set $M \subset E$ is said to be *complete* in E if the collection of all the possible linear combinations of the form $\sum_1^n \alpha_k x_k$ where α_k are numbers and x_k are elements of M is a dense set in E.

Theorem 1. *If a space E contains a complete countable linearly independent system of elements, it is separable.*

Indeed, the space E is then infinite-dimensional since for any natural number n there is a linearly independent system $x_1, \ldots, x_n$ of elements in E consisting of n elements. Further, the set M' of the sums $\sum_1^n r_k x_k$ where n is an arbitrary natural number and r_k $(k = 1, \ldots, n)$ are arbitrary rational numbers is countable (these sums can be numbered in a consecutive order), and, since M is dense in E, for any element $x \in E$ and any $\varepsilon > 0$ there is a sum $\sum_1^n \alpha_k x_k$ such that

$$\left\| x - \sum_1^n \alpha_k x_k \right\| < \frac{\varepsilon}{2}$$

Now we can choose rational numbers r_k which are so close to the corresponding numbers α_k that

$$\left\| \sum_1^n \alpha_k x_k - \sum_1^n r_k x_k \right\| \leqslant \sum_1^n |\alpha_k - r_k| \, \|x_k\| \leqslant K \sum_1^n |\alpha_k - r_k| < \frac{\varepsilon}{2}$$

where $K = \max\limits_{1 \leqslant k \leqslant n} \|x_k\|$.

Therefore

$$\left\| x - \sum_1^n r_k x_k \right\| \leqslant \left\| x - \sum_1^n \alpha_k x_k \right\| + \left\| \sum_1^n \alpha_k x_k - \sum_1^n r_k x_k \right\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

We have thus constructed an arbitrarily accurate approximation for an arbitrary element $x \in E$ with the aid of an element x belonging to M', which proves the separability of E.

The converse of this theorem also holds:

Theorem 2. *A separable space E contains a countable linearly independent system of elements which is complete in E.*

Indeed, by the separability of E, there is a countable sequence of elements dense in E. Moreover, this sequence is complete in E.

Further, every complete countable system of elements $x_1, x_2, \ldots$ contains a linearly independent sequence of elements (generally speaking, not determined uniquely) which in its turn forms a complete system in E.

For instance, let us take the subsequence

$$x_{n_1}, x_{n_2}, x_{n_3}, \ldots \tag{6}$$

(of the sequence $x_1, x_2, \ldots$) in which the indices $n_1, n_2, \ldots$ are chosen according to the following rule. As n_1 we take the smallest index n for which $x_n \neq \theta$. The element x_{n_1} can be regarded as a linearly independent system (consisting of that single element). Further, if the indices $n_1, \ldots, n_k$ have been already determined, we take as n_{k+1} the smallest natural number n for which the elements $x_{n_1}, \ldots, x_{n_k}, x_{n_{k+1}}$ form a linearly independent system.

It is important in this construction that for every k there exists n_{k+1} possessing the indicated property, that is the resultant linearly independent system (6) is in fact infinite (countable). Indeed, suppose that for some k

there is no n_{k+1} with the indicated property. Let us put $z_1 = x_{n_1}, \dots, z_k = x_{n_k}$. Then the system

$$z_1, \dots, z_k \tag{7}$$

where k is fixed, possesses the following properties:

(i) system (7) is linearly independent;

(ii) for any element $x \in E$ and any $\varepsilon > 0$ there are numbers $\alpha_1, \dots, \alpha_k$ such that

$$\left\| x - \sum_1^k \alpha_j z_j \right\| < \varepsilon \tag{8}$$

To show that under the hypotheses assumed properties (i) and (ii) cannot hold we need the following lemma.

Lemma 1. *Properties* (i) *and* (ii) *imply that to each element $x \in E$ there corresponds a system of numbers $\beta_1, \dots, \beta_k$ such that*

$$x = \sum_1^k \beta_j z_j$$

The proof of Lemma 1 will be given below. It follows from the lemma that E is an n-dimensional space, which contradicts the hypothesis that E is separable and therefore, according to the definition of separability, infinite-dimensional.

Consequently, sequence (6) with the required properties can in fact be constructed.

The proof of Lemma 1 is particularly simple when E is a linear space with scalar product (see Theorem 3 in § 14.7). In what follows we shall deal with that very case. For the general case the proof of the lemma is based on the auxiliary lemma below which is itself interesting.

Lemma 2. *If a system of elements $y_1, \dots, y_n$ of the space E is linearly independent then there is a positive number λ such that*

$$\lambda \sum_1^n |\alpha_j| \leq \left\| \sum \alpha_j y_j \right\| \tag{9}$$

for any system of numbers $\alpha_1, \dots, \alpha_n$ (generally speaking, *λ depends on n*).

Proof. Let us consider the function

$$\Phi(\alpha) = \Phi(\alpha_1, \dots, \alpha_n) = \left\| \sum_1^n \alpha_j y_j \right\|$$

of n variables defined throughout the n-dimensional space R_n. This function is continuous because

$$|\Phi(\alpha) - \Phi(\alpha^0)| = \left| \left\| \sum_1^n \alpha_j y_j \right\| - \left\| \sum_1^n \alpha_j^0 y_j \right\| \right| \leq \left\| \sum_1^n \alpha_j y_j - \sum_1^n \alpha_j^0 y_j \right\| =$$

$$= \left\| \sum_1^n (\alpha_j - \alpha_j^0) y_j \right\| \leq \sum_{j=1}^n |\alpha_j - \alpha_j^0| \, \|y_j\| \leq K \sum_{=1}^n |\alpha_j - \alpha_j^0| \to 0, \qquad \alpha \to \alpha^0$$

where

$$K = K_n = \max_{1 \leqslant j \leqslant n} ||y_j||$$

Further, by the linear independence of the elements $y_1, \ldots, y_n$, the function Φ is positive for any numbers $\alpha_1, \ldots, \alpha_n$ except in the case when $\alpha_1 = \ldots = \alpha_n = 0$. Let us also consider the set $\Lambda \subset R_n$ of the points $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ whose coordinates satisfy the equality

$$||\boldsymbol{\alpha}||^* = \sum_1^n |\alpha_j| = 1$$

The set Λ is bounded since the coordinates of its points satisfy the inequality

$$|\alpha_s| \leqslant \sum_{j=1}^n |\alpha_j| = 1$$

Furthermore, this set is closed (see Example 5 in § 7.9). Consequently the function $\Phi(\boldsymbol{\alpha})$ attains its minimum on the set Λ at a point $\boldsymbol{\alpha}^0 \in \Lambda$:

$$\lambda = \Phi(\boldsymbol{\alpha}^0) = \min_{\boldsymbol{\alpha} \in \Lambda} \Phi(\boldsymbol{\alpha})$$

The number λ is obviously positive because the points $\boldsymbol{\alpha} \in \Lambda$ do not coincide with the origin. Thus, we have the inequality

$$0 < \lambda \leqslant \left\| \sum_1^n \alpha_j y_j \right\| \tag{10}$$

which holds for any point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \Lambda$. Thus, we have proved inequality (9) for the special case when $\alpha \in \Lambda$.

Now we shall prove that (9) holds for any point $\boldsymbol{\alpha} \in R_n$. Indeed, if $\boldsymbol{\alpha} = \boldsymbol{\theta} = (0, \ldots, 0)$ inequality (9) becomes trivial. Therefore let us assume that $\boldsymbol{\alpha} \neq \boldsymbol{\theta}$. Then we can consider the new point

$$\boldsymbol{\beta} = \frac{\boldsymbol{\alpha}}{||\boldsymbol{\alpha}||^*} = \left(\frac{\alpha_1}{||\boldsymbol{\alpha}||^*}, \ldots, \frac{\alpha_n}{||\boldsymbol{\alpha}||^*}\right)$$

It is evident that $\boldsymbol{\beta} \in \Lambda$ because

$$\sum_{j=1}^n |\beta_j| = \sum_{j=1}^n \frac{\alpha_j}{||\boldsymbol{\alpha}||^*} = 1$$

and therefore, by virtue of (10), we have

$$\lambda \leqslant \left\| \sum_1^n \frac{\alpha_j}{||\boldsymbol{\alpha}||^*} x_j \right\| = \frac{1}{||\boldsymbol{\alpha}||^*} \left\| \sum_{j=1}^n \alpha_j x_j \right\|$$

whence follows (9).

Next we have to deduce Lemma 1.

Let $\boldsymbol{x} \in E$. If we suppose that the elements

$$\boldsymbol{x}, z_1, \ldots, z_n \tag{11}$$

form a linearly independent system then, by Lemma 2, we must have

$$1 \leqslant 1+\sum_1^n |\alpha_j| \leqslant \frac{1}{\lambda}\left\|x-\sum_1^n \alpha_j z_j\right\| \qquad (\lambda > 0)$$

for any numbers $\alpha_1, \ldots, \alpha_n$, whence

$$0 < \lambda \leqslant \left\|x-\sum_1^n \alpha_j z_j\right\|$$

But this contradicts the condition of Lemma 1 which implies that for $\varepsilon < \lambda$ there is a system $\alpha_1, \ldots, \alpha_n$ of numbers for which inequality (8) holds. Consequently system (11) is linearly dependent and there are some numbers $c, c_1, \ldots, c_n$ which are not equal to zero simultaneously such that

$$cx+c_1z_1+ \ldots +c_nz_n = \theta$$

It follows that $c \neq 0$ because, if otherwise, the system $z_1, \ldots, z_n$ would be linearly dependent; therefore

$$x = \sum_{j=1}^n \beta_j z_j \qquad \left(\beta_j = -\frac{c_j}{c}\right)$$

which completes the proof of Lemma 1.

It should be noted that there holds the inequality

$$\left\|\sum_1^n \alpha_j y_j\right\| \leqslant K \sum_{j=1}^n |\alpha_j| \qquad \left(K = \max_{1 \leqslant j \leqslant n} \|y_j\|\right) \tag{12}$$

(which, in a certain sense, is opposite to (9)) where K, by its definition, is independent of $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Combining inequalities (9) and (12) we can write (for an arbitrary linearly independent system $y_1, \ldots, y_n$) the two inequalities

$$C_1 \sum_1^n |\alpha_j| \leqslant \left\|\sum_1^n \alpha_j y_j\right\| \leqslant C_2 \sum_{j=1}^n |\alpha_j| \tag{13}$$

where the constants C_1 and C_2 ($C_1 = \lambda$ and $C_2 = K$) do not depend on $\alpha = (\alpha_1, \ldots, \alpha_n)$. However, it should be noted that these constants C_1 and C_2 depend on the choice of the norm defined in the space E.

Exercises. Prove the following assertions.

1. The set Π' of all polygonal functions defined on $[a, b]$ such that the corner points of their graphs have rational coordinates is countable.

2. For any function $f(x)$ continuous on $[a, b]$ and any $\varepsilon > 0$ there is a function $\varphi \in \Pi'$ such that

$$|f(x)-\varphi(x)| < \varepsilon$$

This property means that Π' is dense in $C(a, b)$. Now, taking into account that $C(a, b)$ contains the countable linearly independent system of functions $1, x, x^2, \ldots$, we see that $C(a, b)$ is a separable space.

3. The set Π' is dense in $L_p'(a, b)$ $(L_p(a, b))$.

Hint. Use the property that the continuous finite functions defined on (a, b) form a set which is dense in the spaces $L'_p(a, b)$ and $L_p(a, b)$ and also take into account the result established in Exercise 2.

4. The spaces $C(\bar{\Omega})$, $L'_p(\Omega)$ and $L_p(\Omega)$ (where Ω is an open set) are infinite-dimensional.

Hint. Consider a sequence $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \dots$ of pairwise disjoint cubes belonging to Ω and the corresponding sequence of their characteristic functions

$$\varphi_{\Delta_k}(x) = \begin{cases} 1 \text{ on } \Delta_k \\ 0 \text{ on } \Delta_k \end{cases} \qquad (k = 1, 2, \dots)$$

(for the space $C(\bar{\Omega})$ take the functions of the form indicated in § 14.4 (5); in this case Ω is bounded).

5. The result established in Exercise 4 and Theorem 5, § 14.4 imply that the spaces $L'_p(\Omega)$ and $L_p(\Omega)$ are separable.

§ 14.6. Orthogonal Systems in Space with Scalar Product

Let H be a (complex or real) linear space for whose elements $\varphi, \psi, f, \dots$ is defined a scalar product (φ, ψ) $(\varphi, \psi \in H)$ possessing properties (i)-(iii) enumerated in § 6.2.

We shall begin with an arbitrary, not necessarily complete, linear space H with scalar product; as we know, such is the space $L'_2(\Omega)$.

The norm of an element $\varphi \in H$ is equal to the arithmetic square root of its scalar product by itself: $||\varphi|| = (\varphi, \varphi)^{1/2}$. If $||\varphi|| \neq 0$, then the element $\varphi_1 = \dfrac{\varphi}{||\varphi||}$ has unit norm: $||\varphi_1|| = 1$. An element φ with unit norm $||\varphi|| = 1$ is called *normalized.*

Two elements $\varphi, \psi \in H$ are said to be (mutually) *orthogonal* if $(\varphi, \psi) = 0$.

A (finite or infinite) system of elements

$$\varphi_1, \varphi_2, \varphi_3, \dots \tag{1}$$

is called *orthogonal* if its elements are different from zero (have positive norms) and are pairwise orthogonal.

Finally, system (1) is called *orthonormal* if

$$(\varphi_k, \varphi_l) = \delta_{kl} = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

that is, if it is an orthogonal system whose every element has unit norm (is normalized). It is evident that every orthogonal system can be normalized, that is transformed into an orthonormal system.

Every finite orthogonal system $\varphi_1, \dots, \varphi_N$ is linearly independent in H, that is the relation

$$\sum_1^N \alpha_k \varphi_k = \theta$$

where α_k are numbers implies that all $\alpha_k = 0$; $k = 1, \ldots, N$. Indeed, on multiplying both members of this relation scalarly by φ_l ($l = 1, \ldots, N$) and using the linearity of the scalar product we obtain

$$\left(\sum_1^N \alpha_k \varphi_k, \varphi_l\right) = \alpha_l(\varphi_l, \varphi_l) = 0$$

and, since $(\varphi_l, \varphi_l) > 0$, we see that $\alpha_l = 0$ ($l = 1, \ldots, N$).

Given an arbitrary element $f \in H$, the numbers

$$\frac{1}{\|\varphi_k\|^2}(f, \varphi_k) \qquad (k = 1, 2, \ldots)$$

are called the *Fourier* coefficients of f with respect to orthogonal system* (1).

The series $\sum_1^\infty \frac{1}{\|\varphi_k\|^2}(f, \varphi_k)\varphi_k$ (generated by an element $f \in H$) is called the *Fourier series of f with respect to orthogonal system* (1), and we write

$$f \sim \sum_1^\infty \frac{1}{\|\varphi_k\|^2}(f, \varphi_k)\varphi_k \tag{2}$$

If system (1) is orthonormal then $\|\varphi_k\| = 1$ ($k = 1, 2, \ldots$) and the Fourier series of $f \in H$ with respect to such system (1) assumes the simplified form:

$$f \sim \sum_1^\infty (f, \varphi_k)\varphi_k \tag{3}$$

In this case the Fourier coefficients are the numbers (f, φ_k). In what follows we only deal with orthonormal systems (1), however all the results established for them can be modified in a natural way to apply to the general orthogonal systems.

Chapter 15 of our course is devoted to a thorough study of the important special case of Fourier's series with respect to the *trigonometric system*

$$\frac{1}{2}, \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

Here we note that the functions $1/2$, $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, ... forming this system are orthogonal in the space $L_2'(0, 2\pi)$ of *square integrable functions* on $[0, 2\pi]$ (for which the squares of their moduli are integrable on $[0, 2\pi]$). (The same applies to the space $L_2(0, 2\pi)$.) The space $L_2'(0, 2\pi)$ ($L_2(0, 2\pi)$) is a special case of a linear space H with scalar product to which apply all the results established in the present chapter for the general spaces H.

Let there be given an orthonormal system of elements (1) in an arbitrary linear space H with scalar product. We shall discuss the following problem: given an arbitrary element $f \in H$, it is required to find among all the possible systems of numbers $\alpha_1, \alpha_2, \ldots, \alpha_N$ (complex or real depending on whether H is a complex or real space) the one for which the norm

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* J. B. J. Fourier (1768-1830), a French mathematician who laid the foundations of the theory treating of the representation of functions with trigonometric series.

$$\left\| f - \sum_1^N \alpha_k \varphi_k \right\| \tag{4}$$

attains its minimum.

To estimate this norm we write

$$\begin{aligned}
\left\| f - \sum_1^N \alpha_k \varphi_k \right\|^2 &= \left(f - \sum_{k=1}^N \alpha_k \varphi_k,\, f - \sum_{l=1}^n \alpha_l \varphi_l \right) = \\
&= (f,f) - \sum_1^N [\bar{\alpha}_l (f, \varphi_l) + \alpha_l \overline{(f, \varphi_l)}] + \sum_1^N + |\alpha_l|^2 = \\
&= (f,f) + \sum_1^N |\alpha_l - (f, \varphi_l)|^2 - \sum_1^N |(f, \varphi_l)|^2 \geqslant \\
&\geqslant (f,f) - \sum_1^N |(f, \varphi_l)|^2
\end{aligned} \tag{5}$$

We see that (5) becomes an equality if and only if

$$\alpha_l = (f, \varphi_l) \qquad (l = 1, 2, \ldots, N)$$

It is these numbers (f, φ_l) which we called the Fourier coefficients of the element f with respect to the orthonormal system of the functions φ_l.

The result we have established can be written in the form of the equalities

$$\begin{aligned}
E_N(f)_H = \min_{\alpha_k} \left\| f - \sum_1^N \alpha_k \varphi_k \right\| &= \left\| f - \sum_1^N (f, \varphi_k) \varphi_k \right\| = \\
&= \left((f,f) - \sum_{k=1}^N |(f, \varphi_k)|^2 \right)^{1/2}
\end{aligned} \tag{6}$$

The leftmost term in (6) is simply the notation of the minimum of the norm entering in the second term. Relation (6) shows that the *numbers* $\alpha_l = (f, \varphi_l)$; $l = 1, 2, \ldots, N$ *provide the best approximation to the element* $f \in H$ *(with respect to the metric of* H*) with the aid of the linear combinations of the form* $\sum_1^N \alpha_k \varphi_k$ *where* α_k *are arbitrary numbers* (complex or real depending on the nature of the space H). This fact is expressed by the third term of (6). Finally, the rightmost term gives an explicit expression for the magnitude $E_N(f)_H$ (which characterizes the approximation quantitatively) in terms of (f,f) and the Fourier coefficients (f, φ_k) $(k = 1, \ldots, N)$.

It is clear that $E_N(f)_H \geqslant 0$ because this number is the minimum of non-negative norm (4). It is also evident that $E_N(f)_H$ does not increase as the number N grows, which follows from the last term in (6). The same can be shown with the aid of the second term:

$$E_N(f)_H = \min_{\alpha_k} \left\| f - \sum_1^N \alpha_k \varphi_k \right\| \geqslant \min_{\alpha_k} \left\| f - \sum_1^{N+1} \alpha_k \varphi_k \right\| = E_{N+1}(f)_H$$

because the sum $\sum\limits_1^N$ is a special case of the sum $\sum\limits_1^{N+1}$ for $\alpha_{N+1} = 0$.

It follows that for any element $f \in H$ there exists the limit

$$\lambda = \lim_{N \to \infty} E_N(f)_H = \sqrt{(f, f) - \sum_1^\infty |(f, \varphi_k)|^2} =$$

$$= \lim_{N \to \infty} \left\| f - \sum_{k=1}^N (f, \varphi_k) \varphi_k \right\| \geqslant 0 \tag{7}$$

whence, in particular, it follows that *the series of the squares of the moduli of the Fourier coefficients of $f \in H$ is convergent and the inequality*

$$\sum_1^\infty (f, \varphi_k)^2 \leqslant (f, f) \tag{8}$$

*holds. It is known as Bessel's** *inequality (for the element f).*

We called formula (8) an inequality in order to stress that the left-hand side of (8) cannot exceed the right-hand side. However, for some (or for all) elements $f \in H$ relation (8) may become an exact equality. In the latter case it is called *Parseval's relation.*

We shall say that a series

$$u_0 + u_1 + u_2 + \dots$$

of elements $u_k \in H$ is *convergent with respect to the metric of the space H to an element $f \in H$* if its nth partial sum

$$s_n = u_0 + u_1 + \dots + u_n \qquad (s_n \in H;\ n = 1, 2, \dots)$$

satisfies the condition

$$\lim_{n \to \infty} \| f - s_n \| = 0$$

If the series $u_0 + u_1 + u_2 \dots$ converges to f in this sense we write

$$f = u_0 + u_1 + u_2 + \dots = \sum_0^\infty u_k \tag{9}$$

and say that f is *the sum of the series $\sum\limits_0^\infty u_k$ convergent to f relative to the metric of H.*

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* F. W. Bessel (1784-1846), a German astronomer and mathematician.

Now let us suppose that for a given element f the quantity λ (see relation (7)) is equal to zero. Let us discuss the meaning of the fact that the other three terms of (7) are equal to zero in this case.

(i) The relation $\lim\limits_{N\to\infty} E_N(f)_H = 0$ means that, given any $\varepsilon > 0$, there is a natural number N_0 and some numbers $\alpha_1, \ldots, \alpha_{N_0}$ such that

$$\left\| f - \sum_1^{N_0} \alpha_k \varphi_k \right\| < \varepsilon \tag{10}$$

Indeed, if the indicated numbers $N_0, \alpha_1, \ldots, \alpha_{N_0}$ have been found we can fix N_0 and take the minimum of the left-hand member over all α_k's to obtain

$$\varepsilon > E_{N_0}(f)_H \geqslant E_N(f)_H \qquad (N > N_0)$$

which means that $E_N(f)_H \to 0$ $(N \to \infty)$. Conversely, if it is known that $\lim\limits_{N\to\infty} E_N(f)_H = 0$ as $N \to \infty$ then for any $\varepsilon > 0$ there is N such that

$$\varepsilon > E_N(f)_H = \left\| f - \sum_1^{N} \alpha_k \varphi_k \right\| \qquad (\alpha_k = (f, \varphi_k))$$

(ii) The relation $(f,f) - \sqrt{\sum\limits_1^\infty |(f, \varphi_k)|^2} = 0$ indicates that for the given element f there holds Parseval's relation.

(iii) The relation $\lim\limits_{N\to\infty} \left\| f - \sum\limits_{k=1}^{N} (f, \varphi_k)\varphi_k \right\| = 0$ shows that the Fourier series of f with respect to system (6) is convergent to f in the sense of the metric of the space H.

Since properties (i), (ii) *and* (iii) *can only hold simultaneously, each of them implies the other two.*

We remind the reader that property (i), provided it holds for all the elements $f \in H$, expresses (see § 14.5) the fact that the system of elements $\varphi_1, \varphi_2, \varphi_3, \ldots$ *is complete in H.*

As a consequence of what has been established we can state the following important theorem.

Theorem 1. *For an orthonormal system of elements to be complete in H it is necessary and sufficient that one of the following two conditions be fulfilled:*

(a) *The Fourier series*

$$f \sim \sum_1^\infty (f, \varphi_k)\varphi_k$$

of every element $f \in H$ is convergent to f with respect to the metric of H (in this case the sign "$\sim$" can be replaced by "$=$"; see (9)).

(b) *For every element $f \in H$ there holds Parseval's relation*

$$(f, f) = \sum_{1}^{\infty} |(f, \varphi_k)|^2$$

Let us prove the following lemma.

Lemma 1. *Let f, $u_k \in H$ and let*

$$f = u_0 + u_1 + u_2 + \dots$$

where the convergence of the series (to f) is understood in the sense of the metric of H. Then for any element $v \in H$ we have

$$(f, v) = (u_0, v) + (u, v) + (u, v) + \dots$$

that is the number series on the right-hand side converges to the number (f, v).

Indeed,

$$\left|(f, v) - \sum_{0}^{N} (u_k, v)\right| = \left|\left(f - \sum_{0}^{N} u_k, v\right)\right| \leq \left\| f - \sum_{0}^{N} u_k \right\| \|v\| \to 0 \qquad (N \to \infty)$$

Corollary. *If a series*

$$f = \sum_{1}^{\infty} \alpha_k \varphi_k \tag{11}$$

where α_k are some numbers and $\varphi_1, \varphi_2, \dots$ is an orthonormal system is convergent with respect to the metric of H to an element $f \in H$ then the numbers α_s are necessarily the Fourier coefficients of f, that is

$$\alpha_s = (f, \varphi_s) \qquad (s = 1, 2, \dots) \tag{12}$$

which shows that the expansion of f into a series of type (11) *is unique.*

Indeed, on multiplying scalarly both members of equality (11) by φ_s $(s = 1, 2, \dots)$ we obtain, on the basis of Lemma 1, relations (12).

Theorem 2. *If orthonormal system* (1) *is complete in H then for any two elements f, $\varphi \in H$ there holds the numerical equality*

$$(f, \varphi) = \sum_{k=1}^{\infty} (f, \varphi_k)\overline{(\varphi, \varphi_k)} \tag{13}$$

indicating that the number series on the right-hand side converges to the number (f, φ).

Indeed, from the completeness of system (1) and from Theorem 1 it follows that the series

$$f = \sum_{1}^{\infty} (f, \varphi_k)\varphi_k \tag{14}$$

is convergent to f with respect to the metric of the space H. Now to obtain (13) from (14) we simply multiply scalarly all the terms on the right-hand and left-hand sides of (14) by φ, which is legitimate by virtue of Lemma 1:

$$(f, \varphi) = \sum_1^\infty (f, \varphi_k)(\varphi_k, \varphi) = \sum_1^\infty (f, \varphi_k)\overline{(\varphi, \varphi_k)}$$

We see that Parseval's relation is a special case of (13) for $f = \varphi \in H$.

Let us state the following definition. We say that orthonormal system (1) is *closed* if the fact that for an element $\psi \in H$ hold the equalities

$$(\psi, \varphi_k) = 0 \qquad (k = 1, 2, \ldots) \tag{15}$$

implies that ψ coincides with the zero element of $H: \psi = \theta$.

Since for a complete system there holds Parseval's relation we arrive at the following theorem.

Theorem 3. *A complete orthonormal system is closed.*

All the propositions proved in this section up till now apply both to a complete and an incomplete* space H. In particular, they hold for the space $L_2'(\Omega)$ which, as we know, is not complete.

Below we prove a number of propositions in which it is required that the space H in question should be complete.

Let us suppose that H is a complete infinite-dimensional linear space with scalar product, that is a *Hilbert space* (an instance of such a space is the space $L_2(\Omega)$).

Theorem 4. *A series of the form*

$$\sum_1^\infty \alpha_k \varphi_k$$

with respect to an orthonormal system φ_k $(k = 1, 2, \ldots)$ *where*

$$\sum_1^\infty |\alpha_k|^2 < \infty \tag{16}$$

is convergent relative to the metric of H to an element $\varphi \in H$.

Proof. Let

$$s_n = \sum_1^n \alpha_k \varphi_k \qquad (k = 1, 2, \ldots)$$

By the convergence of series (16), for any $\varepsilon > 0$ there is N such that for $n > N$ and any p we have

$$\varepsilon^2 > \sum_{n+1}^{n+p} |\alpha_k|^2 = \left\| \sum_{n+1}^{n+p} \alpha_k \varphi_k \right\|^2 = \| s_{n+p} - s_n \|^2$$

---

* One should distinguish between the notion of a *complete system in H* and that of a *complete space H*. For example, a system φ_k $(k = 1, 2, \ldots)$ can be complete even in an incomplete space H.

This shows that the sequence of the elements $s_n \in H$ satisfies the condition of Cauchy's criterion and therefore, since the space H is complete, there exists an element $\varphi \in H$ to which this sequence converges (in the metric of H), which proves the theorem.

Theorem 5. *The Fourier series*

$$\sum_{1}^{\infty}(f, \varphi_k)\varphi_k \tag{17}$$

of an arbitrary element $f \in H$ is convergent (with respect to the metric of H) to an element $\varphi \in H$, and the element $f-\varphi$ is orthogonal to all φ_k's:

$$(f-\varphi, \varphi_k) = 0 \qquad (k = 1, 2, \dots)$$

Proof. According to Bessel's inequality, the series

$$\sum_{1}^{\infty}|(f, \varphi_k)|^2 \leq (f, f)$$

is convergent. Therefore, by virtue of the foregoing theorem, series (17) converges to an element $\varphi \in H$:

$$\varphi = \sum_{1}^{\infty}(f, \varphi_k)\varphi_k$$

Thus we have

$$f-\varphi = f-\sum_{1}^{\infty}(f, \varphi_k)\varphi_k$$

where the series on the right-hand side is convergent in the metric of H. On multiplying scalarly all the terms of the last equality by φ_s $(s = 1, 2, \dots)$ we obtain

$$(f-\varphi, \varphi_s) = (f, \varphi_s)-(f, \varphi_s) = 0 \qquad (s = 1, 2, \dots)$$

whence follows the desired assertion.

Let us prove the converse of Theorem 3 (on condition that the space H in question is complete).

Theorem 6. *A closed orthonormal system* (1) *in a complete space H is complete.*

Proof. Suppose that system (1) is closed but not complete. Then, by Theorem 1, there is an element $f \in H$ such that its Fourier series does not converge to it. However, as was proved above, this series converges to an element $\varphi \in H$ and the element $f-\varphi$ is orthogonal to all φ_k $(k = 1, 2, \dots)$. By virtue of the closedness of system (1), this implies that $f-\varphi = \theta$, that is $f = \varphi$, and we have thus arrived at a contradiction.

Example 1. l_2 is the set of all (complex or real) number sequences

$$a = (\alpha_1, \alpha_2 \dots)$$

for which the norm

$$\|\boldsymbol{\alpha}\| = \left(\sum_1^\infty |\alpha_k|^2\right)^{1/2} \tag{18}$$

is finite.

If $\boldsymbol{\alpha} \in l_2$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots) \in l_2$ then for any natural n we have (see § 6.2 (6))

$$\sum_1^n |\alpha_j \bar{\beta}_j| \leqslant \left(\sum_{j=1}^n |\alpha_j|^2\right)^{1/2} \left(\sum_1^n |\beta_j|^2\right)^{1/2} \leqslant \|\boldsymbol{\alpha}\|\,\|\boldsymbol{\beta}\|$$

and therefore the series $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_1^\infty \alpha_j \bar{\beta}_j$ is absolutely convergent.

It can readily be verified that $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ possesses properties (i), (ii) and (iii) of a scalar product enumerated in § 6.2; this scalar product generates norm (18), and the corresponding zero element is $\theta = (0, 0, 0, \ldots)$. Consequently *$l_2$ is a linear space with scalar product. Moreover, it is in fact a complete infinite-dimensional space, that is a Hilbert space.* Indeed, let there be given a sequence of elements $\boldsymbol{\alpha}^k = (\alpha_1^k, \alpha_2^k \ldots) \in l_2$ $(k = 1, 2, \ldots)$ satisfying Cauchy's condition, that is let for any $\varepsilon > 0$ there exist N such that

$$\varepsilon > \|\bar{\alpha}^k - \alpha^{k'}\| \geqslant |\alpha_j^k - \alpha_j^{k'}| \qquad (k, k' > N)$$

It follows that for any fixed j the sequence of the numbers α_j^k $(k = 1, 2, \ldots)$ tends to a limit as $k \to \infty$: $\alpha_j^k \to \alpha_j$ $(k \to \infty)$. Let us form the number sequence $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots)$. It belongs to l_2 since

$$\varepsilon \geqslant \left(\sum_{j=1}^n |\alpha_j^k - \alpha_j|^2\right)^{1/2} \qquad (k > N)$$

for any natural number n. Therefore

$$\varepsilon \geqslant \left(\sum_{j=1}^\infty |\alpha_j^k - \alpha_j|^2\right)^{1/2} = \|\boldsymbol{\alpha}^k - \alpha\| \qquad (k > N) \tag{19}$$

Thus, $\boldsymbol{\alpha}^k - \alpha \in l_2$. But $\boldsymbol{\alpha}^k \in l_2$ and consequently $\boldsymbol{\alpha} \in l_2$. Finally, inequality (19) shows that the sequence of the elements $\boldsymbol{\alpha}^1, \boldsymbol{\alpha}^2, \boldsymbol{\alpha}^3, \ldots$ converges to $\boldsymbol{\alpha} \in l_2$ with respect to the metric of l_2, which proves the completeness of l_2.

Let us consider the (countable) set of the elements

$$\boldsymbol{e}^k = (0, 0, \ldots, 0, 1, 0, 0, \ldots) \qquad (k = 1, 2, \ldots)$$

of the space l_2; the kth component of $\boldsymbol{e}^k$ $(k = 1, 2, \ldots)$ is equal to 1 while all the other are equal to zero. These elements form an orthonormal (and, consequently, a linearly independent) system:

$$(\boldsymbol{e}^k, \boldsymbol{e}^l) = \delta_{kl} \qquad (k, l = 1, 2, \ldots)$$

For an arbitrary element $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ of the space l_2 we obviously have

$$a_k = (\boldsymbol{\alpha}, \boldsymbol{e}^k) \qquad (k = 1, 2, \ldots)$$

and

$$a = \sum_{k=1}^{\infty} (a, e^k)e^k$$

where the series on the right-hand side is convergent to a relative to the metric of l_2 since

$$\left\| a - \sum_{k=1}^{N} (a, e^k)e^k \right\| = \|(0, 0, \ldots, 0, \alpha_{N+1}, \alpha_{N+2}, \ldots)\| =$$

$$= \left(\sum_{N+1}^{\infty} |\alpha_k|^2 \right) \to 0 \qquad (N \to \infty)$$

We see that an arbitrary element $a \in l_2$ can be expanded into its Fourier series with respect to the orthonormal system $\{e^k\}$ (that is the series converges to that element). Thus, *the system* $\{e^k\}$ *is complete in* l_2.

Theorem 7. *Let*

$$\varphi_1, \varphi_2, \varphi_3, \ldots \tag{20}$$

be an (infinite) complete orthonormal system of elements in a linear space H with scalar product and let every element $f \in H$ be expanded into its Fourier series with respect to that system

$$f = \sum_{1}^{\infty} \alpha_k \varphi_k \tag{21}$$

with the Fourier coefficients

$$\alpha_k = (f, \varphi_k) \qquad (k = 1, 2, \ldots) \tag{22}$$

convergent to f relative to the metric of H (see Theorem 1). Then if the space H is complete equality (21) *sets up a one-to-one correspondence $f \sim a = (\alpha_1, \alpha_2, \ldots)$ between the elements of H and the elements of l_2 isomorphic with respect to the operations of addition of elements, multiplication of elements by numbers and scalar multiplication of elements, that is if $f \sim a$ and $\varphi \sim \beta$ then $f + \varphi \sim a + \beta$, $cf \sim ca$ and*

$$(f, \varphi)_H = \sum_{1}^{\infty} \alpha_j \bar{\beta}_j = (a, \beta)_{l_2} \tag{23}$$

If the space H is not complete, equality (21) *specifies a (linear and isometric) one-to-one correspondence between the elements of H and the elements of l_2' where l_2' is an incomplete linear subspace of l_2 which however is such that the closure of l_2' coincides with* l_2 (i.e. $\bar{l}_2' = l_2$).

Proof. Equality (21) determines an operator A which associates with every element $f \in H$ the corresponding number sequence $a = (\alpha_1, \alpha_2, \ldots)$. By the completeness of system (20), Parseval's relation holds:

$$\|f\|_H = \|Af\|_{l_2} \qquad (Af = a \in l_2) \tag{24}$$

The operator A is obviously linear:

$$A(cf) = cAf \quad \text{and} \quad A(f+\varphi) = Af+A\varphi$$

for any number c and any elements $f, \varphi \in H$. Moreover, by virtue of Theorem 2, there holds equality (23), more general than (24).

With two different elements f' and f'' of H the operator A associates two different elements $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ belonging to l_2 because the equality $Af' = Af'' = \boldsymbol{\alpha}$ implies $A(f'-f'') = \boldsymbol{\theta}$ whence follows that the Fourier series of $f'-f''$ with respect to system (20) is of the form

$$f'-f'' = 0\cdot\varphi_1+0\cdot\varphi_2+ \dots$$

By the completeness of system (20), this series must be convergent with respect to the metric of the space H to the element $f'-f''$, which means that $f'-f'' = \theta$, that is $f' = f''$.

Now let us assume that the space H is complete. For an arbitrary element $\boldsymbol{\alpha} = (\alpha_1, \alpha_2 \dots) \in l_2$ we can take the formal series

$$\sum_1^\infty \alpha_k\varphi_k \tag{25}$$

By the convergence of the series $\sum\limits_1^\infty |\alpha_k^2| < \infty$ and by the completeness of H (Theorem 4), there is a (uniquely determined) element $\varphi \in H$ to which series (25) is convergent:

$$\varphi = \sum_1^\infty \alpha_k\varphi_k \tag{26}$$

Series (26) is the Fourier series of φ (see the corollary of Lemma 1).

We have shown that for any element $\boldsymbol{\alpha} \in l_2$ there is an element $\varphi \in H$ such that $A\varphi = \boldsymbol{\alpha}$. Combining this property with the properties of the operator A established above we can assert that if H is complete then the operator A establishes a one-to-one correspondence $H \rightleftharpoons l_2$ isomorphic with respect to the operations of addition. multiplication by a number and scalar multiplication. The first part of Theorem 7 has thus been proved.

Now we suppose that the space H is incomplete. Let us denote by l_2' the image of H under the operator A ($l_2' = AH$). According to what was proved above, the operator A specifies a one-to-one correspondence $H \rightleftharpoons l_2'$ isomorphic with respect to the operations of addition, multiplication by a number and scalar multiplication.

The space H contains a sequence of elements $f^1, f^2, f^3, \dots$ satisfying the condition of Cauchy's criterion which is not convergent to any element of H. For this system we have the inequality

$$\varepsilon > ||f^k-f^l||_H = ||\boldsymbol{\alpha}^k-\boldsymbol{\alpha}^l||_{l_2} \qquad (\boldsymbol{\alpha}^k = Af^k)$$

for all k, $l > N$ and arbitrary $\varepsilon > 0$ provided that N is sufficiently large. This inequality shows that the images $\boldsymbol{\alpha}^k = Af^k$ satisfy the condition of

Cauchy's criterion relative to the metric of the space l_2 of number sequences. Since the space l_2 is complete there is an element $\boldsymbol{a} \in l_2$ such that $\|\boldsymbol{a}-\boldsymbol{a}^k\|_{l_2} \to 0$, $k \to \infty$. However, among the elements belonging to H there is no element f for which $Af = \boldsymbol{a}$ because if such an element existed we would have

$$\|\boldsymbol{a}-\boldsymbol{a}^k\|_{l_2} = \|f-f^k\|_H \to 0 \qquad (k \to \infty)$$

which would contradict the hypothesis.

What has been shown indicates that l_2' is not a complete space. But the closure of l_2' coincides with l_2 (i.e. $\bar{l}_2' = l_2$) because, for any element $\boldsymbol{a} = (\alpha_1, \alpha_2, \ldots) \in l_2$, the elements $\boldsymbol{a}^N = (\alpha_1, \alpha_2, \ldots, \alpha_N, 0, 0, \ldots)$ belong to l_2' for any N and at the same time $\|\boldsymbol{a}-\boldsymbol{a}^N\|_{l_2} \to 0$ as $N \to \infty$ $\left(\boldsymbol{a}^N \in l_2'\right.$ because the sums $\sum_1^N \alpha_k\varphi_k$ belong to H since $\varphi_k \in H$ and H is a linear space$\left.\right)$.

We have thus proved the second part of the theorem.

Example 2. Let $\Delta_1, \Delta_2, \Delta_3, \ldots$ be cubes belonging to Ω which can only intersect along some parts of their boundaries and let

$$\varphi_k(x) = \frac{1}{|\Delta_k|^{1/2}}\varphi_{\Delta_k}(x) \qquad (k = 1, 2, \ldots) \tag{27}$$

where

$$\varphi_{\Delta_k}(x) = \begin{cases} 1 & \text{for } x \in \Delta_k \\ 0 & \text{for } x \notin \Delta_k \end{cases}$$

System (27) is apparently orthonormal but incomplete in $L_2'(\Omega)$ (and in $L_2(\Omega)$!) because, for instance, the Fourier series of the function $f(x)$ determined by the equalities

$$f(x) = \begin{cases} \psi(x) & \text{on } \Delta_1 \\ 0 & \text{outside } \Delta_1 \end{cases}$$

where $\psi(x)$ is a continuous function (which is not identically equal to any constant) has the form

$$f(x) \sim \frac{1}{|\Delta_1|^{1/2}} \int_{\Delta_1} \psi(x)\, dx\varphi_{\Delta_1}(x)+0+0+\ldots \tag{28}$$

and the series on the right-hand side of (28) is not convergent in the mean square to the left-hand member.

§ 14.7. Orthogonalization Process

Theorem 1. *Let*

$$\psi_1, \psi_2, \psi_3, \ldots \tag{1}$$

be a linearly independent system of elements belonging to a real linear space H with scalar product. Then there is an orthonormal system of elements

$$\varphi_1, \varphi_2, \varphi_3, \ldots \tag{2}$$

(determined uniquely to within the signs of its members) belonging to H and possessing the following properties:

For any natural k there holds the relation

$$\psi_k = \sum_{j=1}^{k} \alpha_j^{(k)}\varphi_j \qquad (\alpha_k^{(k)} \neq 0) \tag{3}$$

and, conversely,

$$\varphi_k = \sum_{j=1}^{k} \beta_j^{(k)}\psi_j \qquad (\beta_k^{(k)} \neq 0) \tag{4}$$

where $\alpha_j^{(k)}$ and $\beta_j^{(k)}$ are (real) numbers.

If system (1) *is finite and consists of n elements then so is the orthonormal system* (2).

The statement "determined uniquely to within the signs of the members" should be understood in the sense that if there is a system of type (2) satisfying the conditions of the theorem, then the multiplication of all φ_k $(k = 1, 2, \ldots)$ by $\delta_k = \pm 1$ results in a system which again satisfies the conditions of the theorem and that there are no other systems satisfying these conditions.

Proof. By the hypothesis, the element ψ_1 can be regarded as a linearly independent system (consisting of that single element), and therefore

$$||\psi_1|| = (\psi_1, \psi_1)^{1/2} > 0$$

Since there must be $\varphi_1 = \beta_1^{(1)}\psi_1$ and $||\varphi_1|| = 1$, it follows that $\beta_1^{(1)} = \pm\frac{1}{||\psi_1||} \neq$ $\neq 0$. Consequently we also have $\psi_1 = \alpha_1^{(1)}\varphi_1$ where $\alpha_1^{(1)} = \pm||\psi_1|| \neq 0$. This proves the desired assertion for $k = 1$.

Now let us suppose that it is possible to construct an orthonormal system of elements $\varphi_1, \ldots, \varphi_k$ determined uniquely to within the signs of its members so that equalities (3) and (4) are fulfilled. We shall show that this system can be completed with an element φ_{k+1} determined uniquely to within its sign so that the resultant system $\varphi_1, \ldots, \varphi_{k+1}$ is orthonormal and satisfies conditions (3) and (4) in which k is replaced by $k+1$.

The sought-for element must have the form

$$\varphi_{k+1} = \sum_{j=1}^{k+1} \beta_j^{(k+1)}\psi_j = \beta_{k+1}^{(k+1)}\psi_{k+1} + \sum_{j=1}^{k} \gamma_j\varphi_j \tag{5}$$

In the second equality (5) we have replaced the elements $\psi_1, \ldots, \psi_k$ by the corresponding (equal to ψ_l's; $l = 1, \ldots, k$) linear combinations of the elements φ_j with the indices $j \leqslant k$ and then combined the similar terms containing φ_j; $j = 1, \ldots, k$. This is possible because we assumed that our assertion holds for k. The element φ_{k+1} should be constructed so that it is orthogonal to all φ_s $(s = 1, \ldots, k)$; therefore we must have

$$(\varphi_{k+1}, \varphi_s) = \beta_{k+1}^{(k+1)}(\psi_{k+1}, \varphi_s) + \gamma_s = 0 \qquad (s = 1, \ldots, k)$$

On substituting γ_s into (5) we obtain

$$\varphi_{k+1}=\beta_{k+1}^{(k+1)}\left[\psi_{k+1}-\sum_{j=1}^{k}(\psi_{k+1},\varphi_j)\varphi_j\right]$$

The element

$$\psi_{k+1}^{*}=\psi_{k+1}-\sum_{j=1}^{k}(\psi_{k+1},\varphi_j)\varphi_j$$

cannot coincide with the zero element because, if otherwise, the element ψ_{k+1} would be a linear combination of the elements φ_j ($j=1,\ldots,k$) and therefore, by what has already been proved for k, the element ψ_{k+1} would be a linear combination of the elements ψ_j ($j=1,\ldots,k$), which would contradict the linear independence of the system $\psi_1,\ldots,\psi_{k+1}$.

Thus

$$\|\psi_{k+1}^{*}\|>0$$

This allows us to choose the number $\beta_{k+1}^{(k+1)}$ so that the requirement $\|\varphi_{k+1}\|=1$ is satisfied, which implies the formula

$$\beta_{k+1}^{(k+1)}=\pm\frac{1}{\|\psi_{k+1}^{*}\|}$$

determining the coefficient $\beta_{k+1}^{(k+1)}$ to within its sign.

The theorem has been proved.

The construction with the aid of which we have determined orthonormal system (2) (equivalent in the indicated sense to the given linearly independent system (1)) is called the *orthogonalization process*.

Theorem 2. *Two systems of elements*

$$\varphi_1,\varphi_2,\varphi_3,\ldots \tag{6}$$

$$\psi_1,\psi_2,\psi_3,\ldots \tag{7}$$

of H connected for every $k=1,2,\ldots$ by relations (3) *and* (4) *are simultaneously complete or incomplete in H.*

In this theorem H can be an arbitrary normed linear space in which scalar product may not be defined.

Indeed, suppose that system (6) is complete in the space H, and let f be an arbitrary element belonging to H. Then for any $\varepsilon>0$ there is a sum of the form

$$\sum_{1}^{N}\alpha_k\varphi_k \tag{8}$$

where α_k are some numbers such that

$$\varepsilon>\left\|f-\sum_{1}^{N}\alpha_k\varphi_k\right\|$$

Now, since, by equalities (4), expression (8) is a sum of the form

$$\sum_{1}^{N} \beta_k \psi_k$$

where β_k are some numbers, we see that system (7) is complete in H.

The fact that the completeness of system (7) implies that of system (6) is proved in like manner by using equality (3).

Theorem 3. *Let H be a linear space with scalar product possessing the following property: it contains a linearly independent system*

$$\psi_1, \psi_2, \ldots, \psi_n \tag{9}$$

such that for any element $f \in H$ and any positive number ε there are numbers $\alpha_1, \ldots, \alpha_n$ (dependent on ε) such that

$$\left\| f - \sum_{1}^{n} \alpha_k \psi_k \right\| < \varepsilon \tag{10}$$

Then there are also numbers $\beta_1, \ldots, \beta_n$ such that

$$f = \sum_{1}^{n} \beta_k \psi_k \tag{11}$$

that is H is an n-dimensional space.

Proof. On orthogonalizing system (9) we arrive at an orthonormal system

$$\varphi_1, \ldots, \varphi_n \tag{12}$$

It is evident that for any element $f \in H$ and any number $\varepsilon > 0$ there are numbers $a_1, \ldots, a_n$ such that

$$\varepsilon > \left\| f - \sum_{1}^{n} a_k \varphi_k \right\| \geqslant \left\| f - \sum_{1}^{n} (f, \varphi_k) \varphi_k \right\| \tag{13}$$

where the second inequality follows from the minimizing property of an orthonormal system (see § 14.6 (6)). Since the third term in (13) is independent of the number ε which is arbitrary, it must be equal to zero, that is

$$f = \sum_{1}^{n} (f, \varphi_k) \varphi_k \tag{14}$$

To obtain (11) it only remains to replace φ_k in (14) by the corresponding linear combinations of ψ_k.

We have thus proved Lemma 1 of § 14.5 on condition that E is a linear space with scalar product.

§ 14.8. Properties of Spaces $L_2'(\Omega)$ and $L_2(\Omega)$

We defined $L_2'(\Omega)$ as the space of the functions $f(x)$ such that their integrals $\int\limits_{\Omega} f(x)\,dx$ have finite numbers of singularities (provided there are such) and their norms $\|f\|_{L_2}$ are finite:

$$\|f\|_{L_2} = \Big(\int\limits_{\Omega} |f(x)|^2\,dx\Big)^{1/2} < \infty \tag{1}$$

Since the integral in this definition is understood in the Riemann (generally, improper) sense, the space $L_2'(\Omega)$ is not complete (see § 19.7). However, the space $L_2'(\Omega)$ possesses many properties of the (complete) Hilbert space $L_2(\Omega)$ whose definition uses the notion of the Lebesgue integral. We shall enumerate some basic properties of this kind although in the foregoing sections we already mentioned almost all of them.

(i) The scalar product

$$(f, \varphi) = \int\limits_{\Omega} f\bar{\varphi}\,dx$$

generating norm (1) makes sense for any two functions $f, \varphi \in L_2'(\Omega)$.

(ii) In $L_2'(\Omega)$ there is a countable system of functions

$$\lambda_1(x),\ \lambda_2(x),\ \lambda_3(x),\ \ldots \tag{2}$$

(these functions can be constructed so that they are finitely-valued and are determined by rational values of the parameters) dense in $L_2'(\Omega)$ $(L_2(\Omega))$. (See Theorem 5, § 14.4.)

(iii) $L_2'(\Omega)$ is an infinite-dimensional space; it contains an infinite system of linearly independent functions (for instance, the characteristic functions of the cubes $\Delta \subset \Omega$ constructed in § 14.6, Example 2 form such a system).

(iv) By properties (ii) and (iii), the space $L_2'(\Omega)$ is separable. The separability of the space $L_2'(\Omega)$ implies that system (2) (which is dense in $L_2'(\Omega)$) contains a subsystem

$$\psi_1(x),\ \psi_2(x),\ \psi_3(x),\ \ldots \tag{3}$$

obtained by deleting some of the elements of (2), which is linearly independent and complete in $L_2'(\Omega)$ (see the proof of Theorem 2, § 14.5).

(v) Complete linearly independent system (3) can be orthogonalized to obtain a countable orthonormal system of functions

$$\varphi_1(x),\ \varphi_2(x),\ \varphi_3(x),\ \ldots \tag{4}$$

which is also complete in $L_2'(\Omega)$ (see Theorems 1 and 2 in § 14.7). This means that in $L_2'(\Omega)$ there is a complete orthonormal system of functions. There are in fact an infinite number of such systems, which is analogous to the case of the three-dimensional Euclidean space containing an infinite number of triples of pairwise perpendicular unit vectors. In what follows we shall deal with some important systems of this kind.

(vi) Every function $f \in L'_2(\Omega)$ can be expanded into the Fourier series with respect to orthonormal system (4):

$$f(x) = \sum_1^\infty \alpha_k \varphi_k(x)$$

where, by the completeness of the system, the series on the right-hand side converges to $f(x)$ in the mean square. The numbers

$$\alpha_k = (f, \varphi_k); \qquad k = 1, 2, \ldots \tag{5}$$

(the Fourier coefficients of f) satisfy Parseval's relation

$$(f, f) = \sum_1^\infty |\alpha_k|^2 < \infty \tag{6}$$

(Theorem 1, § 14.6).

Equalities (5) set up a one-to-one correspondence

$$L'_2(\Omega) \rightleftarrows l'_2 \tag{7}$$

between the functions $f \in L'_2(\Omega)$ and the sequences of numbers $\alpha = (\alpha_1, \alpha_2, \ldots) \in l'_2 \subset l_2$ where l'_2 is an incomplete linear subspace of the space l_2. Correspondence (7) is an isomorphism with respect to the operations of addition, multiplication by numbers and scalar multiplication (Theorem 7, § 14.6).

By virtue of isomorphism (7), the subspace $L'_2(\Omega)$ corresponding to the incomplete space $l'_2 \subset l_2$ is also incomplete. However, the closure of l'_2 coincides with l_2: $\bar{l}'_2 = l_2$.

Now let us point out some properties of the space $L_2(\Omega)$ of the square integrable functions for which the squares of their moduli are integrable on Ω in the Lebesgue sense.

As was indicated above, $L_2(\Omega)$ is a complete linear space with scalar product. The finitely-valued functions with rational parameters (see § 14.4) form a dense set in $L_2(\Omega)$, which is analogous to the corresponding property of the space $L'_2(\Omega)$ in which these functions also form a dense set. It follows that orthonormal system (4) is complete not only in $L'_2(\Omega)$ but also in $L_2(\Omega)$.

Now, by virtue of Theorem 7, § 14.6, we can assert that equalities (5) establish an isomorphic (with respect to the operations of addition, multiplication by numbers and scalar multiplication) one-to-one correspondence

$$L_2(\Omega) \rightleftarrows l_2 \tag{8}$$

between the elements of $L_2(\Omega)$ and all the elements of l_2. Further, the closure of $L'_2(\Omega)$ in the metric of $L_2(\Omega)$ coincides with $L_2(\Omega)$, and therefore to an arbitrary function $f \in L_2(\Omega)$ there corresponds, according to isomorphism (8), an element $\alpha = (\alpha_1, \alpha_2, \ldots) \in l_2$ and

$$\int_\Omega \left| f(x) - \sum_1^N \alpha_k \varphi_k(x) \right|^2 dx = \sum_{N+1}^\infty |\alpha_k|^2 \to 0 \qquad (N \to \infty) \tag{9}$$

where the sums $\sum_1^N \alpha_k\varphi_k$ belong to $L'_2(\Omega)$ and integral (9) is understood in Lebesgue's sense.

§ 14.9. Complete Systems of Functions in the Spaces C, L'_2 and L' (L_2, L)

Theorem. *Let Ω be a (bounded) measurable open set.*

(i) *If a system of function*

$$\varphi_1, \varphi_2, \varphi_3, \ldots$$

is complete in $C(\bar{\Omega})$ then it is also complete in $L'_2(\Omega)$.

(ii) *If it is complete in $L'_2(\Omega)$ it is also complete in $L'(\Omega)$.*

Proof. We have the obvious inequalities

$$\left(\int_\Omega \left|f(x)-\sum_1^N \alpha_k\varphi_k(x)\right|^2 dx\right)^{1/2} \leqslant \sqrt{\Omega}\max_x \left|f(x)-\sum_1^N \alpha_k\varphi_k(x)\right| \tag{1}$$

and

$$\int_\Omega \left|f-\sum_1^N \alpha_k\varphi_k\right| dx \leqslant \sqrt{\Omega}\left(\int_\Omega \left|f-\sum_1^N \alpha_k\varphi_k\right|^2 dx\right)^{1/2} \tag{2}$$

(see § 14.2 (13)). The first of them holds under the assumption that $\varphi_k, f \in C(\Omega)$ and the second under the assumption that $\varphi_k, f \in L'_2(\Omega)$ (L_2).

If the system φ_k $(k = 1, 2, \ldots)$ is complete in $C(\Omega)$ (in $L'_2(\Omega)$ or in $L_2(\Omega)$) then there is a finite sum $\sum_1^N \alpha_k\varphi_k$ for which the right-hand member of (1) (of (2)) is less than ε, whence it follows that the left-hand member is also less than ε.

Exercise

Prove the following more general proposition: if system (1) is complete in $C(\bar{\Omega})$ then it is also complete in $L'_p(\Omega)$ $(1 \leqslant p < \infty)$; if it is complete in $L'_{p'}(\Omega)$ where $1 \leqslant p < p' < \infty$ then it is also complete in $L'_p(\Omega)$ where Ω is a (bounded) measurable set.

CHAPTER 15

Fourier Series. Approximation of Functions with Polynomials

§ 15.1. Preliminaries

The system of trigonometric functions

$$\frac{1}{2}\cos x, \sin x, \cos 2x, \sin 3x, \dots \tag{1}$$

s orthogonal on the closed interval $[0, 2\pi]$, that is the integral of the product of any two different functions belonging to this system taken over that interval is equal to zero. This follows from the equalities

$$\int_0^{2\pi} \cos kx \cos lx \, dx = 0 \qquad (k \neq l, k, l = 0, 1, \dots)$$

$$\int_0^{2\pi} \sin kx \sin lx \, dx = 0 \qquad (k \neq l; k, l = 1, 2, \dots)$$

$$\int_0^{2\pi} \cos kx \sin lx \, dx = 0 \qquad (k = 0, 1, \dots, l = 0, 1, \dots)$$

$$\int_0^{2\pi} \cos^2 kx \, dx = \int_0^{2\pi} \sin^2 kx \, dx = \pi \qquad (k = 1, 2, \dots)$$

$$\int_0^{2\pi} \left(\frac{1}{2}\right) dx = \frac{\pi}{2}$$

This chapter is devoted to the theory of trigonometric series and to the approximation of functions with trigonometric polynomials.

A function $f(x)$ is said to be *periodic with period* $2\omega > 0$ if it is defined throughout the real axis and satisfies the condition

$$f(x+2\omega) = f(x)$$

for every x. If for such a function the (proper or improper) integral

$$\int_0^{2\omega} f(x)\,dx$$

exists, then the equality

$$\int_0^{a+2\omega} f(x)\,dx = \int_0^{2\omega} f(x)\,dx \tag{2}$$

is fulfilled for any real number a. This is readily seen from Fig. 15.1 where the similarly shaded areas are obviously equal. It is also possible to prove this property purely formally. Indeed, for any given a there exists a uniquely determined natural number k such that $2\,k\omega \leqslant a < 2(k+1)\omega$ and, obviously,

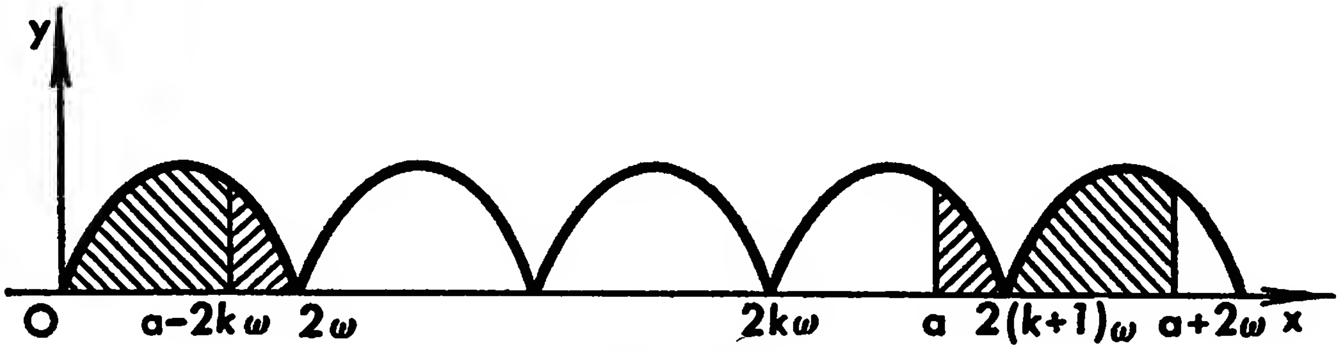

Fig. 15.1

$$\int_a^{2(k+1)\omega} f(x)\,dx = \int_a^{2(k+1)\omega} f(x-2k\omega)\,dx = \int_{a-2k\omega}^{2\omega} f(z)\,dz$$

and

$$\int_{2(k+1)\omega}^{a+2\omega} f(x)\,dx = \int_{2(k+1)\omega}^{a+2\omega} f(x-2(k+1)\omega)\,dx = \int_0^{a-2k\omega} f(z)\,dz$$

On adding together these equalities we arrive at (2).

In what follows, in the case of periodic functions with period 2ω, we shall often use the equality

$$\int_0^{2\omega} f(t-x)\,dt = \int_0^{2\omega} f(t)\,dt$$

in which x is an arbitrary number. This relation follows from (2):

$$\int_0^{2\omega} f(t-x)\,dt = \int_{-x}^{2\omega-x} f(z)\,dz = \int_0^{2\omega} f(t)\,dt$$

The functions forming system (1) are periodic with period 2π. The functions 1, $\cos x$, $\cos 2x$, ... are even while the functions $\sin x$, $\sin 2x$, ... are odd.

For even functions $f(x)$ we have

$$\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx$$

and for odd functions

$$\int_{-b}^{b} f(x)\,dx = 0$$

A sum of the form

$$T_n(x) = \frac{a_0}{2}+\sum_{1}^{n}(a_k\cos kx+b_k\sin kx)$$

where a_k and b_k are constant numbers is termed a *trigonometric polynomial of order n*.

The trigonometric polynomials will be regarded as the simplest periodic functions with period 2π. They will be used for the approximation of more general functions with period 2π.

Given a function $f(x)$ periodic with period 2ω, we can pass to the function $F(u) = f\left(\frac{u\omega}{\pi}\right)$ with the aid of the substitution $x = \frac{u\omega}{\pi}$, the resultant function $F(u)$ being periodic with period 2π. On approximating the latter function with a trigonometric polynomial $T_n(u)$ $\big(F(u) \sim T_n(u)\big)$ we can then return to the variable x to obtain the approximation

$$f(x) \sim T_n\left(\frac{\pi}{\omega}x\right)$$

for the original function.

We shall use the following terminology and notation.

The symbol $C(a, b)$ (see § 14.1) will denote the space (class) of all continuous functions f defined on the closed interval $[a, b]$ equipped with the norm

$$\|f\|_{C(a,\,b)} = \max_{a\leqslant x\leqslant b}|f(x)|$$

By C^* we shall denote the space (class) of the functions f defined and continuous throughout the real axis, having period 2π and equipped with the norm

$$\|f\|_{C^*} = \max_{0\leqslant x\leqslant 2\pi}|f(x)| = \max_{a\leqslant x\leqslant a+2\pi}|f(x)|$$

where a is an arbitrary real number.

A function $f \in C^*$ can be regarded as belonging to $C(0, 2\pi)$ $\big(C^* \subset C(0, 2\pi)\big)$ if we consider it for the values of x ranging over the interval $[0, 2\pi]$. However, not every function belonging to $C(0, 2\pi)$ can be obtained in this way because a function $f \in C^*$ restricted to the interval $(0, 2\pi)$ necessarily has coinciding values at the end points of the period:

$$f(0) = f(2\pi) \tag{3}$$

Conversely, a function $f \in C(0, 2\pi)$ satisfying condition (3) can always be *extended periodically with the period* 2π to the whole real axis to become a member of the class C^*.

By L'^* we shall mean the space (class) of the periodic functions with period 2π which, when restricted to the interval $[0, 2\pi]$, belong to the space $L'(0, 2\pi)$ (see § 14.2) with the norm

$$||f||_{L^*} = ||f||_{L(0, 2\pi)} = \int_0^{2\pi} |f(x)| \, dx$$

We can also say that every function $f(x) \in L'^*$ is periodic (with period 2π) and absolutely integrable over the closed interval $[0, 2\pi]$. We remind the reader that a function f belongs to $L'(0, 2\pi)$ if its integral

$$\int_0^{2\pi} f(x) \, dx \tag{4}$$

exists in Riemann's sense or has a finite number of singularities and is absolutely convergent in the improper sense (see § 13.13).

$L_2'^*$ is the space (class) of the periodic functions f with period 2π which, when restricted to the closed interval $[0, 2\pi]$, belong to the space $L_2'(0, 2\pi)$ (see § 14.3) in which the norm is defined by the equality

$$||f||_{L_2^*} = \left(\int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2}$$

We can also say that every function $f(x) \in L_2'^*$ is periodic (with period 2π) and has the integrable (over the interval $[0, 2\pi]$) square of its modulus; in the case of a real function $f(x) \in L_2'(0, 2\pi)$ we can simply say that its square is integrable over $[0, 2\pi]$. We remind the reader that the space $L_2'(0, 2\pi)$ consists of the functions f such that they are Riemann integrable on $[0, 2\pi]$ or, if their integrals (4) have finite numbers of singularities, the squares of their moduli are absolutely integrable in the improper sense. It should also be stressed that $L_2'^* \subset L'^*$ (see § 14.2 (13)).

In the theory of Fourier's series it appears more natural to deal with the classes (spaces) L^* and L_2^* of periodic functions with period 2π belonging respectively to the spaces $L(0, 2\pi)$ and $L_2(0, 2\pi)$ constructed with the aid of Lebesgue's integral.

The asterisk in all these symbols indicates that the functions forming the corresponding classes are periodic (with period 2π).

A function f belonging to one of these classes depends on one variable x and can be real or complex; in the latter case it is representable as $f(x) = \varphi(x) + i\psi(x)$. Therefore we speak of the "integrability of the square of the modulus" of such a function, which is the same as the "integrability of the square of the function" only when $f(x)$ is real.

System of trigonometric functions (1) is orthogonal and, as will be seen later, is complete in $L_2'^*$ and L_2^* (and even in C^*). Therefore with every function $f \in L_2'^*$ (L_2^*) we can associate its Fourier series (see § 14.6 (2)) with respect to system (1):

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{5}$$

where

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos kt\, dt \qquad (k = 0, 1, \ldots) \tag{6}$$

and

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin kt\, dt \qquad (k = 1, 2, \ldots) \tag{7}$$

The expressions $a_0/2$ $(a_1 \cos x + b_1 \sin x)$, $(a_2 \cos 2x + b_2 \sin 2x)$, ... with coefficients a_k, b_k determined by formulas (6) and (7) entering into the right-hand side of (5) are the *terms of the Fourier series of the function* f (they are also called the *harmonics* of f).

The Fourier coefficients a_k and b_k (see (6) and (7)) make sense not only for the functions $f \in L_2'^*$ but also for the functions $f \in L'^*$ (and, generally, for the functions $f \in L^*$). Indeed, the functions $\cos kx$ and $\sin kx$ are bounded and continuous while every function $f \in L'^*$ is absolutely integrable, and therefore the integrals determining the Fourier coefficients of $f \in L'(0, 2\pi)$ are absolutely convergent:

$$\int_0^{2\pi} |f(x) \cos kx|\, dx \leqslant \int_0^{2\pi} |f(x)|\, dx$$

and

$$\int_0^{2\pi} |f(x) \sin kx|\, dx \leqslant \int_0^{2\pi} |f(x)|\, dx$$

Therefore, for the sake of generality, we shall, when possible, consider the expansions into the Fourier series of functions belonging to L'^* (L^*).

Thus, to every function $f \in L'^*$ (or, generally speaking, to $f \in L^*$) there corresponds its Fourier series irrespective of whether the latter is convergent or divergent at some points x. It should be noted that the addition to a function $f \in L'^*$ of the zero element of L'^* (L^*) which is representable by any function $\theta(x)$ satisfying the condition

$$\int_0^{2\pi} |\theta(x)|\, dx = 0$$

does not alter the Fourier coefficients of f, and hence the Fourier series of f also remains unchanged. For instance, this is the case when a function $f \in L'^*$ (L^*) is redefined at a finite number of points.

The set of the Fourier coefficients of a function is called the *spectrum* of that function. Many vibration processes (oscillations) studied in physics and technical sciences are described by periodic functions $F(u)$ with periods 2ω (which can be different from 2π in the general case). In such a description the variable $u = t$ is time and $y = F(u)$ is the ordinate of an oscillating point or the magnitude of a force or the velocity or the intensity of electric current and the like. If F is a trigonometric polynomial then

$$y = F(u) = \frac{a_0}{2}+\sum_1^n\left(a_k \cos\frac{k\pi}{\omega}u+b_k \sin\frac{k\pi}{\omega}u\right) =$$

$$= \frac{a_0}{2}+\sum_1^n A_k \cos\left(\frac{k\pi}{\omega}u-\varphi_k\right)$$

where $A_k = \sqrt{a_k^2+b_k^2}$ and φ_k $(k = 1, \ldots, n)$ are determined from the relations $a_k = \cos\varphi_k$, $b_k = \sin\varphi_k$ and $0 \leqslant \varphi_k < 2\pi$.

In physics it is said that *the oscillation process $y = F(u)$ decomposes into the simplest harmonic oscillations (harmonics)*

$$A_k \cos\left(\frac{k\pi}{\omega}u-\varphi_k\right) \tag{8}$$

Harmonics (8) have the *frequencies* $\frac{k\pi}{\omega}$, the *amplitudes* A_k and the *initial phases* φ_k. Shown in Fig. 15.2 are the graphs of the three periodic functions with period 2π: $S_2(x) = \sin x-\frac{\sin 2x}{2}$ (the continuous line), $S_3(x) = \sin x-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}$ (the short-dash line) and $S_4(x) = \sin x- \ldots -\frac{\sin 4x}{4}$ (the dotted line).

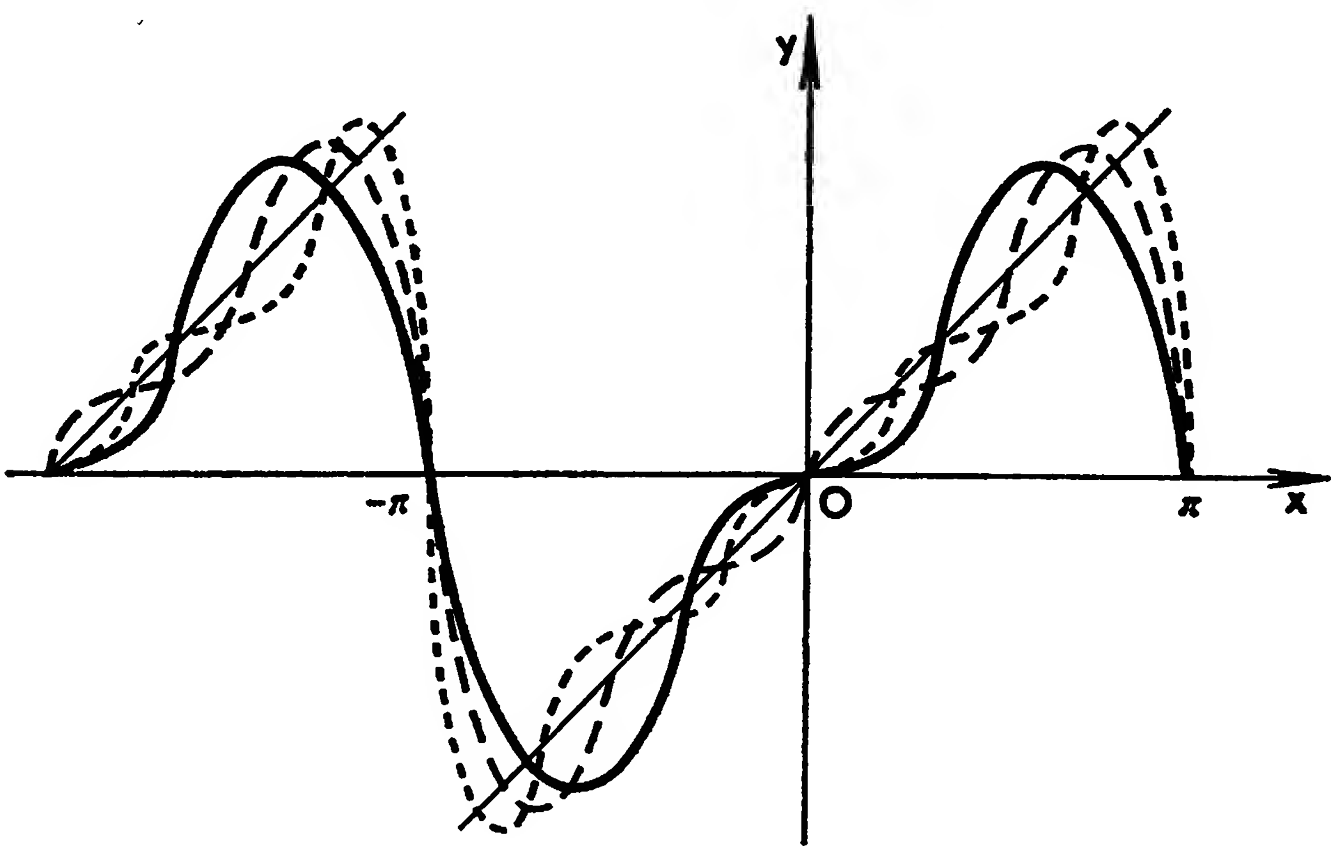

Fig. 15.2

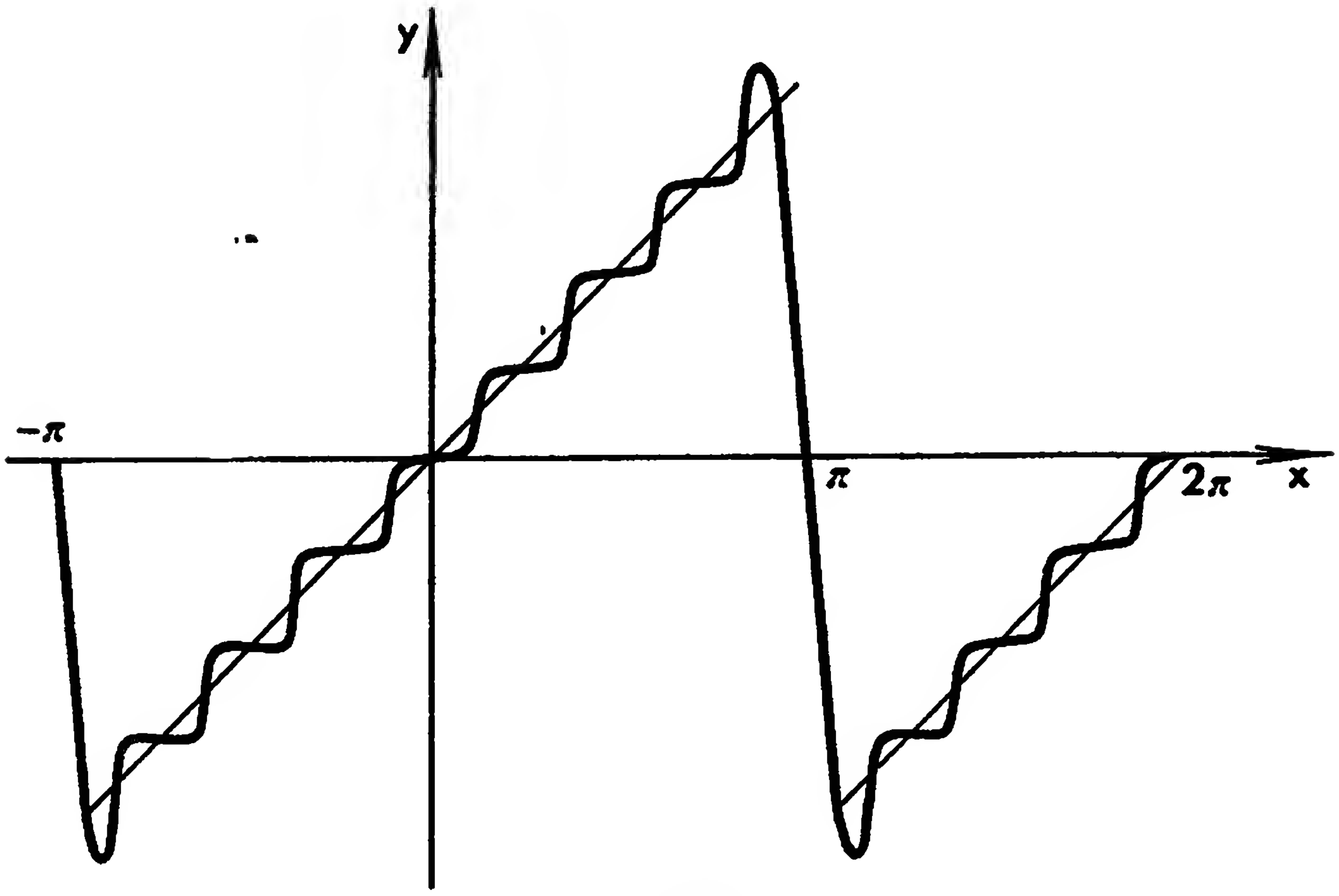

Fig. 15.3

In Fig. 15.3 we see a schematized graph of the sum

$$S_n(x) = \sum_1^n (-1)^{k-1} \frac{\sin kx}{k} \tag{9}$$

constructed for a large value of n. The figure suggests that for $n \to \infty$ the limiting function

$$S(x) = \lim_{n\to\infty} S_n(x) \tag{10}$$

is a periodic function (with period 2π) whose values assumed on the interval $-\pi < x < \pi$ are determined by the relations

$$S(x) = x \quad \text{for} \quad -\pi < x < \pi \quad \text{and} \quad S(\pi) = 0 \tag{11}$$

Since the function $S(x)$ is discontinuous at the points $x_k = (2k+1)\pi$ ($k = 1, 2, \ldots$) the sequence $\{S_n(x)\}$ of the continuous functions $S_n(x)$ ($n = 1, 2, \ldots$) cannot be uniformly convergent to $S(x)$ throughout the real line; however, it is uniformly convergent on any closed interval $[a, b]$ belonging to the interval $(-\pi, \pi)$ and, generally, on any interval of the x-axis on which $S(x)$ has a continuous derivative (see § 15.5).

As is seen in Fig. 15.3, in the vicinity of the points of discontinuity x_k of the limiting function $S(x)$ the graph of $S_n(x)$ (with a large n) has sharp "splashes". This is a characteristic feature of the points of discontinuity of the first kind of the piecewise continuous limiting function of a sequence of smooth functions known as *Gibbs'** *phenomenon* which will be studied in § 15.9.

---

* J. W. Gibbs (1839-1903), an American physicist.

§ 15.2. Dirichlet's Sum

Let there be given a function $f \in L'^*$ (or, more generally, $f \in L^*$) and let $\frac{a_0}{2}+\sum_1^\infty (a_k \cos kx+b_k \sin kx)$ be its Fourier series:

$$f(x) \sim \frac{a_0}{2}+\sum_1^\infty (a_k \cos kx+b_k \sin kx) \tag{1}$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t) \cos kt\, dt \qquad (k = 0, 1, 2, \dots)$$

and

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t) \sin kt\, dt \qquad (k = 1, 2, \dots)$$

The nth partial sum of this series can be transformed as follows:

$$S_n(x) = \frac{a_0}{2}+\sum_1^n (a_k \cos kx+b_k \sin kx) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(t)\, dt+\sum_1^n \frac{1}{\pi}\int_0^{2\pi} f(t)(\cos kt \cos kx+\sin kt \sin kx)\, dt =$$

$$= \frac{1}{\pi}\int_0^{2\pi} \left\{\frac{1}{2}+\sum_1^n \cos k(t-x)\right\} f(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} D_n(t-x) f(t)\, dt \tag{2}$$

where (see § 8.2 (16))

$$D_n(x) = \frac{1}{2}+\sum_1^n \cos kx = \frac{1}{2}\,\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}} \tag{3}$$

We have obtained a compact expression for the nth Fourier sum of the function $f(x)$:

$$S_n(x) = \frac{1}{\pi}\int_0^{2\pi} D_n(t-x) f(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} D_n(u) f(x+u)\, du \tag{4}$$

In the derivation of the last equality (4) we have used the periodicity of the integrand.

Integral (4) is called *Dirichlet's integral* and the trigonometric polynomial $D_n(x)$ is called *Dirichlet's kernel*. It should be noted that for any x and $n =$

$= 0, 1, 2, \dots$ we have

$$\frac{1}{\pi}\int_0^{2\pi} D_n(t-x)\,dt = \frac{1}{\pi}\int_0^{2\pi}\left\{\frac{1}{2}+\sum_1^n +\cos k(t-x)\right\}dt = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}\,dt = 1 \quad (5)$$

since

$$\int_0^{2\pi}\cos k(t-x)\,dt = \int_0^{2\pi}\cos kt\,dt = 0 \qquad (k = 1, 2, \dots)$$

In the derivation of the last equality (5) we have used the periodicity (with period 2π) of the function $\cos kt$ and the fact that it is orthogonal on the interval $[0, 2\pi]$ to the function identically equal to unity.

Any two Lebesgue (absolutely) integrable functions belonging to L^* which are equal almost everywhere have one and the same Fourier series, that is their Fourier coefficients coincide.

Any series of the form

$$\frac{\alpha_0}{2}+\sum_1^{\infty}(\alpha_k\cos kx+\beta_k\sin kx) \tag{6}$$

where α_k and β_k (the coefficients of the series) are some constants is called a *trigonometric series.*

A trigonometric series is a Fourier series only if there is a function $f \in L^*$ whose Fourier coefficients coincide with the numbers α_k and β_k, that is $a_k = \alpha_k$ and $b_k = \beta_k$. For instance, if it is known that series (6) converges in the mean square to a function $f \in L_2'^*$ (or to $f \in L_2^*$) on $[0, 2\pi]$ then it is the Fourier series of that function (see the corollary of Lemma 1, § 14.6).

The product of two even or two odd functions is an even function while the product of an even function by an odd function is an odd function. Consequently, if a function $f \in L'^*$ (or $f \in L^*$) is even its Fourier series has the form

$$f(x) \sim \frac{a_0}{2}+\sum_1^{\infty}a_k\cos kx \qquad \left(a_k = \frac{2}{\pi}\int_0^{\pi}f(t)\cos kt\,dt\right)$$

because its all coefficients b_k are equal to zero, and if it is odd its Fourier series is of the form

$$f(x) \sim \sum_{k=1}^{\infty}b_k\sin x \qquad \left(b_k = \frac{2}{\pi}\int_0^{\pi}f(t)\sin kt\,dt\right)$$

because in this case all the coefficients a_k are zero.

If the coefficients a_k and b_k of Fourier's nth sum of a periodic function $f(x)$ with period 2π are computed approximately with the aid of the rectangle rule (see § 10.6) using $2n+1$ equidistant points of division

$$x_k = \frac{2\pi k}{2n+1} \qquad (k = 0, 1, \dots, 2n) \tag{7}$$

the resultant trigonometric polynomial is of the form

$$S_n(f, x) = \frac{a_0^{(n)}}{2} + \sum_1^n (a_k^{(n)} \cos kx + b_k^{(n)} \sin kx)$$

where

$$a_k^{(n)} = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j) \cos kx_j \qquad (k = 0, 1, \ldots, n)$$

and

$$b_k^{(n)} = \frac{2}{2n+1} \sum_{j=0}^{2n} f(x_j) \sin kx_j \qquad (k = 0, 1, \ldots, n)$$

A remarkable property of this trigonometric polynomial of order n is that it provides an interpolation formula with nodes (7) for the function f. We thus have

$$f(x_j) = S_n(f, x_j) \qquad (j = 0, 1, \ldots, 2n)$$

This property can easily be verified by taking into account that

$$\frac{2}{2n+1} \sum_{j=0}^{2n} \cos kx_j \cos lx_j = \delta_{kl}$$

$$\frac{2}{2n+1} \sum_{j=0}^{2n} \sin kx_j \sin lx_j = \delta_{kl}$$

and

$$\frac{2}{2n+1} \sum_{j=0}^{2n} \sin kx_j \cos lx_j = 0 \qquad (k, l = 0, 1, \ldots, n)$$

§ 15.3. Formulas for the Remainder of Fourier's Series

Formulas (4) and (5) of the foregoing section imply that for a function $f \in L'^*$ (or, generally, $f \in L^*$) we have

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{2\pi} D_n(u) [f(x+u) - f(x)]\, du = \frac{1}{\pi} \int_{-\pi}^{\pi} D_n(u)\, \Delta_u f(x)\, du \quad (1)$$

where

$$\Delta_u f(x) = f(x+u) - f(x) \qquad (2)$$

is the first finite difference of f at the point x with step u.

In these transformations we have used the periodicity of the integrand function.

Equality (1) gives a compact expression for the remainder of the Fourier series. The question of whether or not the Fourier series of a function f

converges to $f(x)$ at a given point x and the estimation of the speed of convergence reduce to the investigation of the behaviour of integral (1) as $n \to \infty$.

For any number $\eta > 0$ satisfying the inequalities $0 < \eta \leqslant \pi$ expression (1) can also be written in the form

$$S_n(x) - f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{\sin nu}{2 \tan \frac{u}{2}} + \frac{1}{2} \sin nu \right) \Delta_u f(x)\, du =$$

$$= \frac{1}{\pi} \int_{-\eta}^{\eta} \frac{\sin nu}{u} \Delta_u f(x)\, du + \varrho_n(x) \tag{3}$$

where

$$\varrho_n(x) = \int_{-\infty}^{\infty} \sin nu\, g(u)\, \Delta_u f(x)\, du + \int_{-\infty}^{\infty} \cos nu\, \mu(u)\, \Delta_u f(x)\, du \tag{4}$$

$$g(u) = \begin{cases} \dfrac{1}{\pi} \left(\dfrac{1}{2 \tan \frac{u}{2}} - \dfrac{1}{u} \right) & \text{for } 0 < |u| < \eta \\[2ex] \dfrac{1}{2\pi \tan \frac{u}{2}} & \text{for } \eta \leqslant |u| \leqslant \pi \\[2ex] 0 & \text{outside } [-\pi, \pi] \end{cases} \tag{5}$$

and

$$\mu(u) = \begin{cases} \dfrac{1}{2\pi} & \text{for } 0 < |u| < \pi \\ 0 & \text{outside } (-\pi, \pi) \end{cases} \tag{6}$$

In what follows we shall essentially use the fact that both functions g and μ are bounded throughout the real axis:

$$|g(u)| \leqslant M \quad \text{and} \quad |\mu(u)| \leqslant M$$

because

$$\left| \frac{1}{2 \tan \frac{u}{2}} - \frac{1}{u} \right| = \frac{\left| u - 2 \tan \frac{u}{2} \right|}{2u \tan \frac{u}{2}} \leqslant \frac{O(u^3)}{u \tan \frac{u}{2}} \leqslant C \qquad (0 < |u| < \eta) \tag{7}$$

where C is a constant independent of u. It is also obvious that the functions g and μ have compact supports.

In the next section we shall prove a proposition (Lemma 2) which implies that for any x we have the relation

$$\varrho_n(x) = o(1), \qquad n \to \infty$$

which holds uniformly with respect to the values of x belonging to any closed interval $[a, b]$ on which the function f is bounded. It follows from this property that for any x we have the equality

$$S_n(x)-f(x)=\frac{1}{\pi}\int_{-\eta}^{\eta}\frac{\sin nu}{u}\Delta_u f(x)\,du+o(1),\qquad n\to\infty \tag{8}$$

which holds uniformly with respect to $x \in [a, b]$ where $[a, b]$ is an arbitrary interval on which the function f is bounded.

For a fixed point x formula (8) always holds provided that the function f is defined at that point. To find out whether or not the difference $S_n(x)-f(x)$ tends to zero (as $n \to \infty$) at the point x it is sufficient to investigate the first (principal) term on the right-hand side of (8) because the second term is sure to tend to zero.

If it is known that the function f in question is bounded on a closed interval $[a, b]$ then in order to find out whether or not $S_n(x)$ uniformly tends to $f(x)$ as $n \to \infty$ on that interval or on its part it is sufficient to answer this question for the principal term on the right-hand side of (8) since it is already known that the second term does tend to zero uniformly on $[a, b]$.

Of course, if a function f is unbounded on an interval $[a, b]$ then it is discontinuous at some of its points, and therefore if the Fourier series of f converges to f on $[a, b]$ the convergence must be nonuniform because the terms of the series are uniformly continuous functions on $(-\infty, \infty)$.

Let us dwell on an important property of Fourier's series known as the *principle of localization: to answer the question as to whether or not the Fourier series of a function $f \in L'^*$ (L^*) converges to that function on a closed interval $[a', b']$ it is sufficient to know the properties of the function on any closed interval $[a, b]$ containing $[a', b']$ strictly inside.* Indeed, let us put $\eta = \min\{a'-a, b-b'\}$. Then for the points $x \in [a', b']$ at which we intend to investigate the convergence of the Fourier series the values of the integrand on the right-hand side of (8) depend solely on the values of f assumed on $[a, b]$ (because if $x \in [a', b']$ and $0 < u < \eta$ then the points x, $x+u$ and $x-u$ belong to $[a, b]$).

§ 15.4. Oscillation Lemmas

Let $f \in L'(-\infty, \infty)$ (or, generally, $f \in L(-\infty, \infty)$); then for any real λ we have

$$\left.\begin{aligned}\left|\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx\right| &\leq \frac{1}{2}\int_{-\infty}^{\infty}\left|f\left(x+\frac{\pi}{\lambda}\right)-f(x)\right|dx\\ \left|\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx\right| &\leq \frac{1}{2}\int_{-\infty}^{\infty}\left|f\left(x+\frac{\pi}{\lambda}\right)-f(x)\right|dx\end{aligned}\right\} \tag{1}$$

Indeed,

$$\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = \int_{-\infty}^{\infty} f\left(u+\frac{\pi}{\lambda}\right)\sin\lambda\left(u+\frac{\pi}{\lambda}\right)du =$$

$$= -\int_{-\infty}^{\infty} f\left(u+\frac{\pi}{\lambda}\right)\sin\lambda u\,du = -\int_{-\infty}^{\infty} f\left(x+\frac{\pi}{\lambda}\right)\sin\lambda x\,dx$$

Therefore

$$\left|\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx\right| = \frac{1}{2}\left|\int_{-\infty}^{\infty}\left[f(x)-f\left(x+\frac{\pi}{\lambda}\right)\right]\sin\lambda x\,dx\right| \leqslant$$

$$\leqslant \frac{1}{2}\int_{-\infty}^{\infty}\left|f\left(x+\frac{\pi}{\lambda}\right)-f(x)\right|dx$$

For the cosine the argument is quite analogous.

Lemma 1. *If* $f \in L'(-\infty, \infty)\ (L(-\infty, \infty))$ *then*

$$\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx = 0 \quad\text{and}\quad \lim_{\lambda\to\infty}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = 0 \quad (2)$$

Relations (2) are a direct consequence of inequalities (1) because the right-hand sides of (1) tend to zero as $\lambda \to \infty$ (see Theorem 6, § 14.4).

From Lemma 1 it follows immediately that, for a function $f \in L'^{*}\ (L^{*})$, at any fixed point x where the value of $f(x)$ is finite, there holds equality (8), § 15.3, that is the term $\varrho_n(x)$ tends to zero as $n \to \infty$ since $\varrho_n(x)$ is the sum of two integrals for one of which we can write

$$\int_{-\infty}^{\infty}\sin nu\, g(u) f(x+u)\,du - f(x)\int_{-\infty}^{\infty}\sin nu\, g(u)\,du \to 0 \qquad (n\to\infty) \quad (3)$$

(the other integral possesses an analogous property; see below).

Here it should be taken into account that $g(u)$ is a bounded piecewise smooth function with support belonging to the interval $[-\pi, \pi]$, and therefore the functions $g(u)f(x+u)$ and $g(u)$ belong to $L'(-\infty, \infty)\ (L(-\infty, \infty))$, which low salus to apply Lemma 1. As we have mentioned, the similar property of the other integral entering into $\varrho_n(x)$ can also be easily proved; to this end one should take into account that $\mu(u)$ is a bounded piecewise smooth function with compact support.

The proof of the properties concerning the uniform convergence of $\varrho_n(x)$ to zero is based on the following more sophisticated lemma.

Lemma 2. *Let $f, g \in L'(-\infty, \infty)$ $(L(-\infty, \infty))$ and, in addition, let g be bounded. Then we have the relations*

$$\left.\begin{aligned} \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \cos \lambda u g(u) f(u+x)\, du = 0 \\ \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \sin \lambda u g(u) f(u+x)\, du = 0 \end{aligned}\right\} \tag{4}$$

which hold uniformly with respect to all $x \in (-\infty, \infty)$.

Proof. Let $|g(u)| \leq K$. On setting an arbitrary $\varepsilon > 0$ we can choose a continuous finite function φ such that

$$\int_{-\infty}^{\infty} |f(u)-\varphi(u)|\, du < \frac{\varepsilon}{2K}$$

Then

$$\begin{aligned} &\left| \int_{-\infty}^{\infty} \cos \lambda x g(u) f(u+x)\, du \right| \leq \\ &\quad \leq \frac{1}{2} \int_{-\infty}^{\infty} \left| g\left(u+\frac{\pi}{\lambda}\right) f\left(u+\frac{\pi}{\lambda}+x\right) - g(u) f(u+x) \right| du \leq \\ &\quad \leq \frac{1}{2} \int_{-\infty}^{\infty} \left| g\left(u+\frac{\pi}{\lambda}\right)\right| \left| f\left(u+\frac{\pi}{\lambda}+x\right) - f(u+x) \right| du + \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \left| g\left(u+\frac{\pi}{\lambda}\right) - g(u) \right| |\varphi(u+x)|\, du + \\ &\quad + \frac{1}{2} \int_{-\infty}^{\infty} \left| g\left(u+\frac{\pi}{\lambda}\right) - g(u) \right| |f(u+x)-\varphi(u+x)|\, du < \\ &\quad < \frac{K}{2} \int_{-\infty}^{\infty} \left| f\left(u+\frac{\pi}{\lambda}+x\right) - f(u+x) \right| du + \\ &\quad + \frac{M}{2} \int_{-\infty}^{\infty} \left| g\left(u+\frac{\pi}{\lambda}\right) - g(u) \right| du + \\ &\quad + \frac{2K}{2} \int_{-\infty}^{\infty} |f(u+x)-\varphi(u+x)|\, du < \varepsilon \qquad (|\lambda| > \lambda_0 > 0) \end{aligned} \tag{5}$$

provided that λ is sufficiently large (here M is a constant bounding $|\varphi|$: $M > |\varphi(u)|$; also take into account that the substitution $v = u+x$ in the second integral on the right-hand side eliminates x).

It is quite evident that $\cos \lambda x$ on the left-hand side of (5) can be replaced by $\sin \lambda x$.

Lemma 3. *Equality* (4) *continues to hold under the assumption that* $f \in L'$ *and* $g(u) = g(\alpha, u)$ *is a bounded function* (i. e. $|g(\alpha, u)| \leqslant K$ *for* $\alpha_1 \leqslant \alpha \leqslant \alpha_2$) *dependent continuously on* (α, u) *where* α *is a parameter. Moreover, equality* (4) *holds uniformly with respect to* $\alpha \in [\alpha_1, \alpha_2]$ *and* x *belonging to any finite interval.*

Indeed, let us consider the third term in inequalities (5). It consists of three summands the first and the third of which can be estimated as was done in (5) while the second can be estimated on the basis of the following inequality (see the explanations below):

$$\frac{1}{2}\int_{-N}^{N}\left|g\left(\alpha, u+\frac{\pi}{\lambda}\right)-g(\alpha, u)\right| |\varphi(u+x)|\, dx \leqslant$$

$$\leqslant \frac{M}{2}\int_{-N}^{N}\left|g\left(\alpha, u+\frac{\pi}{\lambda}\right)-g(\alpha, u)\right| du < \varepsilon \qquad (\lambda > \lambda_0)$$

provided that λ_0 is sufficiently large.

Since φ is a finite function, there is N such that $\varphi(u+x) = 0$ for all $x \in [a, b]$ and u satisfying the inequality $|u| > N$, and therefore we have $\int_{-\infty}^{\infty} = \int_{-N}^{N}$ for the summand in question.

From Lemma 2 follows equality (8) of § 15.3 (it holds uniformly on the interval $[a, b]$ where $f(x)$ is bounded) if we take into account that the remainder term on its right-hand side is determined by equality (4), § 15.3 where the functions $g(u)$ and $\mu(u)$ are finite and integrable. Indeed, the second integral on the left-hand side of (3) (it is independent of x) tends to zero as $n \to \infty$, and its product by $f(x)$ tends to zero uniformly with respect to $x \in [a, b]$ if f is bounded on $[a, b]$ (i. e. $|f(x)| \leqslant M$). For the first integral on the left-hand side of (3) we use the following argument. This integral is a periodic function with period 2π because so is f. Consequently, if we prove that it is uniformly convergent on $[0, 2\pi]$, this will imply its uniform convergence throughout the real axis. Let us consider the auxiliary function

$$F(x) = \begin{cases} f(x) & \text{for } \ 0 \leqslant x \leqslant 3\pi \\ 0 & \text{for } \ x \notin [0, 3\pi]. \end{cases}$$

Since $f \in L'^{*}$ (L^{*}), we have $F \in L' = L'(-\infty, \infty)$ $\big(L(-\infty, \infty)\big)$, and, since the support of g belongs to $[0, \pi]$, we obtain, for $x \in [0, 2\pi]$, the relation

$$\int_{-\infty}^{\infty} \sin nu\, g(u) f(x+u)\, du = \int_{0}^{\pi} \sin nu\, g(u)\, F(x+u)\, du =$$

$$= \int_{-\infty}^{\infty} \sin nu\, g(u)\, F(x+u)\, du \to 0, \qquad n \to \infty$$

which holds uniformly on $[0, 2\pi]$ (see Lemma 2).

The following theorem is proved on the basis of Lemma 1.

Theorem 1. *The Fourier coefficients a_k and b_k of a function $f \in L'^*$ (L^*) tend to zero as $k \to \infty$.*

To deduce this theorem from Lemma 2 it is sufficient to consider the restriction of the function $f \in L'^*$ (L^*) to the interval $[0, 2\pi]$ and then extend it to $(-\infty, \infty)$ by putting it equal to zero outside $[0, 2\pi]$. Let the extended function be $f_1 \in L'(-\infty, \infty)$ $(L(-\infty, \infty))$. Then by Lemma 1,

$$a_k = \frac{1}{\pi}\int_{0}^{2\pi} f(t) \cos kt\, dt = \frac{1}{\pi}\int_{\infty}^{\infty} f_1(t) \cos kt\, dt \to \infty \qquad (k \to \infty)$$

(for the coefficient b_k in the function $\sin kt$ the argument is the same).

We also note that if $f \in L_2'^*$ (or $f \in L_2^*$) then the fact that the Fourier coefficients a_k and b_k of f tend to zero follows from Bessel's inequality, by virtue of which

$$\sum_{1}^{\infty} (|a_k|^2 + |b_k|^2) < \infty$$

The fact that integrals (2) tend to zero can be elucidated by the following consideration. Let us take, for instance, the integral involving the sine. Although the given function $f \in L'^*$ (L^*) may have a finite or even an infinite number of discontinuities it nevertheless possesses many properties of a continuous function. These properties exhibit themselves in the oscillation lemmas proved above. The multiplication of the function $f(x)$ by the function $\sin \lambda x$ transforms the graph of the former into a "wave-like" curve. Each "wave" consists of two "half-waves" which almost compensate each other in the mean when the integration is performed. The result of such a compensation due to the "oscillatory" behaviour of the integrand is that integral (2) tends to zero when $\lambda \to \infty$.

§ 15.5. Test for Convergence of Fourier Series. Completeness of Trigonometric System of Functions

A function $f(x)$ is said to satisfy a *Hölder condition of order* α $(0 < \alpha \leq 1)$ *on an interval* $[a, b]$ $((a, b))$ if for any $x, x' \in [a, b]$ $((a, b))$ there holds the inequality

$$|f(x') - f(x)| \leq M|x' - x|^{\alpha}$$

where M is a constant independent of x and x'. This condition is also spoken of as a *Lipschitz** *condition of order* α. In the case $\alpha = 1$ we have

$$|f(x')-f(x)| \leq M|x'-x|^{\alpha}$$

and simply speak of a *Lipschitz condition* for the function f on $[a,b]$ $((a, b))$, the number M being referred to as a *Lipschitz constant.*

For instance, if f is a continuous and piecewise smooth function on $[a, b]$ it satisfies a Lipschitz condition on $[a, b]$ because

$$|f(x')-f(x)| = \left|\int_{x}^{x'} f'(t)\,dt\right| \leq M\,|x'-x| \qquad (|f'(t)| \leq M)$$

If a function f possesses a bounded derivative ($|f'| \leq M$) in an interval (a, b) and is continuous on $[a, b]$ we can apply Lagrange's theorem on finite increments to obtain

$$|f(x')-f(x)| = |f'(\xi)(x'-x)| \leq M|x'-x|, \qquad \xi \in (x, x')$$

which shows that f satisfies a Lipschitz condition on $[a, b]$.

The function $|x|^{\alpha}$ $(0 < \alpha \leq 1)$ satisfies a Hölder condition of order α throughout the real axis (and, moreover, on any finite interval) because if $0 < |x| < |x'|$ we can write, on putting $\left|\frac{x'}{x}\right| = t, 1 < t < \infty$, the inequalities

$$\frac{||x'|^{\alpha}-|x|^{\alpha}|}{|x'-x|^{\alpha}} \leq \frac{||x'|^{\alpha}-|x|^{\alpha}|}{||x'|-|x||^{\alpha}} = \frac{t^{\alpha}-1}{(t-1)^{\alpha}} \leq 1 \qquad (M = 1)$$

These inequalities are quite obvious for $\alpha = 1$. For $\alpha < 1$ they follow from the fact that the leftmost function of the argument t has a limit equal to zero as $t \to 1$ and equal to 1 as $t \to \infty$ and possesses a positive derivative on $(1, \infty)$, which means that it increases on $(1, \infty)$.**

Theorem 1. *Let a function f belong to L'^{*} (or to L^{*}) and satisfy a Hölder condition of (an arbitrary) order α on a closed interval $[a, b]$ (in particular, this is the case when f is a continuous piecewise smooth function on $[a, b]$). Then for any a' and b' satisfying the inequalities $a < a' < b' < b$ the Fourier series of f is convergent on $[a', b']$, the convergence being uniform.*

Proof. Let $\delta = \min\{a'-a,\ b-b'\}$, $0 < \eta < \delta$ and $\eta < \pi$. Then, for $x \in [a', b']$ and $0 \leq |u| \leq \eta$, the points x and $x+u$ belong to $[a, b]$, and therefore

$$|f(x+u)-f(x)| \leq |f(x+u)-f(x)| \leq M\,|u|^{\alpha} \tag{1}$$

Now for the chosen value of η we use formula § 15.3 (8):

$$S_n(x)-f(x) = \frac{1}{\pi}\int_{0}^{\eta} \sin nu \frac{f(x+u)-f(x)}{u}\,du + o(1) \qquad n \to \infty$$

---

* R. O. S. Lipschitz (1832-1903), a German mathematician.

** In the case $|x| = |x'|$ we have $||x'|^{\alpha}-|x|^{\alpha}| = 0 \leq |x'-x|^{\alpha}$.

This relation holds uniformly with respect to $x \in [a, b]$. Consequently, for any $\varepsilon > 0$ we obtain, by virtue of (1), the uniform estimate

$$|S_n(x)-f(x)| \leqslant \frac{1}{\pi} \int\limits_{-\eta}^{\eta} \frac{2M|u|^{\alpha}}{u}\, du+|o(1)| \leqslant$$

$$\leqslant \frac{2M}{\pi\alpha}\eta^{\alpha}+\frac{\varepsilon}{2} < \varepsilon \qquad (n > N,\ x \in [a', b']) \tag{2}$$

where we first choose η so that the inequality $2M\eta^{\alpha}/\pi\alpha < \varepsilon/2$ is fulfilled and then take N so large that $|o(1)| < \varepsilon/2$ for $n > N$. The theorem has been proved.

Theorem 2. *If a function $f \in C^*$ is piecewise smooth on the interval $[0, 2\pi]$, the Fourier series of f converges to it throughout the real axis, the convergence being uniform.*

Indeed, the function f regarded on an interval $[-\varepsilon, 2\pi+\varepsilon]$ (where $\varepsilon > 0$) is continuous and piecewise smooth, and therefore, by the foregoing theorem, its Fourier series is uniformly convergent to it on $[0, 2\pi]$, whence it follows, by the periodicity of f and of the terms of the Fourier series, that this applies to the whole real axis.

Theorem 3 (Weierstrass). *The systems of functions*

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots \tag{3}$$

$$1, \cos x, \cos 2x, \ldots \tag{4}$$

and

$$\sin x, \sin 2x, \ldots \tag{5}$$

are complete in the following spaces respectively:

(i) in *the space C^*,*

(ii) in *the subspace of C^* consisting of all even functions and also the space $C(0, \pi)$, and*

(iii) in *the subspace of C^* consisting of all odd functions and also the class of the functions belonging to $C(0, \pi)$ and satisfying the condition $f(0) = f(\pi) = 0$.*

Proof. Let f be an arbitrary function belonging to the class C^*. It is uniformly continuous on the closed interval $[-\pi, \pi]$ and has the period 2π. Therefore, given any $\varepsilon > 0$, there is a polygonal function $\Pi(x)$ with period 2π such that

$$|f(x)-\Pi(x)| < \frac{\varepsilon}{2} \tag{6}$$

for all x. If f is an even (odd) function then the function $\Pi(x)$ can be chosen so that it is also even (odd). For instance, if we join with a polygonal line the points of the graph of f having the abscissas $x_j = jh$; $j = 0, \pm 1, \pm 2, \ldots$, $h = \frac{2\pi}{N}$ where N is a sufficiently large natural number, this line can serve as the graph of the desired function $\Pi(x)$. The function $\Pi(x)$ satisfies the conditions of the foregoing theorem and consequently its nth Fourier sum satisfies the inequality

$$|\Pi(x)-S_n(x)|<\frac{\varepsilon}{2}\quad\text{for all } x \tag{7}$$

when n is sufficiently large. Besides, if $\Pi(x)$ is an even (odd) function the sum $S_n(x)$ is also even (odd).

It follows from (6) and (7) that $|f(x)-S_n(x)|<\varepsilon$ for all x. This proves the theorem because $S_n(x)$ is a trigonometric polynomial, that is a finite linear combination of functions belonging to (3), (4) or (5) depending on the character of the function f in question. Note that S_n is Fourier's sum of Π and not of f.

The assertion of Theorem 3 does not contradict the fact that there are functions $f \in C^*$ whose Fourier series are divergent at some points (in this connection see the remark at the end of the present section). It should also be noted that if a function continuous on $[0, \pi]$ (belonging to $C(0, \pi)$) is extended to $(-\pi, 0)$ in an even fashion and then is periodically extended (with period 2π) to the whole real axis, the resulting even function is a function of class C^*. If a function continuous on $[0, \pi]$ and satisfying the condition $f(0) = f(\pi) = 0$ is extended to $[-\pi, 0]$ in an odd fashion and then extended periodically to the real axis, the result is an odd function of class C^*.

It should also be taken into account that Theorem 3 implies the following property: *for any continuous periodic function f ($f \in C^*$) with period 2π there is a sequence of trigonometric polynomials $T_n(x)$ ($n = 1, 2, \ldots$) uniformly convergent to that function (throughout the real axis)* whence it follows that *the function f is representable as the sum of a uniformly convergent series of trigonometric polynomials:*

$$f(x)=\sum_{0}^{\infty}Q_k(x),\ Q_0=T_0,\ Q_k=T_k-T_{k-1}\qquad(k=1,2,\ldots)$$

Theorem 4. *The Fourier series of a function $f \in L_2'^*$ (and, generally, of $f \in L_2^*$) is convergent to it in the mean square on the period.*

Proof. Indeed, by the foregoing theorem, system (3) of trigonometric functions is complete in C^*. Then, moreover, it is complete in L_2^* (see the theorem proved in § 14.9). Therefore the theorem stated follows from Theorem 1, § 14.6 and from the general theory of series with respect to orthogonal systems.

By virtue of the theorem we have referred to, for the complete orthonormal system of trigonometric functions (§ 15.1 (1)), Parseval's relation holds for any function $f \in L_2'^*$ (and, moreover, for $f \in L_2^*$):

$$\int_{-\pi}^{\pi}|f|^2\,dx=\frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f\,dx\right|^2+\frac{1}{\pi}\sum_{k=1}^{\infty}\left(\left|\int_{-\pi}^{\pi}f(t)\cos kt\,dt\right|^2+\right.$$
$$\left.+\left|\int_{-\pi}^{\pi}f(t)\sin nt\,dt\right|^2\right)$$

or, which is the same,

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|f|^2\,dx=\frac{|a_0|^2}{2}+\sum_{k=1}^{\infty}(|a_k|^2+|b_k|^2) \tag{8}$$

Example. The function $\psi(x)$ with period 2π defined by the relations*

$$\psi(x)=\begin{cases}\dfrac{\pi-x}{2} & \text{for } 0<x<2\pi\\ 0 & \text{for } x=0\end{cases} \tag{9}$$

obviously belongs to L'^{*}. Its Fourier series is of the form

$$\psi(x)=\sum_{1}^{\infty}\frac{\sin kx}{k} \tag{10}$$

because the function ψ is odd, and

$$b_k=\frac{2}{\pi}\int_{0}^{\pi}\frac{\pi-t}{2}\sin kt\,dt=\frac{1}{k}\qquad (k=1,2,\dots)$$

Any closed interval $[a', b']$ not containing the points $x_k=2k\pi$ $(k=0,\pm1,\pm2,\dots)$ lies strictly inside another closed interval $[a,b]$ $(a<a'<b'<b)$ possessing the same property, and the function ψ is continuous on that interval $[a,b]$ together with its derivative, and consequently it is smooth. Therefore, by Theorem 1, Fourier series (10) of the function ψ converges to it uniformly on $[a', b']$. Hence, it converges at any point $x\neq 2k\pi$ $(k=0,\pm1,\pm2,\dots)$. Moreover, it is also convergent to ψ at these points $2k\pi$ because at the points $2k\pi$ the function ψ and all the members of series (10) vanish. However, there is no uniform convergence in any neighbourhoods of the points $x_k=2k\pi$ $(k=0,\pm1,\pm2,\dots)$.

Further, it is evident that $\psi\in L_2^{*}$, and therefore, by Theorem 4, Fourier series (10) of the function ψ is convergent to ψ in the mean square on the interval $[-\pi,\pi]$:

$$\int_{-\pi}^{\pi}\left|\psi(x)-\sum_{1}^{n}\frac{\sin kx}{k}\right|^2dx\to 0\qquad (n\to\infty)$$

The function $\psi(x)$ is a simple example of a discontinuous periodic function with period 2π having a single point of discontinuity (of the first kind) on the interval $0\leqslant x<2\pi$. Its jumps at the points of discontinuity are equal to $\psi(0+0)-\psi(0-0)=\pi$.

It is apparent that the function $\psi(x-x^0)$ whose graph is shifted by a distance of x_0 along the x-axis relative to that of $\psi(x)$ has the discontinuities at the points $x^0+2k\pi$ ($k = 0, \pm 1, \pm 2, \ldots$) with jumps equal to π. It can be expanded into the trigonometric series

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* This function $\psi(x)$ and the function $S(x)$ mentioned in § 15.1 (see § 15.1 (11)) are connected by the equality $-\psi(x)=\frac{1}{2}S(x-\pi)$ and therefore the graphs of $-\psi$ and $\frac{1}{2}S$ and of their partial Fourier sums are obtained from each other by a shift by π along the x-axis.

$$\psi(x-x^0) = \sum_1^\infty \frac{\sin k(x-x^0)}{k} = \sum_1^\infty \left(\frac{-\sin kx^0}{k}\cos kx + \frac{\cos kx^0}{k}\sin kx\right)$$

which is its Fourier series because it is convergent in the mean square to $\psi(x-x^0)$ on $(0, 2\pi)$ (see the corollary of Lemma 1 proved in § 14.6).

Now, proceeding from the properties of the function ψ, we shall prove the following theorem.

Theorem 5. *Let $f \in L'^*$ (L^*) be a piecewise smooth function on a closed interval $[a, b]$ having a single point of discontinuity $x^0 \in (a, b)$ on that interval. Then the Fourier series of f is convergent at the point x^0 to the arithmetic mean of the right-hand and left-hand limits of f at that point:*

$$\frac{f(x^0+0)+f(x^0-0)}{2} = \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx^0 + b_k \sin kx^0) \tag{11}$$

Proof. Let us assume that the value of f at the point x^0 is equal to the arithmetic mean of the right-hand and left-hand limits of f at that point:

$$f(x^0) = \frac{1}{2}[f(x^0-0)+f(x^0+0)] \tag{12}$$

If otherwise, we can redefine f at x^0 so that equality (12) holds, which does not alter the Fourier coefficients of f and, consequently, its Fourier series and its sum at the point x^0 either.

Let us put

$$f(x) = f_1(x)+f_2(x) \tag{13}$$

where

$$f_2(x) = \frac{1}{\pi}[f(x^0+0)+f(x^0-0)]\,\psi(x-x^0)$$

and ψ is the function defined by the above formula (10). The coefficient in $\psi(x-x^0)$ is chosen so that

$$f_2(x^0+0)-f_2(x^0-0) = f(x^0+0)-f(x^0-0)$$

It obviously follows that $f_1(x^0+0)-f_1(x^0-0) = 0$, and since $\psi(x^0) = \frac{1}{2}[\psi(x^0+0)+\psi(x^0-0)] = 0$, we have $f_1(x^0) = \frac{1}{2}[f_1(x^0+0)+f_1(x^0-0)]$; consequently, $f(x^0) = f_1(x^0) = f_1(x^0+0) = f_1(x^0-0)$.

Thus, the function f_1 is continuous at the point x^0 and, since it belongs to L'^* and is piecewise smooth on $[a, b]$, Theorem 1 implies that its Fourier series is convergent to $f_1(x^0)$ at x^0. As we know, the Fourier series of f_2 is also convergent to $f_2(x^0)$ at x^0. Consequently the Fourier series of f is convergent to $f(x^0)$ at the point x^0 because the sum of the Fourier series of the functions f_1 and f_2 is equal to the Fourier series of $f = f_1+f_2$.

Taking into account (12), we thus complete the proof of the theorem.

The Fourier series of the function f (described in Theorem 5) which has a jump at the point $x = x^0$ is convergent in a neighbourhood of the point x^0 and at that point itself but the convergence is nonuniform, and its speed is low. The speed of convergence of the Fourier series of $f_1(x)$ is higher, and the convergence is uniform in some neighbourhood of x^0. On the other hand, the function $f_2(x)$ is represented by an extremely simple formula and can even be studied thoroughly without expanding it into the Fourier series. It should also be noted that the Fourier series of the function ψ is thoroughly studied in the special literature devoted to trigonometric series.

Let us mention some further facts concerning the question of convergence and divergence of Fourier series.

A. N. Kolmogorov[*] constructed a function belonging to the class L^* of Lebesgue (absolutely) integrable functions whose Fourier series is divergent throughout the real axis.

L. Carleson[**] showed that the Fourier series of any function belonging to the class L_2^* is convergent almost everywhere. Since $C^* \subset L_2^*$, the proposition of Carleson also applies to any continuous periodic function with period 2π defined on the whole real axis. This proposition also holds for the functions $f \in L_p^*$ $(1 < p < \infty)$[***].

On the other hand there are examples (constructed by Du Bois-Reymond and Fejér) of continuous periodic functions $f (f \in C^*)$ whose Fourier series are divergent on the set of all rational points. These examples indicate that if it is only known that a function f is continuous this is not sufficient for its Fourier series to be convergent. For the convergence to take place the function f should satisfy some additional requirements. In the theorems proved above the role of such an additional requirement was played by a Hölder (Lipschitz) condition of order α. In some other more sophisticated theorems this condition is replaced by weaker sufficient conditions.

The properties of the Fourier series of 2π-periodic functions established in this section can be automatically extended to the Fourier series of functions with an arbitrary period 2ω. In this more general case we have

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty}\left(a_k \cos \frac{k\pi}{\omega} x + b_k \sin \frac{k\pi}{\omega} x\right) \tag{14}$$

where

$$a_k = \frac{1}{\omega}\int_0^{2\omega} \cos \frac{k\pi}{\omega} t f(t)\, dt \quad \text{and} \quad b_k = \frac{1}{\omega}\int_0^{2\omega} \sin \frac{k\pi}{\omega} t f(t)\, dt \tag{15}$$

Thus, if a periodic function $f \in L'(0, 2\omega)$ $(L(0, 2\omega))$ with period 2ω satisfies a Hölder condition of order α $(0 < \alpha \leqslant 1)$ on a closed interval $[a, b]$, its

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[*] Academician A. N. Kolmogorov (born 1903), a prominent Soviet mathematician.

[**] L. Carleson, a noted modern Swedish mathematician.

[***] This was proved by the American mathematician R. A. Hunt.

Fourier series (14) is uniformly convergent to it on any closed interval $[a', b'] \subset (a, b)$; if f is piecewise smooth on $[a, b]$ then its Fourier series converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ at its points of discontinuity x belonging to (a, b).

Finally, if a function $y = f(x)$ describes a periodic vibration process with period 2ω (in physics or in some technical science) which is a sum of a finite or infinite number of simple harmonic oscillations corresponding to the frequencies $\frac{k\pi}{\omega}$ $(k = 0, \pm 1, \dots)$ then the kth harmonic $u_k(x) = a_k \cos \frac{k\pi}{\omega} x + b_k \sin \frac{k\pi}{\omega} x$ can readily be found since the numbers a_k and b_k are nothing but the Fourier coefficients of f computed by formulas (15). On the other hand, as is known, a harmonic $u_k(x) = a_k \cos \frac{k\pi}{\omega} x + b_k \sin \frac{k\pi}{\omega} x$ $(k = 0, \pm 1, \pm 2, \dots)$ can be obtained physically from the compound real oscillation $y = f(x)$ with the aid of special devices (resonators), and the experimental results thus obtained are in coherence with the corresponding data obtained mathematically.

Examples. The functions below are considered as periodic with period 2π.

1. $f_1(x) = \operatorname{sgn} x(|x| < \pi)$; $\quad f_1(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{2k+1}$;

2. $f_2(x) = x(|x| < \pi)$; $\quad f_2(x) = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \sin kx}{k}$;

3. $f_3(x)$ is an even function equal to $\frac{\pi - x}{2}$ on $[0, \pi]$,

$$f_3(x) = \frac{\pi}{4} + \frac{2}{\pi} \sum_{0}^{\infty} \frac{\cos(2k+1)x}{(2k+1)^2}$$

4. $f_4(x)$ is an even function equal to 1 on $(0, h)$ and equal to zero on (h, π), $0 < h < \pi$,

$$f_2(x) = \frac{2h}{\pi} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{\sin kh}{kh} \right) \cos kx \right]$$

5. $f_5(x)$ is a continuous even function equal to zero on $(2h, \pi)$, $0 < h \leqslant \frac{\pi}{2}$, equal to 1 for $x = 0$ and linear on $(0, 2h)$,

$$f_3(x) = \frac{2h}{\pi} \left[\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{\sin kh}{kh} \right)^2 \cos kx \right]$$

Exercise. Find out on what intervals (finite, infinite or coinciding with the whole real axis) the series in Examples 1-5 are uniformly convergent.

§ 15.6. Complex Form of Fourier Series

Let a_k and b_k be the Fourier coefficients of a function $f \in L'^*$ (or $f \in L^*$). By Euler's formulas, we have

$$a_k \cos kx + b_k \sin kx = a_k \frac{e^{ikx}+e^{-ikx}}{2} + b_k \frac{e^{ikx}-e^{-ikx}}{2i} = c_k e^{ikx} + c_{-k} e^{-ikx}$$

where

$$c_k = \frac{a_k - ib_k}{2} \quad \text{and} \quad c_{-k} = \frac{a_k + ib_k}{2} \tag{1}$$

It follows that

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} (\cos kt - i \sin kt) f(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-ikt}\, dt$$

and

$$c_{-k} = \frac{1}{2\pi} \int_0^{2\pi} (\cos kt + i \sin kt) f(t)\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{ikt}\, dt$$

Hence, the numbers

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-ikt}\, dt \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{2}$$

can be computed with the aid of unified formula (2) applicable to all k's (including $k = 0$ for which $c_0 = a_0/2$).

It is important to note that if f is a real function then a_k and b_k are real numbers while the numbers c_k and c_{-k} may be complex in the general case but are mutually complex conjugate:

$$c_{-k} = \bar{c}_k; \qquad k = 0, \pm 1, \pm 2, \ldots \tag{3}$$

Conversely, if $c_k = \bar{c}_{-k}$ for some k then the corresponding Fourier coefficients a_k and b_k of the function f are real; if (3) holds for all k's then all the Fourier coefficients and, together with them, the function f itself are real.

Indeed, if, for instance, $f \in L_2'^*$ then the Fourier series of f converges to f in the mean square, and, if the terms of the series are real then the function $f(x)$ is also real. In the general case when $f \in L'^*$ the same follows from Theorem 2, § 15.11.

The nth sum of the Fourier series of f can obviously be written in the form

$$S_n(x) = \frac{a_0}{2} + \sum_1^n (a_k \cos kx + b_k \sin kx) = \sum_{-n}^{n} c_k e^{ikx} \tag{4}$$

and the Fourier series of f itself in the form of the two-way series

$$f(x) \sim \frac{a_0}{2} + \sum_1^\infty (a_k \cos kx + b_k \sin kx) \sim \sum_{-\infty}^{\infty} c_k e^{ikx} \tag{5}$$

We shall say that the series on the right-hand side of (5) is convergent for a given value of x if the limit

$$\lim_{n\to\infty} \sum_{-n}^{n} c_k e^{ikx}$$

exists.

We thus have defined the convergence of series (5) in the sense of the principal value: for the convergence in the ordinary sense it would be natural to consider series (5) convergent when the limit

$$\lim_{n,\, m\to\infty} \sum_{-m}^{n} c_k e^{ikx}$$

exists as n and m tend to infinity independently of each other.

The (complex) functions

$$\frac{1}{\sqrt{2\pi}} e^{ikx} \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{6}$$

form an orthonormal system on the closed interval $[0, 2\pi]$ because

$$\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} e^{ikx} \frac{1}{\sqrt{2\pi}} e^{-ilx}\, dx = \frac{1}{2\pi}\int_0^{2\pi} e^{i(k-l)x}\, dx = \delta_{kl}$$

Since the trigonometric functions $\cos kx$, $\sin kx$ $(k = 0, 1, 2, \ldots)$ form a complete system in C^* and, moreover, in L_2^* and L^* (Theorem 3, § 15.5), the same property holds for the system e^{ikx} $(k = 0, \pm 1, \pm 2, \ldots)$ because

$$\cos kx = 1/2\,(e^{ikx}+e^{-ikx}) \quad \text{and} \quad \sin kx = 1/2i(e^{ikx}-e^{-ikx})$$

The numbers c_k $(k = 0, \pm 1, \pm 2, \ldots)$ *determined by formulas* (2) *are the Fourier coefficients of* f *with respect to the system of functions* e^{ikx} (see § 14.6 (2)).

If follows that *the series*

$$f(x) \sim \sum_{-\infty}^{\infty} c_k e^{ikx}$$

obtained in (5) *from the ordinary trigonometric Fourier series of a function* f *is the Fourier series of* f *with respect to the functions* e^{ikx} $(k = 0, \pm 1, \pm 2, \ldots)$. *It is called the Fourier series of the function* f *in complex form.*

By virtue of the completeness of system (6) in L_2^*, for any function $f \in L_2^*$ there holds Parseval's relation

$$\int_0^{2\pi} |f(x)|^2\, dx = \sum_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(t)\, e^{ikt}\, dt \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f(t)\, e^{-ikt}\, dt\right)$$

which can also be rewritten as

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_k|^2 \tag{7}$$

§ 15.7. Differentiation and Integration of Fourier Series

Let $f(x)$ be a continuous piecewise smooth periodic function with period 2π.* Taking an arbitrary coefficient c_k different from c_0 we can perform integration by parts in its expression to obtain

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt = \frac{1}{2\pi}\left(f(t)\frac{e^{-ikt}}{-ik}\Big|_0^{2\pi} + \frac{1}{ik2\pi}\int_0^{2\pi} f'(t)\,e^{-ikt}\,dt\right) =$$

$$= \frac{1}{ik}c'_k \qquad (k = \pm 1, \pm 2, \ldots) \tag{1}$$

where

$$c'_k = \frac{1}{2\pi}\int_0^{2\pi} f'(t)\,e^{-ikt}\,dt \tag{2}$$

Here we have used the periodicity of the functions $f(t)$ and e^{ikt}, by virtue of which

$$f(t)\,e^{-ikt}\,\Big|_0^{2\pi} = 0$$

The derivative $f'(t)$ is periodic piecewise continuous function with period 2π which may have a finite number of points of discontinuity of the first kind on the interval $[0, 2\pi]$. Of course, it belongs to L'^*, and the numbers c'_k (the complex Fourier coefficients of f') make sense for it.

If a periodic function $f(x)$ with period 2π is continuous and possesses a continuous piecewise smooth derivative of the $(l-1)$th order, integration by parts (see (1)) can be carried out l times, which results in the equalities

$$c_k = \frac{1}{(ik)^l}c_k^{(l)} \qquad (k = \pm 1, \pm 2, \ldots) \tag{3}$$

where

$$c_k^{(l)} = \frac{1}{2\pi}\int_0^{2\pi} f^{(l)}(t)\,e^{-ikt}\,dt$$

are the Fourier coefficients of the derivative $f^{(l)}(x)$ of the lth order of the function f.

There holds the following important theorem.

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* What is established below also applies to a periodic function f with period 2π which is absolutely continuous on every closed interval (see § 19.5 (11)).

Theorem 1. *If the Fourier series*

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} \tag{4}$$

of a continuous piecewise smooth periodic function $f(x)$ with period 2π is differentiated termwise the resultant series is the Fourier series of the derivative $f'(x)$ of the function $f(x)$:*

$$f'(x) \sim \sum_{-\infty}^{\infty}{}' c'_k e^{ikx} = \sum_{-\infty}^{\infty}{}' (ik) c_k e^{ikx} \tag{5}$$

Here the symbol $\sum'$ indicates that the summation is extended over all the values of the index k except $k = 0$, that is the term involving c_0 is not contained in series (5).

Proof. In the general case the function $f'(x)$ is piecewise continuous and has discontinuities at those points where the derivative of f is discontinuous; however, at the points x_k where f' is discontinuous the limits $f'(x_k \pm 0)$ exist. To a function of this kind there corresponds its Fourier series $\sum' c'_k e^{ikx}$:

$$f'(x) \sim \sum{}' c'_k e^{ikx} \tag{6}$$

Series (6) may not be convergent to $f'(x)$ for some x. In series (6) we have

$$c'_0 = \frac{1}{2\pi}\int_0^{2\pi} f'(t)\,dt = \frac{1}{2}[f(2\pi)-f(0)] = 0$$

since f is a continuous periodic function with period 2π. On the other hand, for all $k \neq 0$ there holds equality (1), and consequently (6) implies (5). Series (4) is uniformly convergent to $f(x)$, which follows from Theorem 2, § 15.5.

Theorem 2. *If the Fourier series*

$$\varphi(x) \sim \sum{}' c'_k e^{ikx} \qquad (c'_0 = 0) \tag{7}$$

*of a piecewise continuous function** $\varphi(x)$ (whose all discontinuities are of the first kind!) is integrated termwise (by taking as an antiderivative of e^{ikx}, $k = \pm 1, \pm 2, \ldots$, the function $(ik)^{-1}e^{ikx}$) the resultant series is the uniformly convergent Fourier series of the continuous piecewise smooth function $f(x) - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$:*

$$f(x) - \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \sum_{-\infty}^{\infty}{}' c_k e^{ikx} \tag{8}$$

---

* The theorem also applies to an absolutely continuous function with period 2π (see § 19.5 (11)).

** The theorem also applies to $\varphi \in L^*$. In that case f is an absolutely continuous function (§ 19.5 (11)) whose Fourier series is uniformly convergent to it (this is proved in comprehensive courses in the theorem of the Fourier series).

where

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, e^{-ikt}\, dt \qquad (k = \pm 1, \pm 2, \ldots)$$

and

$$f(x) = \int_0^x \varphi(t)\, dt \tag{9}$$

Indeed, by virtue of (9), the function $f(x)$ as integral with variable upper limit of the piecewise continuous function $\varphi(t)$ is continuous and piecewise smooth on $[0, 2\pi]$. Besides, we have

$$f(2\pi) - f(0) = \int_0^x \varphi(t)\, dt \Big|_0^{2\pi} = 0$$

because

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, dt = c_0' = 0$$

and, consequently, the Fourier series of f is uniformly convergent to f, whence follows (8). On the other hand, the right-hand side of (8) can be regarded, by virtue of (1), as the result of the termwise integration (in the indicated sense) of the right-hand side of (7).

Remark. Theorems 1 and 2 essentially extend, for the case of the Fourier series, the sufficient tests for termwise integrability and differentiability of general series known to the reader. These theorems also admit of further generalizations which however not only involve the notion of the Lebesgue integral but also that of a generalized function (see § 16.11).

Exercise 1. Prove that if a periodic function $f(x)$ with period 2π possesses the continuous piecewise smooth derivative $f^{(l-1)}(x)$ of the $(l-1)$th order, it is representable in the form

$$f(x) = \frac{a_0}{2} + \frac{1}{\pi}\int_{-\pi}^{\pi} B_l(t-x)\, f^{(l)}(t)\, dt$$

where

$$B_l(u) = \sum_1^\infty \frac{\cos\left(ku + \frac{l\pi}{2}\right)}{k^l} \qquad (l = 1, 2, \ldots)$$

Exercise 2. Using the relation

$$B_1(u) = -\sum_1^\infty \frac{\sin ku}{k}$$

from which it follows that $B_1(u) = \frac{u-\pi}{2}$ $(0 < u < 2\pi)$ show that the function $B_l(u)$ (for any $l = 1, 2, 3, \ldots$) regarded on the closed interval $[0, 2\pi]$ is a polynomial of degree l possessing the property that its integral over $[0, 2\pi]$ is equal to zero. These polynomials are called *Bernoulli's** *polynomials.*

§ 15.8. Estimating the Remainder of Fourier's Series

Theorem 1. *Let a periodic function f with period 2π have a continuous piecewise smooth derivative $f^{(l-1)}$ of the $(l-1)$th order throughout the real axis and let its derivative $f^{(l)}$ satisfy the inequality*

$$\frac{1}{2\pi}\int_0^{2\pi} |f^{(l)}|^2\,dx \le M^2 \tag{1}$$

Then the deviation of the function $f(x)$ from its $(N-1)$th Fourier sum satisfies the following inequality:

$$|f(x)-S_{N-1}(x)| \le M\left(2\sum_N^{\infty}\frac{1}{k^{2l}}\right)^{1/2} < \sqrt{\frac{2}{2l-1}}\,\frac{M}{(N-1)^{l-1/2}} \tag{2}$$

Proof. The condition of the theorem implies that the Fourier series of the function $f(x)$ is convergent to it throughout the real axis. The deviation of $f(x)$ from $S_{N-1}(x)$ can be written in the form

$$\begin{aligned} f(x)-S_{N-1}(x) &= \sum_N^{\infty}(c_k e^{ikx}+c_{-k}e^{-ikx}) = \\ &= \sum_N^{\infty}\left[\left(\frac{1}{ik}\right)^l c_k^{(l)} e^{ikx} + \left(\frac{1}{-ik}\right)^l c_{-k}^{(l)} e^{-ikx}\right] \end{aligned} \tag{3}$$

where c_k are the complex Fourier coefficients of f which are expressed (in the third member of (3)) in terms of the Fourier coefficients $c_k^{(l)}$ of the derivative $f^{(l)}$ with the aid of formula (3) of the foregoing section.

On taking into account that

$$|e^{ikx}| = 1$$

(because x is real) and using Parseval's relation for $f^{(l)}$ we obtain

$$\begin{aligned} |f(x)-S_{N-1}(x)| &\le \sum_N^{\infty}\frac{1}{k^l}\left(|c_k^{(l)}|+|c_{-k}^{(l)}|\right) \le \\ &\le \left(2\sum_N^{\infty}\frac{1}{k^{2l}}\right)^{1/2}\left(\sum_N^{\infty}\left(|c_k^{(l)}|^2+|c_{-k}^{(l)}|^2\right)\right)^{1/2} \le \\ &\le \left(2\sum_N^{\infty}\frac{1}{k^{2l}}\right)^{1/2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^{(l)}(x)|^2\,dx\right)^{1/2} \le M\left(2\sum_N^{\infty}\frac{1}{k^{2l}}\right)^{1/2} \end{aligned} \tag{4}$$

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* After D. Bernoulli (1700-1782), a Swiss mathematician and physicist.

The first inequality (2) has thus been proved. The second cruder estimate in (2) for the deviation of f from S_{N-1} follows from the inequality

$$\sum_N^\infty \frac{1}{k^{2l}} \leqslant \int_{N-1}^\infty \frac{dx}{x^{2l}} = \frac{1}{2l-1}\,\frac{1}{(N-1)^{2l-1}}$$

Remark. Estimate (2) continues to hold (and the proof remains essentially the same) when the periodic function $f(x)$ possesses an absolutely continuous derivative of the $(l-1)$th order and the derivative of order l exists almost everywhere and satisfies inequality (1) in which the integral is understood in Lebesgue's sense (see the footnotes concerning Theorems 1 and 2 in the foregoing section).

We also note that it is possible to prove the estimate

$$|f(x)-S_n(x)| < C\frac{\ln n}{n^l}\sup_x |f^{(l)}(x)|$$

where C is a constant independent of n, but this involves a rather lengthy argument.

Exercise. Confining yourself, for simplicity, to the case when l is divisible by 4, show that the first inequality (2) provides a precise estimate.

Hint. From (3) it follows that for $x = 0$ we have

$$f(0)-S_n(0) = \sum_{n+1}^\infty \frac{1}{k^l}\left(c_k^{(l)}+c_{-k}^{(l)}\right)$$

and therefore the first two inequalities (4) become equalities if the numbers $c_k^{(l)}$ are chosen so that they are proportional to $\frac{1}{k^l}$ (see the remark after § 6.2 (6)).

§ 15.9. Gibbs' Phenomenon

In this section we shall study Gibbs' phenomenon which was mentioned in § 15.1.

The function

$$\psi(x) = \sum_1^\infty \frac{\sin kx}{k} \tag{1}$$

which is equal to

$$\frac{\pi-x}{2} = \sum_1^\infty \frac{\sin kx}{k} \tag{2}$$

on the interval $(0, \pi)$ has as the nth partial sum of its Fourier series the expression

$$\psi_n(x) = \sum_1^n \frac{\sin kx}{k} \tag{3}$$

On the interval $0 < x < \pi$ we can write for this function (see the explanations below) the relations

$$\begin{aligned}\frac{x}{2}+\psi_n(x) &= \int_0^x \left(\frac{1}{2}+\sum_1^n \cos kt\right) dt = \\ &= \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\,dt = \int_0^x \frac{\sin nt}{2\tan\frac{t}{2}}\,dt+\frac{1}{2}\int_0^x \cos nt\,dt = \\ &= \int_0^x \frac{\sin nt}{t}\,dt+\int_0^x \sin nt\left(\frac{1}{2\tan\frac{t}{2}}-\frac{1}{t}\right)dt+\frac{1}{2}\int_0^x \cos nt\,dt = \\ &= \int_0^x \frac{\sin nt}{t}\,dt+o(1) \qquad (n \to \infty)\end{aligned} \tag{4}$$

which hold uniformly with respect to $x \in (0, \pi)$. Here only the estimations of the second and third summands in the fifth member of (4) require explanation. Let us, for example, estimate the second summand. It can be written as

$$\begin{aligned}A_n &= \int_0^x \sin ntg(t)\,dt = \\ &= \frac{1}{2}\int_0^x \sin ntg(t)\,dt-\frac{1}{2}\int_{\pi/n}^{x+\pi/n} \sin ntg\left(t+\frac{\pi}{n}\right)dt = \\ &= \frac{1}{2}\int_0^{\pi/n} \sin ntg(t)\,dt-\frac{1}{2}\int_x^{x+\pi/n} \sin ntg\left(t+\frac{\pi}{n}\right)dt+ \\ &\quad +\frac{1}{2}\int_{\pi/n}^x \sin nt\left[g(t)-g\left(t+\frac{\pi}{n}\right)\right]dt\end{aligned}$$

whence, since $|g(t)| \leq M$ and $g(t) = 0$ outside $(0, \pi)$ (see § 15.3 (7)), follows

$$|A_n| \leq \frac{1}{2}\,\frac{\pi}{n}M+\frac{1}{2}\,\frac{\pi}{n}M+\frac{1}{2}\int_{-\infty}^{\infty}\left|g(t)-g\left(t+\frac{\pi}{n}\right)\right|dt \to 0, \qquad n \to \infty \tag{5}$$

where the right-hand side is independent of $x \in (0, \pi)$, and therefore the left-hand side tends to zero uniformly with respect to these x's.

On putting $x = \frac{\pi}{n}$ in (4) and passing to the limit as $n \to \infty$ we obtain (see the explanations below)

$$d_+ = \lim_{n\to\infty} \psi_n\left(\frac{\pi}{n}\right) = \int_0^\pi \frac{\sin t}{t}\,dt > \frac{\pi}{2} \tag{6}$$

(see relations (9) and (12) below). At the same time we have

$$\psi(0+0) = \lim_{\substack{x>0\\ x\to 0}} \frac{\pi-x}{2} = \frac{\pi}{2} \tag{7}$$

The direct computation shows that

$$\frac{d_+}{\frac{1}{2}(\psi(0+0)-\psi(0-0))} = \frac{d_+}{\psi(0+0)} = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt = 1.089490\ldots \tag{8}$$

The fact that this ratio is greater than 1 (and *is not equal to* 1) is termed Gibbs' phenomenon.

Fig. 15.3 shows schematic graphs of the function $f(x) = -\psi(x-\pi)$ and of its nth partial sum $S_n(x)$. We see that the function $S_n(x)$ oscillates in the vicinity of $f(x)$ on the interval $[-\pi+\delta, \pi-\delta]$ (where $\delta > 0$) for sufficiently large n. Indeed, the sum $S_n(x)$ uniformly tends to $f(x)$ on this interval as $n \to \infty$. On the other hand, near the point $x = \pi$ the graph of $S_n(x)$ sharply deviates upward from the graph of $f(x)$; it is this way in which the sum $S_n(x)$ exhibits Gibbs' phenomenon. Then the graph of $S_n(x)$ sharply goes downward to the point $x = \pi$ of the x-axis. An analogous behaviour takes place on the interval $[-\pi, 0]$.

It should be noted that the same phenomenon takes place in the case of an arbitrary function $f \in L^*$ which is continuous together with its derivative on half-open intervals $[a, x^0)$ and $(x^0, b]$ and has, together with its derivative, a discontinuity of the first kind at x^0. This follows from the possibility of representing the function f in the form of the sum $f(x) = f_1(x)+f_2(x)$ (see § 15.5 (12)) where f_1 is a continuous* piecewise smooth function on $[a, b]$ and $f_2(x) = \frac{\varkappa}{\pi}\psi(x-x^0)$ where $\varkappa$ is the jump of f at x^0. The Fourier series of f_1 uniformly converges to f_1 on $[a', b'] \subset (a, b)$ while the function f_2 is nothing but the result of the multiplication of the function ψ by a constant and by the shift of the argument by x^0. For the function f_2 (and, consequently, for f as well) Gibbs' phenomenon takes place with the same value of ratio (8):

$$\frac{d_+(x^0)}{\frac{1}{2}(f(x^0+0)-f(x^0-0))} = \frac{2}{\pi}\int_0^\pi \frac{\sin t}{t}\,dt$$

where in this case

$$d_+(x^0) = \lim_{n\to\infty}\left[S_n\left(x^0+\frac{\pi}{n}\right) - \frac{f(x^0+0)+f(x^0-0)}{2}\right]$$

and S_n is the nth Fourier sum of f.

Gibbs' phenomenon and, generally, Fourier's expansions of functions, should be regarded as an exhibition of the law of nature; for instance, the former was first discovered by Gibbs purely experimentally.

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* More precisely, the function f_1 may have removable discontinuities.

There hold the relations

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx \tag{9}$$

For the second of them see § 13.15 (12). The first of them can be obtained as follows (see the explanations below):

$$\frac{\pi}{2} = \int_0^\pi \frac{\sin (n+1/2)t}{2 \sin t/2}\, dt = \int_0^\pi \frac{\sin nt}{t}\, dt + \int_0^\pi \sin nt \left(\frac{1}{2 \tan t/2} - \frac{1}{t}\right) dt +$$

$$+\frac{1}{2}\int_0^\pi \cos nt\, dt = \int_0^\pi \frac{\sin nt}{t}\, dt + o(1) =$$

$$= \int_0^{n\pi} \frac{\sin x}{x}\, dx + o(1) \to \int_0^\infty \frac{\sin x}{x}\, dx \qquad (n \to \infty) \tag{10}$$

The first equality in (10) follows from § 15.2 (3) and (5) and the third from Theorem 1, § 15.4 and also from § 15.3 (7).

The integral on the right-hand side of (10) can also be written in the form of the series

$$\int_0^\infty \frac{\sin x}{x}\, dx = \sum_0^\infty \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x}\, dx = \sum_0^\infty \int_0^\pi \frac{\sin (k\pi+u)\, du}{k\pi+u} = \sum_0^\infty (-1)^k a_k,$$

$$a_k = \int_0^\pi \frac{\sin u\, du}{k\pi+u} \tag{11}$$

It is clear that the numbers a_k are nonnegative and monotonically tend to zero, and hence the right-hand member of (11) is a Leibniz series. It follows, in particular, that

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin x}{x}\, dx < a_0 = \int_0^\pi \frac{\sin x}{x}\, dx \tag{12}$$

Remark. The function

$$\lambda_n(x) = \int_0^x \frac{\sin nt}{t}\, dt = \int_0^{nx} \frac{\sin u}{u}\, du \tag{13}$$

attains its maximum value on $[0, \infty)$ (which is equal to $\pi/2$) at the point $x_1 = \pi/2$.

Indeed, since the derivative

$$\lambda_n'(x) = n\frac{\sin nx}{x}$$

vanishes only at the points $x_k = k\pi/n$ $(k = 1, 2, \dots)$ of the interval $(0, \infty)$, the points of extremum of the function $\lambda_n(x)$ on that interval can only be among these points. But we have

$$\lambda_n(x_k) = \int_0^{\pi} \frac{\sin u}{u}\, du - \int_{\pi}^{2\pi} \frac{\sin u}{u}\, du + \dots + (-1)^{k-1} \int_{(k-1)\pi}^{k\pi} \frac{\sin u}{u}\, du <$$

$$< \int_0^{\pi} \frac{\sin u}{u}\, du = \lambda(x_1) = \frac{\pi}{2}$$

whence follows the desired assertion if we take into account inequality (12) and the fact that $\lambda_n(0) = 0$.

It follows that the number d_+ (defined originally in (6)) can aso be defined with the aid of the equality

$$d_+ = \lim_{x\to 0} \max_{0\leq t\leq x} \psi_n(t) \tag{14}$$

Indeed, let us choose $x \in (0, \pi)$ and then find a natural number n such that $\frac{\pi}{n} \leq x < \frac{\pi}{n-1}$. Then $\psi_n\left(\frac{\pi}{n}\right) \leq \max\limits_{0\leq t\leq x} \psi_n(t)$, and, since $n \to \infty$ when $x \to 0$, we see that, by virtue of (6), $d_+ \leq \underline{\lim}_{x\to 0} \max\limits_{0\leq t\leq x} \psi_n(t)$. On the other hand, by virtue of (4), we can write the relations

$$\psi_n(x) < \lambda_n(x) + o(1) = \lambda_n(\pi/n) + o(1) = d_+ + o(1) \qquad (n \to \infty)$$

holding uniformly with respect to $x \in (0, \pi)$, whence, taking into account that $n \to \infty$ for $x \to 0$, we obtain $\overline{\lim}_{x\to 0} \max\limits_{0\leq t\leq x} \psi_n(t) \leq d_+$.

Now we see that d_+ is in fact equal to the right-hand member of (14).

§ 15.10. Fejér's* Sums

As was mentioned in § 15.5, there are examples of continuous periodic functions f with period 2π belonging to C^* whose Fourier series are divergent at some points and even at an infinite number of points of an arbitrarily chosen countable set, for instance, at all rational points. In this connection of extreme importance is the fact that the Fourier series of an arbitrary function $f \in C^*$ is uniformly summable to f throughout the real axis with the aid of the method of arithmetic means (see § 11.10).

Let us take a function $f \in L'^*$ (or, generally, $f \in L^*$) and form its Fourier series

$$f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_k \cos kx + b_k \sin kx)$$

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* L. Fejér (1880-1959), a Hungarian mathematician.

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\,dt \qquad (k = 0, 1, \ldots)$$

and

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\,dt \qquad (k = 1, 2, \ldots)$$

Denoting the partial sum of the Fourier series as

$$S_n = \frac{a_0}{2}+\sum_1^n (a_k\cos kx+b_k\sin kx)$$

we form the arithmetic means of the sums $S_0, S_1, \ldots, S_n$:

$$\sigma_n = \frac{S_0+S_1+\ldots+S_n}{n+1} \qquad (n = 0, 1, 2, \ldots) \tag{1}$$

We shall begin with deriving a compact expression for σ_n. To this end we write

$$\sigma_n = \frac{1}{n+1}\left\{\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt+\sum_{k=1}^n \frac{1}{\pi}\int_0^{2\pi} D_k(t-x)\,f(t)\,dt\right\} =$$

$$= \frac{1}{(n+1)\pi}\int_0^{2\pi}\left\{\frac{1}{2}+\sum_0^n D_k(t-x)\right\}f(t)\,dt$$

On taking the imaginary parts on both sides of the equality

$$\sum_{k=0}^n e^{i(k+1/2)x} = \frac{e^{ix/2}-e^{i(n+3/2)x}}{1-e^{ix}} = \frac{1-e^{i(n+1)x}}{e^{-ix/2}-e^{ix/2}} \tag{2}$$

we receive

$$\sum_{k=0}^n \sin\left(k+\frac{1}{2}\right)x = \frac{1-\cos(n+1)x}{2\sin\frac{x}{2}} = \frac{\sin^2\frac{n+1}{2}x}{\sin\frac{x}{2}} \tag{3}$$

Consequently,

$$\frac{1}{2}+\sum_1^n D_k(x) = \frac{1}{2}+\sum_1^n\frac{\sin\left(k+\frac{1}{2}\right)x}{2\sin\frac{x}{2}} =$$

$$= \frac{1}{2\sin\frac{x}{2}}\sum_0^n \sin\left(k+\frac{1}{2}\right)x = \frac{1}{2}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}\right)^2$$

and we obtain

$$\sigma_n(x) = \frac{1}{\pi}\int_0^{2\pi} F_n(t-x) f(t)\, dt = \frac{1}{\pi}\int_0^{2\pi} F_n(u) f(x+u)\, du \tag{4}$$

where

$$F_n(x) = \frac{1}{2(n+1)}\left(\frac{\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}\right)^2 \tag{5}$$

In the derivation of the last equality (4) we have used the periodicity of the integrand.

The function $\sigma_n(x)$ is called *Fejér's sum* (*of order* n) and the function $F_n(x)$ is *Fejér's kernel*. It can readily be seen that

$$F_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \frac{n+1-k}{n+1}\cos kx \tag{6}$$

Therefore the sum $\sigma_n(x)$ can also be written as

$$\sigma_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \frac{n+1-k}{n+1}(a_k \cos kx + b_k \sin kx) \tag{7}$$

Let us enumerate the following properties of the kernel $F_n(x)$:

(i) $F_n(x)$ is a nonnegative even trigonometric polynomial of order n (see (5) and (6));

(ii) $$\frac{1}{\pi}\int_0^{2\pi} F_n(x)\, dx = 1 \tag{8}$$

(see (6); take into account that $\cos kx$ is orthogonal to 1; $k = 1, 2, \ldots, n$).

(iii) $$\int_0^{\pi} F_n(x)\, dx \leq \frac{1}{2(n+1)}\int_0^{\pi}\frac{dx}{\left(\sin\frac{\delta}{2}\right)^2} = \frac{\pi-\delta}{2(n+1)\left(\sin\frac{\delta}{2}\right)^2} \to 0 \quad (n \to \infty)$$

By property (ii), the deviation of $\sigma_n(x)$ from $f(x)$ is expressed by the formula

$$\sigma_n(x)-f(x) = \frac{1}{\pi}\int_0^{2\pi} F_n(t-x)[f(t)-f(x)]\, dt =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} F_n(t)[f(x+t)-f(x)]\, dt \tag{9}$$

In the derivation of the last equality we have used the periodicity of the integrand function.

Now we proceed to the following theorem.

Theorem 1 (Fejér). *Fejér's sum of order n of an arbitrary 2π-periodic function $f(x)$ continuous throughout the real axis (i.e. $f \in C^*$) uniformly tends to that function as $n \to \infty$:*

$$\|f-\sigma_n\|_{C(0,\,2\pi)} = \max_x |f(x)-\sigma_n(x)| \to 0 \qquad (n \to \infty) \tag{10}$$

Proof. Let $\omega(\delta)$ be the modulus of continuity of the function f; $\omega(\delta)$ is a continuous function of δ (see § 7.10, Example 2). Therefore, by virtue of (9) and properties (i), (ii) and (iii), we have

$$|\sigma_n(x)-f(x)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} F_n(t)\,\omega(|t|)\,dt = \frac{2}{\pi}\int_{0}^{\pi} F_n(t)\,\omega(|t|)\,dt =$$

$$= \frac{2}{\pi}\int_{0}^{\delta} F_n(t)\,\omega(|t|)\,dt + \frac{2}{\pi}\int_{\delta}^{\pi} F_n(t)\,\omega(|t|)\,dt \leqslant$$

$$\leqslant \omega(\delta) + K\frac{2}{\pi}\int_{\delta}^{\pi} F_n(t)\,dt < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \qquad (n > n_0,\ K \geqslant \omega(t)) \tag{11}$$

where first δ is chosen so small that $\omega(\delta) < \frac{\varepsilon}{2}$ and then n_0 is taken so large that the second summand in the fifth member of (11) becomes less than $\frac{\varepsilon}{2}$ for $n > n_0$.

Later on we shall prove some other properties of Fejér's sums (Theorems 2 and 5 in § 15.11) for the general case of an arbitrary dimension n which are important for the theory of Fourier's series.

We already know that it may happen that the arithmetic means of the partial sums of a number series converge to a limit while the series itself is divergent. That very situation takes place in the case of the Fourier series of an arbitrary continuous function. There exist continuous functions whose Fourier series are divergent on any preassigned countable set, for instance, on the set of all rational numbers. But, as has been shown, this does not prevent the arithmetic means of the partial Fourier sums of any continuous function f from converging (even uniformly!) to f.

Note that, in particular, Fejér's theorem implies the completeness of the system of trigonometric functions

$$1, \cos x, \sin x, \ldots \tag{12}$$

in the space C^*. For, if $f \in C^*$ and $\varepsilon > 0$ is an arbitrary number then there is n such that there holds inequality (11) where $\sigma_n(x)$ is a trigonometric polynomial, that is a finite linear combination of functions belonging to system (12).

§ 15.11. Elements of the Theory of Fourier Series for Functions of Several Variables

The functions

$$\frac{1}{(2\pi)^{n/2}} e^{ikx}, \quad k = (k_1, \ldots, k_n); \qquad kx = \sum_1^n k_j x_j$$

$$(k_j = 0, \pm 1, \pm 2, \ldots; j = 0, \ldots, n) \tag{1}$$

are periodic with period 2π with respect to each variable x_j.

They are orthonormal on the n-dimensional cube ("n-dimensional period")

$$\Delta_* = \{-\pi \leqslant x_j \leqslant \pi;\ j = 1, \ldots, n\} \tag{2}$$

(and also on any other cube with edges of length 2π parallel to the coordinate axes) because

$$\int_{\Delta_*} \frac{1}{(2\pi)^{n/2}} e^{ikx} \frac{1}{(2\pi)^{n/2}} e^{-ilx}\, dx =$$

$$= \frac{1}{(2\pi)^n} \int_{-\pi}^{\pi} e^{i(k_1-l_1)x_1}\, dx_1 \ldots \int_{-\pi}^{\pi} e^{i(k_n-l_n)x_n}\, dx_n = \begin{cases} 0 & \text{for } k \neq l \\ 1 & \text{for } k = l \end{cases}$$

For, if $k \neq l$, then there is j for which $k_j - l_j \neq 0$, and hence we obtain

$$\int_{-\pi}^{\pi} e^{i(k_j-l_j)\,dx_j}\, dx_j = 0$$

if $k = l$, the integrands of all the integrals entering into the product are equal to unity. It is clear that system (1) is countable.

We shall deal with the following classes of (complex or real) functions with period 2π (with respect to each variable x_j) defined in R_n.

The class C^* consists of the continuous functions and is equipped with the norm

$$\|f\|_{C^*} = \max_x |f(x)|$$

The class L^* consists of the functions (Lebesgue) integrable on the period Δ_*, the norm being defined as

$$\|f\|_{L^*} = \int_{\Delta_*} |f(x)|\, dx$$

The class L_2^* consists of the functions which are Lebesgue measurable and have (Lebesgue) integrable squares of their moduli. The corresponding norm is

$$\|f\|_{L_2^*} = \left(\int_{\Delta_*} |f(x)|^2\, dx\right)^{1/2}$$

The Fourier series of a function $f \in L^*$ *in complex form* is written as

$$f(x) \sim \sum c_k e^{ikx} \tag{3}$$

where

$$c_k = \frac{1}{2(\pi)^n} \int\limits_{\Delta^*} f(t)\, e^{-ikt}\, dt \tag{4}$$

and the sum extends over all the possible integral vectors (i.e. having integral components) $k = (k_1, \ldots, k_n)$ $(k_j = 0, \pm 1, \pm 2, \ldots; j = 1, \ldots, n)$.

The numbers c_k are called the *Fourier coefficients of* f. The Nth Fourier sum is written in the form

$$S_N(x) = \sum_{|k_j| \leq N} c_k e^{ikx} = \frac{1}{(2\pi)^n} \int\limits_{\Delta_*} \sum_{|k_j| \leq N} e^{-ik(t-x)} f(t)\, dt =$$

$$= \frac{1}{(2\pi)^n} \int\limits_{\Delta_*} \sum_{|k_1| \leq N} e^{-ik_1(t_1-x_1)} \ldots \sum_{|k_n| \leq N} e^{-ik_n(t_n-x_n)} f(t)\, dt =$$

$$= \frac{1}{\pi^n} \int\limits_{\Delta_*} \prod_{j=1}^{n} D_N(t_j - x_j) f(t)\, dt \tag{5}$$

where

$$D_N(u) = \frac{1}{2} + \sum_{1}^{N} \cos ku$$

is Dirichlet's sum (of order N).

The multi-dimensional analogue of Fejér's Nth sum has the form

$$\sigma_N(x) = \frac{1}{\pi^n} \int\limits_{\Delta_*} \Phi_N(t-x) f(t)\, dt = \frac{1}{\pi^n} \int\limits_{\Delta_*} \Phi_N(u) f(x+u)\, du \tag{6}$$

where

$$\Phi_N(u) = \prod_{j=1}^{n} F_N(u_j), \quad F_N(t) = \frac{1}{2(N+1)} \left(\frac{\sin \frac{N+1}{2} t}{\sin \frac{t}{2}} \right)^2 \tag{7}$$

Let

$$\Delta_\varepsilon = \{|u_j| \leq \varepsilon;\ j = 1, \ldots, n\}$$

Then for the function $\Phi_N(u)$ $(\Phi_N(u) \geq 0)$ we have

$$\frac{1}{\pi^n} \int\limits_{\Delta_\varepsilon} \Phi_N(u)\, du \leq \frac{1}{\pi^n} \int\limits_{\Delta_*} \Phi_N(u)\, du = 1 \tag{8}$$

and

$$\frac{1}{\pi^n} \int\limits_{\Delta_* - \Delta_\varepsilon} \Phi_N(u)\, du \to 0 \qquad (N \to \infty) \tag{9}$$

For the planes in which the faces of Δ_ε lie divide $\Delta_* - \Delta_\varepsilon$ into a finite number of rectangles (i.e. rectangular parallelepipeds with edges parallel to the coordinate axes) on each of which at least one coordinate u_{j_0} satisfies the inequality $|u_{j_0}| \geqslant \varepsilon$. Therefore if the integral with element of integration $\Phi_N \, du$ is written as the product of the integrals with elements of integration $F_n(u_j)\, du_j$, one of the factors (corresponding to $j = j_0$) tends to zero as $N \to \infty$ while the other factors (which are positive) do not exceed 1 (see § 15.10, properties (i)-(iii) of the kernel F_N).

Let us prove the following theorem generalizing Theorem 1, § 15.10 to the n-dimensional case.

Theorem 1. *The Nth Fejér sum σ_N of a function $f \in C^*$ satisfies the condition*

$$\|f - \sigma_N\|_{C^*} \to 0 \qquad (N \to \infty)$$

Proof. Taking into account (7)-(9) and denoting by $\omega(\delta)$ the modulus of continuity of f we can write

$$|\sigma_N(x) - f(x)| = \frac{1}{\pi^n} \left| \int_{\Delta_*} \Phi_N(u)\,[f(x+u) - f(x)]\, du \right| \leqslant \frac{1}{\pi^n} \int_{\Delta_*} \Phi_N(u)\, \omega(|u|)\, du \leqslant$$

$$\leqslant \frac{1}{\pi^n} \left(\omega(2\delta\sqrt{n}) \int_{\Delta_\delta} \Phi_N(u)\, du + \frac{K}{\pi^n} \int_{\Delta_* - \Delta_\delta} \Phi_N(u)\, du \right) \leqslant$$

$$\leqslant \omega(2\delta\sqrt{n}) + \frac{K}{\pi^n} \int_{\Delta_* - \Delta_\delta} \Phi_N(u)\, du < \varepsilon$$

$$(N > N_0,\ K = \max|\omega(t)|)$$

Here we first choose δ so that $\omega(2\delta\sqrt{n}) < \frac{\varepsilon}{2}$ ($2\delta\sqrt{n}$ is the diameter of Δ_δ) and then take a sufficiently large N_0 so that the second summand in the fifth member of these relations becomes less than $\frac{\varepsilon}{2}$.

Theorem 2. *The Nth Fejér sum σ_N of a function $f \in L^*$ satisfies the condition*

$$\|f - \sigma_N\|_{L^*} = \int_{\Delta_*} |f(x) - \sigma_N(x)|\, dx \to 0 \qquad (N \to \infty)$$

Indeed, we have (see the explanations below)

$$\|f - \sigma_N\|_{L^*} \leqslant \frac{1}{\pi^n} \int_{\Delta_*} dx \left| \int_{\Delta_*} \Phi_N(u)\,[f(x+u) - f(x)]\, du \right| \leqslant$$

$$\leqslant \frac{1}{\pi^n} \int_{\Delta_*} \Phi_N(u)\, du \int_{\Delta_*} |f(x+u) - f(x)|\, dx =$$

$$= \frac{1}{\pi^n} \int_{\Delta_*} \Phi_N(u)\, \lambda(u)\, du \to 0, \qquad N \to \infty \tag{10}$$

where

$$\lambda(u) = \int_{\Delta_*} |f(x+u)-f(x)|\,dx$$

is a continuous (see below) periodic function with period 2π (with respect to each scalar argument) which is equal to zero for $u = 0$. According to Theorem 1, the last integral in (10) involving this function tends to zero as $n \to \infty$ for this integral is nothing but the value of the Nth Fejér sum of the function $\lambda(u)$ at the point $u = 0$.

In the second relation (10) we have changed the order of integration, which is legitimate because in the theory of the Lebesgue integral it is proved that, for any nonnegative measurable function, the integrations with respect to different variables can be interchanged without altering the result (see Fubini's theorem in § 19.3, property 19).

The continuity of $\lambda(u)$ is established as follows:

$$|\lambda(u)-\lambda(u^0)| \le \int_{\Delta_*} |f(x+u)-f(x+u^0)|\,dx \to 0 \qquad (u \to u^0)$$

The last property (the convergence to zero) follows from Theorem 6, § 14.4, in which we should assume that $f = 0$ outside a finite cube with sufficiently large dimensions containing the cube Δ_* (with the same centre).

Theorem 3. *System of functions* (1) *is complete in* C^* (*and, consequently, in* L^* *and* L_2^* *as well; see* § 14.9).

The assertion of this theorem follows from Theorem 1 if we take into account that, for any N, Fejér's sum $\sigma_N(x)$ is an expression of the form

$$\sum_{|k_j| \le N} \alpha_k e^{ikx} = T_N(x) \tag{11}$$

where α_k are some numbers, that is σ_N is a finite linear combination of functions belonging to system (1) (in other words, σ_N is a trigonometric polynomial of order N). Indeed, the one-dimensional Fejér kernel can be written in the form (see § 15.10 (6))

$$F_N(x) = \frac{1}{2}+\sum_{k=1}^{N} \frac{N+1-k}{N+1}\cos kx = \frac{1}{2}\left[1+\sum_{1}^{N} \frac{N+1-k}{N+1}\left(e^{ikx}+e^{-ikx}\right)\right]$$

whence it is seen that the kernel $\Phi_N(t-x) = \prod_{j=1}^{n} F_N(t_j-x_j)$ is a trigonometric polynomial in x of order N with vector parameter t. The multiplication of $\Phi_N(t-x)$ by $f(t)$ and the integration of that product with respect to the parameter $t \in \Delta_*$ (see (6)) lead to a trigonometric polynomial in x of order N.

Theorem 4. *Fourier series* (3) *of a function* $f \in L_2^*$ *is convergent in the mean square to* f (i.e. in the sense of $L_2 = L_2(\Delta_*)$).

This follows from the completeness in L_2^* of the system of functions (1) orthonormal on Δ_* and from Theorem 1, § 14.6 proved in the general theory of series with respect to orthogonal systems.

Any series of the form

$$\sum c_k e^{ikx} \tag{12}$$

(where c_k are numbers) extending over all the possible integral vectors $\boldsymbol{k}$ is called a *trigonometric series (in complex form).*

Theorem 5. *If series* (12) *is the Fourier series of a function f belonging to L^*, this function is determined uniquely to within its values on a set of measure zero.*

Indeed, let (12) be the Fourier series of a function $f \in L^*$. Then

$$c_k = \frac{1}{(2\pi)^n} \int_{\Delta_*} f(t)\, e^{-ikt}\, dt$$

This formula uniquely specifies, for any N, Fejér's sum $\sigma_N(f, x)$ of the function f in question because the kernel $\Phi_N(u)$ of this sum is a definite linear combination of the functions e^{-iku} $(|k_j| \leqslant N)$. By Theorem 2, the sum $\sigma_N(x)$ tends to $f(x)$ as $N \to \infty$ in the sense of the convergence in the space $L(\Delta_*)$. This obviously determines the function f uniquely to within its values assumed on a set of measure zero.

At the end of § 15.5 we already mentioned Kolmogorov's theorem asserting the existence of a function $f_0 \in L^*$ dependent on one variable whose Fourier series is divergent throughout the real axis. Hence, from the point of view of the pointwise convergence the Fourier series of the function f_0 has nothing in common with f_0 and does not represent it in this sense. But this does not mean that there is no connection at all between f_0 or, generally, between a function $f \in L^*$ and its Fourier series. On the contrary, they are closely related but in the sense different from that of the pointwise convergence. Indeed, as has been shown, the Fejér sum σ_N of order N of any function $f \in L^*$ which is closely connected with the Fourier series of f possesses the property that (Theorem 2) it converges to f with respect to the metric of L^*. Theorem 5 also establishes a relationship between the functions $f \in L^*$ and their Fourier series; according to this theorem any two different (not coinciding almost everywhere) functions belonging to L^* possess different Fourier's series. As will be seen later (§ 16.11), it is this property that provides a foundation for the generalization of the notion of functions $f \in L^*$ to more general objects (the so-called periodic generalized functions).

Formula (5) for the Nth sum of the Fourier series of $f(x)$ can also be written in the form

$$\begin{aligned} S_N(x) &= \frac{1}{\pi^n} \int_{\Delta_*} \prod_{j=1}^{n} \left(\frac{1}{2} + \cos(t_j - x_j) + \ldots + \cos N(t_j - x_j)\right) f(t)\, dt = \\ &= \frac{1}{\pi^n} \int_{\Delta_*} \sum_{k_1=0}^{N}{}' \ldots \sum_{k_n=0}^{N}{}' \cos k_1(t_1 - x_1) \ldots \cos k_n(t_n - x_n)\, f(t)\, dt \end{aligned} \tag{13}$$

where the prime indicates that the functions $\cos k_j(t_j - x_j)$ with $k_j = 0$ $(j = 1, \ldots, n)$ under the summation sign $\sum'$ should be replaced by $1/2$.

Expression (13) admits of some further useful modifications; we shall limit ourselves to presenting them for the two-dimensional case ($n = 2$). For $n > 2$ they are quite analogous.

On putting

$$\left.\begin{aligned}
a_{kl} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos ku\cos lv\, f(u, v)\, du\, dv\\
b_{kl} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sin ku\sin lv\, f(u, v)\, du\, dv\\
c_{kl} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\cos ku\sin lv\, f(u, v)\, du\, dv\\
d_{kl} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sin ku\cos lv\, f(u, v)\, du\, dv
\end{aligned}\right\}\qquad(14)$$

and

$$\begin{aligned}
A_{kl} = a_{kl}\cos kx\cos ly+b_{kl}\sin kx\sin ly+&\\
+c_{kl}\cos kx\sin ly+d_{kl}\sin kx\cos ly&\qquad(15)
\end{aligned}$$

we obtain

$$\begin{aligned}
S_N(x, y) &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\sum_{k=0}^{N}{}'\sum_{l=0}^{N}{}'\cos k(u-x)\cos l(v-y) f(u, v)\, du\, dv =\\
&= \sum_{0}^{N}{}'\sum_{0}^{N}{}' A_{kl}(x, y)\qquad(16)
\end{aligned}$$

where the prime in the second sum indicates that A_{00}, A_{k0} and A_{0l} $(k, l \neq 0)$ must be in fact replaced by $\frac{A_{00}}{4}$, $\frac{A_{k0}}{2}$ and $\frac{A_{0l}}{2}$ respectively, and the same convention applies to the first sum. Thus,

$$\begin{aligned}
S_N(x, y) = \frac{a_{00}}{4}+\frac{1}{2}\sum_{1}^{N}(a_{k0}\cos kx+d_{k0}\sin kx)+&\\
+\frac{1}{2}\sum_{1}^{N}(a_{0l}\cos ly+c_{0l}\sin ly)+\sum_{1}^{N}\sum_{1}^{N}A_{kl}(x, y)&\qquad(16')
\end{aligned}$$

Fourier series (3) of the function f with respect to system (1) can be transformed formally into the series

$$\begin{aligned}
f(x, y) \sim \frac{a_{00}}{4}+\frac{1}{2}\sum_{1}^{\infty}(a_{k0}\cos kx+d_{k0}\sin kx)+&\\
+\frac{1}{2}\sum_{1}^{\infty}(a_{0l}\cos ly+c_{0l}\sin ly)+\sum_{1}^{\infty}\sum_{1}^{\infty}A_{kl}(x, y)&\qquad(17)
\end{aligned}$$

It should be noted that series (17) can also be derived in a direct manner. The matter is that the trigonometric functions

$$\cos kx \cos ly,\ \sin kx \sin ly,\ \cos kx \sin ly,\ \sin kx \cos ly \qquad (18)$$
$$(k, l = 0, +1, +2, \dots)$$

form, as can readily be shown, an orthogonal system on the rectangle $\Delta_* = \{-\pi \leqslant x, y \leqslant \pi\}$. The expansion of f into the Fourier series with respect to this system exactly results in series (17).

Thus, series (17) in which the numbers a_{kl}, b_{kl}, c_{kl} and d_{kl} are computed by formulas (14) is the Fourier series of f with respect to system (18), and numbers (14) are the Fourier coefficients of f relative to system (18).

It follows that

$$A_{kl} = a_{kl} \cos kx \cos ly + b_{kl} \sin kx \sin ly +$$
$$+ c_{kl} \cos kx \sin ly + d_{kl} \sin kx \cos ly =$$
$$= c^*_{kl} e^{i(kx+ly)} + c^*_{k,-l} e^{i(kx-ly)} + c^*_{-k,l} e^{i(-kx+ly)} + c^*_{-k,-l} e^{-i(kx+ly)}$$

where (in distinction to the foregoing formulas) c^*_{kl} denotes the corresponding complex Fourier coefficient (in order to distinguish between such a coefficient and c_{kl} in (14)).

We also note that system (18) is complete in C^* (and, consequently, in L^* and L^*_2 as well), which follows from the completeness of system (1) and from the fact that the functions forming system (1) are finite linear combinations of functions belonging to system (18).

Finally, we also mention that if $f \in L^*$ is a real function, its Fourier coefficients a_{kl}, b_{kl}, c_{kl} and d_{kl} are real while, generally speaking, the coefficients c^*_{kl} are complex but satisfy the condition that $c^*_{-k,-l}$ and $\bar{c}^*_{kl}$ are mutually complex conjugate: $c^*_{-k,-l} = \bar{c}^*_{k,l}$.

Now we return to the general n-dimensional case using the former notation. If a function $f \in C^*$ possesses the partial derivative $\frac{\partial f}{\partial x_n} \in C^*$, the expression of its any Fourier coefficient c_k with $k_n \neq 0$ admits of integration by parts (see § 15.7):

$$c_k(f) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} e^{-i \sum_1^{n-1} k_j t_j}\, dt_1 \dots dt_{n-1} \int_0^{2\pi} e^{-ik_n t_n} f(t_1, \dots, t_n)\, dt_n =$$
$$= \frac{1}{(2\pi)^n} \int_0^{2\pi} \dots \int_0^{2\pi} e^{-i \sum_1^{n-1} k_j t_j}\, dt_1 \dots dt_{n-1} \frac{1}{ik_n} \times$$
$$\times \int_0^{2\pi} e^{-ik_n t_n} \frac{\partial f}{\partial x_n}(t_1, \dots, t_n)\, dt_n = \frac{1}{ik_n} c_k\left(\frac{\partial f}{\partial x_n}\right)$$

Generally, if $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ is a given nonnegative integral vector (i.e. λ_j are nonnegative integers; $j = 1, \dots, n$) and $f^{(s)} \in C^*$ for every nonnegative integral vector $\boldsymbol{s} \leq \boldsymbol{\lambda}$ (i.e. $s_j \leq \lambda_j$; $j = 1, \ldots, n$), the integration by parts performed the corresponding number of times yields

$$c_k(f) = \frac{1}{i^{|\lambda|}k^{\lambda}}\, c_k(f^{(\lambda)}) \qquad \left(|\boldsymbol{\lambda}| = \sum_{1}^{n} \lambda_j,\ k^{\lambda} = k_1^{\lambda_1}\ldots k_n^{\lambda_n}\right) \tag{19}$$

where we should put $\lambda_j = 0$ and $k_j^{\lambda_j} = 0^0 = 1$ when $k_j = 0$. As to the numbers $c_k(f^{(\lambda)})$, they are the Fourier cfficients ooef the derivative $f^{(\lambda)}$.

Formula (19) does in fact hold in the more general case when the function $f \in L^*$ has the (Sobolev) generalized derivatives (see § 19.5) $f^{(k)} \in L^*$ $(k_j \leq \lambda_j)$.

Theorem 6. *Let $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda)$ be a vector with positive integral components λ equal to each other and let a function $f(\boldsymbol{x}) = f(x_1, \ldots, x_n)$ belong to C^* together with its partial derivatives $f^{(k)}$ of order $\boldsymbol{k} \leq \boldsymbol{\lambda}$ $(k_j \leq \lambda)$. Also, let the inequalities*

$$\frac{1}{(2\pi)^n} \int_{\Delta_*} |f^{(l)}(x)|^2\, dx \leq M^2 \tag{20}$$

hold for any vector $\boldsymbol{l} = (l_1, \ldots, l_n)$ whose components l_j are equal to 0 or to $\lambda > 0$.

Then the deviation of the Nth Fourier sum $S_N(\boldsymbol{x})$ of the function $f(\boldsymbol{x})$ from $f(\boldsymbol{x})$ satisfies the inequality

$$|f(x) - S_N(x)| \leq \frac{CM}{N^{\lambda - 1/2}}$$

where the constant C depends solely on λ and is independent of M and N.

Proof. The remainder of the Fourier series of f corresponding to the partial sum S_N is

$$\varrho_N(x) = \sum_{\max|k_j| > N} c_k e^{ikx} \tag{21}$$

Let us choose a natural number m satisfying the inequalities $0 \leq m < n$ and consider the set Ω_m of the integral vectors $\boldsymbol{k} = (k_1, \ldots, k_n)$ whose coordinates satisfy the relations

$$\left.\begin{array}{ll} |k_j| = 0 & (j = 1, \ldots, m \text{ if } m > 0) \\ |k_{m+1}| > N,\ |k_j| \geq 1 & (j = m+2, \ldots, n \text{ if } m < n-1) \end{array}\right\} \tag{22}$$

The same symbol Ω_m will also be used for denoting any set of vectors $\boldsymbol{k}$ which can be transformed into the above-described set by the corresponding permutation of the indices j. To each given m there obviously corresponds a finite system of such sets Ω_m; the set of all $\boldsymbol{k}$'s over which sum (21) extends is

$$\{\boldsymbol{k} \colon \max|k_j| > N\} = \sum_{m=0}^{n-1} \sum \Omega_m \tag{23}$$

where the second sum extends, for every m, over all the different sets Ω_m.

Let us estimate the sum of the moduli of only those terms of series (21) which correspond to the vectors $\boldsymbol{k}$ belonging to some Ω_m. For definiteness, let us assume that Ω_m is described as in (22). For the other sets of the type of Ω_m the calculations are quite analogous and lead to the same estimates. We have

$$\sum_{k\in\Omega_m} |c_k| = \sum_{k\in\Omega_m} \left|\frac{1}{k_{m+1}^{\lambda}\dots k_n^{\lambda}} c_k\left(\frac{\partial^{(n-m)\lambda}f}{\partial x_{m+1}^{\lambda}\dots\partial x_n^{\lambda}}\right)\right| \leqslant$$

$$\leqslant \left(\sum_{k\in\Omega_m} \frac{1}{k_{m+1}^{2\lambda}\dots k_n^{2\lambda}}\right)^{1/2} \left(\sum_{k\in\Omega_m} \left|c_k\left(\frac{\partial^{(n-m)\lambda}f}{\partial x_{m+1}^{\lambda}\dots\partial x_n^{\lambda}}\right)\right|^2\right)^{1/2} \leqslant$$

$$\leqslant c'M\left(\sum_{k_{m+1}=N}^{\infty} \frac{1}{k_{m+1}^{2\lambda}} \sum_{k_{m+2}=1}^{\infty} \frac{1}{k_{m+2}^{2}} \dots \sum_{k_n=1}^{\infty} \frac{k}{k_n^{2}}\right)^{1/2} \leqslant \frac{c''M}{N^{\lambda-1/2}} \tag{24}$$

(see (19) and (20)). Consequently

$$|\varrho_N(x)| \leqslant \frac{CM}{N^{\lambda-1/2}} \tag{25}$$

where C is a constant independent of M and N.

It follows from (25) that the Fourier series of f is uniformly convergent; as is known, it is also convergent to f in the mean square. Therefore it converges uniformly to that very function $f(x)$ (see the lemma below), and consequently

$$|\varrho_N(x)| = |f(x)-S_N(x)| \leqslant \frac{CM}{N^{\lambda-1/2}}$$

which is what we set out to prove.

Lemma 1. *If a series of the form*

$$u_0(x)+u_1(x)+u_2(x)+\dots$$

consisting of functions $u_j(x)$ $(j=0, 1, 2, \dots)$ *continuous in a domain* Ω *is convergent in the mean square to a continuous function* $S(x)$ *and, at the same time, converges uniformly on* Ω *to a function* $\sigma(x)$, *then* $S(x)=\sigma(x)$ *on* Ω.

Proof. Let $S_N(x)$ be the sum of the first N terms of the series and let

$$\varkappa_N = \max_{x\in V} |\sigma(x)-S_N(x)|$$

where $V \subset \Omega$ is an arbitrary ball. By the hypothesis, $\varkappa_N \to 0$ (§ 11.7). Therefore

$$\left(\int_V |S(x)-\sigma(x)|^2\,dx\right)^{1/2} \leqslant$$

$$\leqslant \left(\int_V |S(x)-S_N(x)|^2\,dx\right)^{1/2} + \left(\int_V |S_N(x)-\sigma(x)|^2\,dx\right)^{1/2} \leqslant$$

$$\leqslant \left(\int_V |S(x)-S_N(x)|^2\,dx\right)^{1/2} + \varkappa_N\sqrt{|V|} \to 0 \qquad (N\to\infty)$$

Consequently, the leftmost member of these relations is equal to zero, and, since the functions $S(x)$ and $\sigma(x)$ are continuous, they are indentically equal to each other.

Theorem 6 has been proved. It does in fact hold (and the proof remains the same) under the more general assumption that the partial derivatives in (20) are understood as generalized ones in Sobolev's sense.

Now, for an arbitrary $\eta > 0$, we form the set K_η (a "cross-shaped" region) which is the union of the n sets $\{|u_j| < \eta\}$ $(j = 1, \ldots, n)$ and proceed to the following theorem.

Theorem 7. *For any function $f \in L'^*$ (or $f \in L^*$) and an arbitrary $\eta > 0$, the equality*

$$S_N(x)-f(x) = \frac{1}{\pi^n}\int_{K_\eta}\prod_{j=1}^{n} D_N(u_j)\,[f(x+u)-f(x)]\,du+o(1) \qquad (N \to \infty) \quad (26)$$

holds uniformly on any set Ω of points x on which f is bounded.

Proof. We shall confine ourselves to the two-dimensional case. We have the relation

$$\int_{\eta}^{\pi}\int_{\eta}^{\pi} D_N(u)\,D_N(v)\,[f(x+u,y+v)-f(x,y)]\,du\,dv =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sin\left(N+\frac{1}{2}\right)u\,g(u)\,\sin\left(N+\frac{1}{2}\right)v\,g(v)\times$$

$$\times[f(x+u,y+v)-f(x,y)]\,du\,dv \to 0 \qquad (N \to \infty) \quad (27)$$

which holds uniformly on the domain Ω where f is bounded. Here

$$g(u) = \begin{cases} \dfrac{1}{2\sin\dfrac{u}{2}} & \text{for } \eta < u < \pi \\ 0 & \text{outside } [\eta, \pi] \end{cases}$$

Property (27) follows from a simple lemma generalizing Lemma 2 of § 15.4 to the two-dimensional case.

A property analogous to (27) obviously also holds for the integral on the left-hand side of (27) when its domain of integration is replaced by a domain symmetric to the former with respect to any of the coordinate axes and with respect to the origin.

Remark. A thorough investigation shows that in the general case the cross-shaped region K_η in formula (26) cannot be replaced by the cube $\Delta_\eta = \{|u_j| \leq \eta, j = 1, \ldots, n\}$; here lies an essential distinction between the Fourier series for functions of many variables and those for functions of one variable (cf. § 15.3 (8); see also § 16.8 (17)).

§ 15.12. Algebraic Polynomials. Chebyshev's Polynomials

To establish a relationship between algebraic and trigonometric polynomials (more precisely, even trigonometric polynomials) let us resort to the equality

$$\cos n\theta+i\sin n\theta = (\cos\theta+i\sin\theta)^n =$$
$$= (\cos\theta)^n+iC_n^1(\cos\theta)^{n-1}\sin x+i^2C_n^2(\cos\theta)^{n-2}\sin^2\theta+\ldots$$

The terms on the right-hand side involving even powers of $\sin\theta$ are real while those with odd powers of $\sin\theta$ are imaginary. Besides, $(\sin\theta)^{2m} = (1-\cos^2\theta)^m$ $(m = 1, 2, \ldots)$. It follows that for any natural n we have

$$\cos n\theta = Q_n(\cos\theta)$$

where $Q_n(x) = \cos n\,\mathrm{arc}\cos x = \alpha_0^{(n)}+\alpha_1^{(n)}x+\ldots+\alpha_n^{(n)}x^n$ is an algebraic polynomial of degree n with real coefficients. It is called *Chebyshev's polynomial of degree n.*

Obviously,

$$Q_0(x) \equiv 1$$
$$Q_1(x) = \cos\mathrm{arc}\cos x = x$$
$$Q_2(x) = 2(\cos\mathrm{arc}\cos x)^2-1 = 2x^2-1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

It follows that every even trigonometric polynomial

$$T_n(\theta) = \frac{\alpha_0}{2}+\sum_1^n\alpha_k\cos k\theta$$

is transformed with the aid of the substitution $\theta = \mathrm{arc}\cos x$ (or $x = \cos\theta$) into the corresponding algebraic polynomial of degree n (this substitution specifies a homeomorphic, that is continuous and one-to-one, mapping of the closed interval $0 \leqslant \theta \leqslant \pi$ on the closed interval $-1 \leqslant x \leqslant 1$):

$$P_n(x) = T_n(\mathrm{arc}\cos x) = \frac{\alpha_0}{2}+\sum_1^n\alpha_k\cos k\,\mathrm{arc}\cos x$$

It is important that, conversely, the substitution $x = \cos\theta$ $(0 \leqslant \theta \leqslant \pi,$ $-1 \leqslant x \leqslant 1)$ transforms an arbitrary algebraic polynomial

$$P_n(x) = a_0+a_1x+\ldots+a_nx^n$$

of degree n into the corresponding even trigonometric polynomial

$$T_n(\theta) = P_n(\cos\theta) = \frac{\alpha_0}{2}+\sum_1^n\alpha_k\cos k\theta$$

(see § 8.11 (8)) where the numbers α_k $(k = 0, 1, \ldots, n)$ depend on P_n.

§ 15.13. Weierstrass' Theorem

Theorem 1 (Weierstrass). *The system of functions*

$$1, x, x^2, \ldots \tag{1}$$

is complete in the space $C(a, b)$ of continuous functions. In other words, for any function $f(x)$ continuous on $[a, b]$ and any $\varepsilon > 0$ there is an algebraic polynomial $P_n(x)$ such that

$$|f(x)-P_n(x)| < \varepsilon \qquad \text{for all } x \in [a, b] \tag{2}$$

We shall first carry out the proof for the interval $[-1, +1]$. Let $f(x)$ be a continuous function defined on $[-1, +1]$. Then $f(\cos t)$ is a continuous function on the interval $[0, \pi]$, and, since the system of the functions

$$1, \cos t, \cos 2t, \ldots$$

is complete in $C(0, \pi)$ (see Theorem 3, § 15.5), given any $\varepsilon > 0$, there is a trigonometric polynomial $T_n(t)$ such that $|f(\cos t)-T_n(t)| < \varepsilon$.

The trigonometric polynomial $T_n(t)$ can be written in the form $T_n(t) = P_n(\cos t)$ where P_n is an algebraic polynomial of degree n. Hence,

$$|f(\cos t)-P_n(\cos t)| < \varepsilon \qquad (0 \leq t \leq \pi)$$

and therefore

$$|f(x)-P_n(x)| < \varepsilon \qquad (-1 \leq x \leq 1)$$

which proves the theorem for the closed interval $[-1, +1]$.

In the general case when we are given a continuous function $f(x)$ defined on a closed interval $[a, b]$ we can make the substitution

$$x = a+\frac{b-a}{2}(z+1)$$

which specifies a linear one-to-one mapping of the interval $[-1, +1]$ of variation of z on the interval $[a, b]$ of variation of x. Then the function $f(x)$ goes into the function

$$F(z) = f\left(a+\frac{b-a}{2}(z+1)\right)$$

which is continuous on $[-1, +1]$; by what has already been proved, for the latter function there is a polynomial $P_n(z)$ such that

$$|F(z)-P_n(z)| < \varepsilon, \quad z \in [-1, +1]$$

The inverse transformation leads to the inequality

$$|f(x)-R_n(x)| < \varepsilon, \qquad x \in [a, b]$$

where $R_n(x) = P_n\left(\frac{2(x-a)}{b-a}-1\right)$ is obviously a polynomial.

The theorem has been proved.

It should be stressed that the degree n of the polynomial $P_n(x)$ (for which inequality (2) holds with a given ε) depends on ε. Generally speaking, when $\varepsilon \to 0$, the degree n tends to infinity.

§ 15.14. Legendre's Polynomials

Let us take the functions

$$L_0(x) \equiv 1,\ L_n(x) = \frac{1}{2^n n!}\,\frac{d^n(x^2-1)^n}{dx^n} \qquad (n = 1, 2, \ldots) \qquad (1)$$

We shall consider them on the closed interval $[-1, +1]$. For each $l = 0, 1, 2, \ldots$, the function $L_l(x)$ is obviously a polynomial exactly of degree l. On differentiating the expression $(x^2-1)^n = (x-1)^n(x+1)^n$ according to Leibniz' rule we obtain

$$\frac{d^n(x^2-1)^n}{dx^n} = n!(x+1)^n + \ldots$$

where the terms denoted by the dots contain the factor $x-1$. Therefore $L_l(1) = 1$ $(l = 0, 1, \ldots)$. Putting $m < n$ and integrating by parts we receive

$$2^n n! \int_{-1}^{+1} L_n(x) x^m\, dx = \int_{-1}^{+1} x^m \frac{d^n(x^2-1)^n}{dx^n}\, dx =$$

$$= x^m \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\Bigg|_{-1}^{1} - m \int_{-1}^{+1} x^{m-1} \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\, dx =$$

$$= -m \int_{-1}^{+1} x^{m-1} \frac{d^{n-1}(x^2-1)}{dx^{n-1}}\, dx = \ldots = 0$$

The first summand in the third member is equal to zero because $+1$ and -1 are the roots of multiplicity n of $(x^2-1)^n$ and, consequently, the derivative $\frac{d^m}{dx^m}(x^2-1)^n$ $(m = 0, 1, \ldots, n-1)$ vanishes at the points ± 1. To the last of the integrals written explicitly in the above relations which contains x^{m-1} (instead of the original power x^m) we again apply integration by parts, which again reduces the power of x by unity, etc. The continuation of this process brings the power of x involved to zero.

The equality we have obtained shows that system (1) is orthogonal on $[-1, +1]$.

Let us evaluate the integral of the square of $L_n(x)$ over the interval $[-1, +1]$. To this end we put $u_n(x) = (x^2-1)^n$. Then

$$\int_{-1}^{+1} u_n^{(n)}(x)\, u_n^{(n)}(x)\, dx = -\int_{-1}^{+1} u_n^{(n-1)}(x)\, u_n^{(n+1)}(x)\, dx =$$

$$= \int_{-1}^{+1} u_n^{(n-2)}(x)\, u_n^{(n+2)}(x)\, dx = \ldots = (-1)^n \int_{-1}^{+1} u_n u_n^{(2n)}\, dx =$$

$$= (2n)! \int_{-1}^{+1} (1-x)^n (1+x)^n\, dx$$

But we have

$$\int_{-1}^{+1}(1-x)^n(1+x)^n\,dx=\frac{n}{n+1}\int_{-1}^{+1}(1-x)^{n-1}(1+x)^{n+1}\,dx=\dots$$

$$\dots=\frac{n(n-1)\dots 1}{(n+1)(n+2)\dots 2n}\int_{-1}^{+1}(1+x)^{2n}\,dx=\frac{(n!)^2}{(2n)!\,(2n+1)}\,2^{2n+1}$$

and therefore $\int_{-1}^{+1}L_n^2(x)\,dx=\frac{2}{2n+1}$. Consequently the normalized polynomials are of the form

$$\varphi_n(x)=\sqrt{\frac{2n+1}{2}}\,L_n(x)=\sqrt{\frac{2n+1}{2}}\,\frac{1}{2^n n!}\,\frac{d^n(x^2-1)^n}{dx^n}\qquad(n=0,1,\dots)\quad(2)$$

On the other hand, if we apply the orthogonalization process developed in § 14.7 to the system $1, x, x^2, \dots$ on the interval $[-1, +1]$ we shall obtain the complete orthonormal system $P_0(x), P_1(x), P_2(x), \dots$ on $[-1, +1]$.

At the nth stage of this process ($n = 0, 1, 2, \dots$) the polynomial $P_n(x)$ of degree n is uniquely determined to within its sign by imposing the conditions that, in the first place, it is normalized $\left(\text{i.e.}\int_{-1}^{+1}P_n^2\,dx=1\right)$ and, in the second place, it is orthogonal to $P_0, P_1, \dots, P_{n-1}$. But, as is readily seen, the polynomial $\varphi_n(x)$ possesses all the indicated properties, and therefore it is identically equal to one of the polynomials $+P_n$ or $-P_n$ depending on which of them has a positive coefficient in x^n because $\varphi_n(x)$ possesses this property.

The system $P_0, P_1, \dots$ being complete in $C(-1, +1)$, we have thus proved that the system of functions

$$\varphi_0, \varphi_1, \varphi_2 \dots \quad (3)$$

is not only orthonormal on $[-1, +1]$ but also complete in $C(-1, +1)$ (and, moreover, in $L_2(-1, +1)$).

The functions $\varphi_n(x)$ are called *Legendre's polynomials* (orthonormal on the closed interval $[-1, +1]$). The functions $L_n(x)$ are also called Legendre's polynomials (uniquely determined by the conditions $L_n(1) = 1$; $n = 0, 1, \dots$).

It follows that to Legendre's polynomials applies the general theory of orthogonal systems of functions. In particular, *any function* $f(x) \in$ $\in L_2(-1, +1)$ *can be expanded into the Fourier series*

$$f(x)=\sum_0^\infty (f,\varphi_k)\varphi_k(x)$$

with respect to Legendre's polynomials φ_k *which is convergent to* f *in the mean square on the interval* $[-1, +1]$.

Series with respect to Legendre's polynomials can be tested for the ordinary or uniform convergence by analogy with the theory of trigonometric

Fourier series whose theory was presented here. For instance, if it is known that a function f possesses a continuous derivative of the second order on the closed interval $[-1, +1]$, its Fourier series with respect to the system of Legendre's polynomials is uniformly convergent to that function on the interval. As in the theory of trigonometric Fourier series, the estimates for the remainder of the Fourier series of f with respect to Legendre's polynomials depend on the differentiability properties of f. Generally speaking, the smoother the function, the higher the speed of convergence of the series. It should also be mentioned that, as a rule, a series in Legendre's polynomials converges faster strictly inside the interval $[-1, +1]$ and slower at its end points.

CHAPTER 16

Fourier Integral. Generalized Functions

§ 16.1. Notion of Fourier Integral

In the foregoing chapter we dealt with periodic functions of period 2π belonging to the class L'^* (or, generally, to L^*). For each such function there exists its Fourier series

$$f(x) = \frac{a_0}{2}+\sum_1^\infty (a_k \cos kx+b_k \sin kx) = \sum_{-\infty}^{\infty} c_k e^{ikx} \tag{1}$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt \qquad (k = 0, 1, \ldots) \tag{2}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt \qquad (k = 1, 2, \ldots) \tag{3}$$

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} \, dt \qquad (k = 0, \pm 1, \pm 2, \ldots) \tag{4}$$

and

$$c_k = \frac{a_k - ib_k}{2}, \quad c_{-k} = \frac{a_k + ib_k}{2} \qquad (k = 0, 1, 2, \ldots; b_0 = 0)$$

In this chapter we shall be concerned with functions nonperiodic in the general case, which are defined throughout the real axis and belong to the class $L' = L'(-\infty, \infty)$ or to the more general class $L = L(-\infty, \infty)$ of Lebesgue integrable functions.

Every function $f \in L'$ is absolutely integrable in Riemann's improper sense (see § 14.2) on $(-\infty, \infty)$ while every function $f \in L$ is absolutely integrable in Lebesgue's sense over the real axis. All the results that will be established for $f \in L'$ also apply to $f \in L$ but in the latter case their justification involves the notion of the Lebesgue integral.

For a function $f \in L'$ the following integrals make sense for any real s:

$$a(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos st \, dt \tag{5}$$

$$b(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \sin st \, dt \tag{6}$$

and

$$c(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-ist} \, dt \tag{7}$$

These integrals exist because

$$|f(t) \cos st| \leqslant |f(t)| \in L' \tag{8}$$

for the integrand in integral (5), and similar estimates hold for (6) and (7).

It can readily be shown that the functions $a(s)$, $b(s)$ and $c(s)$ are continuous. Indeed, if f has a finite number of points of discontinuity, this follows from the uniform convergence of integrals (5), (6) and (7) because the integrand functions in these integrals are continuous with respect to (s, t) except at the points t where f is discontinuous. For the general proof of the continuity see Lemma 1 in § 16.2.

The functions $a(s)$, $b(s)$ and $c(s)$ are respectively analogues of the Fourier coefficients a_k, b_k and c_k of a periodic function; the latter are defined for discrete values of k while the functions $a(s)$, $b(s)$ and $c(s)$ depend on the continuously varying argument s.

The functions $a(s)$, $b(s)$ and $c(s)$ possess the properties

$$a(s) \to 0, b(s) \to 0 \quad \text{and} \quad c(s) \to 0 \qquad (s \to \infty)$$

analogous to the corresponding properties of the Fourier coefficients (see Lemma 1, § 15.4).

It would be natural to call the functions $a(s)$, $b(s)$ and $c(s)$ Fourier's cosine transform, Fourier's sine transform and Fourier's (complex) transform of the function f respectively; however, for the sake of symmetry, these names are used for the integrals differing from (5), (6) and (7) in some constant coefficients.

As an analogue of the term of a Fourier series it is natural to consider the function

$$a(s) \cos sx + b(s) \sin sx = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \cos s(t-x) \, dt =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[e^{is(t-x)} + e^{-is(t-x)}\right] dt = c(s)e^{isx} + c(-s)e^{-isx}$$

dependent on x and on the parameter s. For a real function $f(t)$ we have $c(-s) = \overline{c(s)}$.

The expression

$$S_N(x) = \int_0^N (a(s)\cos sx + b(s)\sin sx)\,ds =$$

$$= \frac{1}{\pi}\int_0^N ds \int_{-\infty}^{\infty} f(t)\cos s(t-x)\,dt =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\,dt \int_0^N \cos s(t-x)\,ds =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin N(t-x)}{t-x}\,dt = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x+t)\frac{\sin Nt}{t}\,dt =$$

$$= \frac{1}{\pi}\int_{-\eta}^{\eta} f(x+t)\frac{\sin Nt}{t}\,dt + o(1) \qquad (N \to \infty,\ \eta > 0) \tag{9}$$

is an analogue of the Nth Fourier sum; the summand $o(1)$ on the right-hand side of (9) uniformly tends to zero with respect to all x belonging to any interval $[a, b]$. As will be shown, under some general assumptions, expression (9) approximates the given function f in the sense that

$$f(x) = \lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \int_{-\infty}^{+\infty} f(x+t)\frac{\sin Nt}{t}\,dt$$

The latter relation is known as *Fourier's single-integral formula*. We shall call the expression $S_N(x)$ (determined by formula (9)) Fourier's single integral.

Since the integrand in the first integral (9) is continuous in s, this integral exists. In the second equality (9) we have changed the order of integration. By Theorem 2, § 13.14, this is legitimate when f has a finite number of discontinuities because in this case the integral

$$\int_{-\infty}^{\infty} f(t)\cos s(t-x)\,dt$$

is uniformly convergent with respect to $s \in [0, N]$ and the integrand is continuous in (t, s) except at a finite number of the points t. For a more general function f this is justified by the lemma which will be proved in § 16.2. Finally, the number $\eta > 0$ in the last integral (9) can be chosen quite arbitrarily for if, for instance, some $\eta > 0$ has been chosen, we can write

$$\int_{\eta}^{\infty} f(x+t)\frac{\sin Nt}{t}\,dt = \int_{\eta}^{l} f(x+t)\frac{1}{t}\sin Nt\,dt + \lambda_N(x) \tag{10}$$

provided that $l > \eta$ is taken so large that

$$|\lambda_N(x)| = \left|\int_l^\infty f(x+t)\frac{1}{t}\sin Nt\,dt\right| < \frac{1}{l}\int_l^\infty |f(x+t)|\,dt <$$

$$< \frac{1}{l}\int_{l+a}^\infty |f(u)|\,du < \varepsilon \tag{11}$$

for x ranging over $[a, b]$. Then the integral on the right-hand side of (10) also becomes less than ε for $N > N_0$ provided that N_0 is sufficiently large (see Lemma 2, § 15.4 where we should put $g(t) = \frac{1}{t}$ for $t \in [\eta, l]$ and $g(t) = 0$ outside $[\eta, l]$).

Thus, the integral on the left-hand side of (10) becomes less than ε in its absolute value for all $x \in [a, b]$ when $N > N_0$ and therefore it tends to zero as $N \to \infty$ uniformly on $[a, b]$.

The function $S_N(x)$ can also be written in the complex form

$$S_N(x) = \int_0^N \left(c(s)e^{isx}+c(-s)e^{-isx}\right) ds = \int_{-N}^N c(s)e^{isx}\,ds =$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-N}^N e^{isx}\,ds\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(t)e^{-ist}\,dt \tag{12}$$

§ 16.2. Lemma on Change of Order of Integration

Lemma 1. *Let f and φ be functions belonging to $L'(0, \infty)$ (or to $L(0, \infty)$) and let $\lambda(s, t)$ $(0 \leq s, t < \infty)$ be a bounded continuous function $(|\lambda(s, t)| \leq K)$. Then the integral*

$$\mu(s) = \int_0^\infty \lambda(s, t) f(t)\,dt \tag{1}$$

is a bounded continuous function of s, and the equality

$$\int_0^\infty \varphi(s)\,ds\int_0^\infty \lambda(s, t) f(t)\,dt = \int_0^\infty f(t)\,dt\int_0^\infty \lambda(s, t)\,\varphi(s)\,ds \tag{2}$$

holds.

Proof. For an arbitrary $\varepsilon > 0$ we can take finite continuous functions f_1 and φ_1 such that

$$\int_0^\infty |f(t)-f_1(t)|\,dt < \frac{\varepsilon}{K} \quad \text{and} \quad \int_0^\infty |\varphi(t)-\varphi_1(t)|\,dt < \frac{\varepsilon}{K}$$

Then

$$\mu(s) = \int_0^N \lambda(s,\, t) f_1(t)\, dt + \eta(s) \tag{3}$$

where N is chosen so large that the closed interval $[0, N]$ contains the support of f_1 and

$$|\eta(s)| = \left| \int_0^\infty [f(t) - f_1(t)]\, \lambda(s,\, t)\, dt \right| \leqslant K \int_0^\infty |f(t) - f_1(t)|\, dt < \varepsilon \tag{4}$$

Since the integral on the right-hand side of (3) is a continuous and bounded function of s, Lemma 1, § 13.14 implies that the function $\mu(s)$ is continuous and also bounded ($|\mu(s)| \leqslant NK \max |f_1(t)| + \varepsilon$). Consequently, the outer integral on the left-hand side of (2) (taken with respect to s) makes sense. The existence of the integral on the right-hand side of (2) is proved in like manner.

Now we can write (see the explanations below)

$$\begin{aligned}\int_0^\infty \varphi(s)\, ds \int_0^\infty \lambda(s,\, t) f(t)\, dt &= \int_0^N \varphi(s)\, ds \int_0^N \lambda(s,\, t) f(t)\, dt + o(1) = \\ &= \int_0^N f(t)\, dt \int_0^N \varphi(s) \lambda(s,\, t)\, ds + o(1) = \\ &= \int_0^\infty f(t)\, dt \int_0^\infty \varphi(s) \lambda(s,\, t)\, dt \qquad (N \to \infty)\end{aligned}$$

The first of the above equalities follows from the estimation of the integral $\int_N^\infty \varphi(s)\, ds \int_0^N \lambda(s,\, t) f(t)\, dt$:

$$\left| \int_N^\infty \int_0^N + \int_0^N \int_N^\infty + \int_N^\infty \varphi(s)\, ds \int_N^\infty f(t) \lambda(s,\, t)\, dt \right| \leqslant$$

$$\leqslant K \left(\int_N^\infty \int_0^N + \int_0^N \int_N^\infty + \int_N^\infty |\varphi(s)|\, ds \int_N^\infty |f(t)|\, dt \right) = o(1) \qquad (N \to \infty)$$

The last equality is justified in a similar way. It should be taken into account that the integral in the third member in the equalities differs from the rightmost integral by $o(1)$ ($N \to \infty$) and therefore tends to the latter as $N \to \infty$.

Remark 1. Lemma 1 also applies to the case when the scalar variables s and t are replaced by vector variables $\boldsymbol{s} = (s_1, \ldots, s_m)$ and $\boldsymbol{t} = (t_1, \ldots, t_n)$ $(0 \leq s_j, t_k < \infty)$.

Remark 2. Equality (2) also follows from Fubini's theorem proved in the theory of the Lebesgue integral (see §§ 19.3 and 19.4, property 19).

§ 16.3. Convergence of Fourier's Single Integral

An important property of Fourier's single integral of a function f is that, under some rather general assumptions, it converges to the latter as $N \to \infty$, that is

$$f(x) = \lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t+x)\frac{\sin Nt}{t}\,dt \tag{1}$$

This property is implied by the following important lemma establishing a close relationship between Fourier's integrals and series.

Lemma 1. *Let two functions $f \in L' = L'(-\infty, \infty)$ or $f \in L$ and $f_* \in L'^*$ or $f_* \in L^*$ (this means that f_* is periodic with period 2π) coincide on a closed interval $[a, b]$ (i.e. $f(x) = f_*(x)$ for $x \in [a, b]$).*
Then, for any $x \in (a, b)$, there holds the relation

$$\lim_{N\to\infty} [S_N(x) - S_N^*(x)] = 0 \tag{2}$$

which is uniform with respect to x belonging to any closed interval $[a', b'] \subset \subset (a, b)$, the function $S_N(x)$ being Fourier's single integral of f and $S_N^*(x)$ being the Nth partial Fourier sum of the function f_*.

Proof. Let us set a value $x \in (a, b)$ and let the closed interval $[a', b'] \subset \subset (a, b)$ contain x. On putting

$$\eta = \min\{a'-a, b-b'\}$$

we can write the relation (see § 16.1 (9))

$$S_N(x) = \frac{1}{\pi} \int_{-\eta}^{\eta} f(x+t)\frac{\sin Nt}{t}\,dt + o(1) \qquad (N \to \infty) \tag{3}$$

holding uniformly with respect to $x \in [a, b]$. On the other hand,

$$S_N^*(x) = \frac{1}{\pi} \int_{-\eta}^{\eta} f_*(x+t)\frac{\sin Nt}{t}\,dt + o(1) \qquad (N \to \infty) \tag{4}$$

holds uniformly with respect to all x (see § 15.3 (8) where S_n and f should be replaced by S_n^* and f_* respectively). Now, since $x+t \in [a, b]$ in integrals (3) and (4), the condition of the lemma implies $f(x+t) = f_*(x+t)$ whence follows the equality

$$S_N(x) - S_N^*(x) = o(1) - o(1) = o(1) \qquad (N \to \infty)$$

which holds uniformly on $[a', b']$.

A real or complex function $f(x)$ defined on the real axis $(-\infty, \infty)$ will be called *locally piecewise smooth* if it is piecewise smooth on any finite closed interval $[a, b]$ (i.e. $f(x)$ and its derivative $f'(x)$ have a finite number of points of discontinuity of the first kind on every such interval). For our further aims it will be convenient to assume that for all x there holds the condition $2f(x) = f(x+0)+f(x-0)$ although a locally piecewise smooth function (understood in the sense of the ordinary terminology) must not necessarily satisfy this additional requirement.

With the aid of Lemma 1 *the sufficient tests for convergence of Fourier's series established earlier can be automatically extended to Fourier's single-integral formula.* For example, there holds the following theorem (see Theorems 1 and 5 in § 15.5).

Theorem 1. *Fourier's single integral of a locally piecewise smooth function $f \in L'$ (or $f \in L$) converges to that function as $N \to \infty$, the convergence being uniform on any closed interval $[a', b']$ lying strictly inside an interval $[a, b]$ $(a < a' < b' < b)$ at whose all points the function f is continuous.*

Proof. Let us choose $x \in (-\infty, \infty)$ and let $[a, b]$ be a closed interval containing x, the function f being continuous on $[a, b]$. We shall begin with the case when $b-a < 2\pi$. Let us embed this interval in an open interval $(\alpha, \alpha+2\pi)$ $(\alpha < a < b < \alpha+2\pi)$ and construct the auxiliary periodic function f_* with period 2π which is equal to f on $[\alpha, \alpha+2\pi)$. It is obvious that $f_* \in L'^*$ is a piecewise smooth periodic function. For its Nth Fourier sum S_N^* we have

$$S_N^*(x) \to f_*(x), \qquad x \in [a, b] \tag{5}$$

Consequently, by virtue of Lemma 1, for Fourier's single integral of the function f we can write the relation

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} S_N^*(x) + \lim_{N\to\infty} [S_N(x)-S_N^*(x)] = f_*(x) = f(x) \tag{6}$$

As is already known from the theory of Fourier's series, property (5) holds uniformly on $[a', b'] \subset (a, b)$ provided that f is continuous on $[a, b]$. Therefore, by Lemma 1, property (6) also holds uniformly on $[a', b']$.

To complete the proof we now consider the case when $b-a \geqslant 2\pi$. On dividing $[a', b']$ into a finite number of closed intervals whose lengths are less than 2π we see that the relation $S_N(x) \to f(x)$ holds uniformly on each of the subintervals and hence it holds uniformly on $[a', b']$.

Equality (6) has been proved for the case when $N \to \infty$ running over the set of natural numbers. In order to extend this result to the case of an arbitrary variable N tending to infinity let us put $N = [N]+\alpha$ (for positive values of N) where $[N]$ is the integral part of N; then the relation

$$\int_{-\infty}^{\infty} \frac{\sin Nt}{t} f(x+t)\, dt - \int_{-\infty}^{\infty} \frac{\sin [N]t}{t} f(x+t)\, dt =$$

$$= 2\int_{-\infty}^{\infty} \cos\left([N]+\frac{\alpha}{2}\right) t\, \frac{\sin (\alpha/2)\, t}{t} f(x+t)\, dt \to 0 \qquad (N \to \infty)$$

holds uniformly with respect to x belonging to any finite interval of continuity of f. This follows from § 15.4, Lemma 3 in which we should put

$$g(\alpha, t) = \frac{\sin(\alpha/2)\,t}{t} \qquad (0 \leqslant \alpha < 1)$$

§ 16.4. Fourier Transform and Its Inverse. Iterated Fourier Integral. Fourier Cosine and Sine Transforms

A real or complex function $f(x)$ defined throughout the real axis is said to be *locally integrable* if $f \in L'(a, b)$ (or $f \in L(a, b)$) for any finite closed interval $[a, b]$.

If $f \in L' = L'(-\infty, \infty)$ then also $f \in L'(a, b)$ but, generally speaking, the converse is not true. For instance, a continuous function defined throughout the real axis is always locally integrable but it must not necessarily belong to $L'(-\infty, \infty)$.

If f is a locally integrable function, the integrals

$$\tilde{f}^N(x) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t)\, e^{-ixt} dt \quad \text{and} \quad \hat{f}^N(x) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t)\, e^{ixt}\, dt \tag{1}$$

make sense for any real x and any $N > 0$.

The limits

$$\lim_{N\to\infty} \tilde{f}^N(x) = \tilde{f}(x) \quad \text{and} \quad \lim_{N\to\infty} \hat{f}^N(x) = \hat{f}(x) \tag{2}$$

(provided they exist) will be called, respectively, *Fourier's transform* and *Fourier's inverse transform of the function f*. We shall write them in the form

$$\tilde{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-ixt}\, dt \quad \text{and} \quad \hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{ixt}\, dt \tag{3}$$

but it should be kept in mind that, in the general case, the integrals in (3) are understood in the sense of the principal value $\left(\text{that is } \int_{-\infty}^{\infty} = \lim_{N\to\infty} \int_{-N}^{N}\right)$.

For the functions $f \in L'$ (or $f \in L$) their Fourier transforms (3) always make sense and moreover integrals (3) are ordinary *absolutely convergent* improper integrals for which the symbol $\int_{-\infty}^{\infty}$ can be understood as $\lim_{N, N'\to\infty} \int_{-N}^{N'}$ where N and N' tend to infinity independently of each other.

The definitions stated imply the relation (see § 16.1 (9) and (12))

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} ds \int_{-\infty}^{\infty} f(t) \cos s(t-x)\, dt = \hat{\tilde{f}}(x) \tag{4}$$

As was proved in § 16.3, equality (4) is sure to hold for any *locally piecewise smooth function* $f \in L'$. The inner integral (with respect to t) in (4) is absolutely convergent for such a function while the outer integral (with respect to s) is convergent but is not necessarily absolutely convergent.

The double integral in (4) will be referred to as *Fourier's iterated integral of the function f.*

Thus, we see that *Fourier's iterated integral of a locally piecewise smooth function $f \in L'$ is equal to that function f.*

As to the rightmost member of (4), it indicates that f can be regarded as the result of the two operations, namely, *Fourier's* (direct) *transformation* (the operation symbolized by the sign "$\sim$") and *Fourier's inverse transformation* (the operation symbolized by the sign "$\frown$") performed on f.

Under the same conditions imposed on f there also holds the equality $f(x) = \tilde{\hat{f}}(x)$ because

$$\tilde{\hat{f}}(x) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} e^{-isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt =$$

$$= \lim_{N\to\infty} \frac{1}{2\pi} \int_{-N}^{N} e^{isx}\, ds \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt = \hat{\tilde{f}}(x)$$

(here we have replaced s by $-s$).

The equalities

$$f(x) = \hat{\tilde{f}}(x) = \tilde{\hat{f}}(x) \tag{5}$$

do in fact hold under some considerably more general conditions on f, particularly when the operations $\sim$ and $\frown$ symbolizing Fourier's transformations are generalized in an appropriate manner (see the following sections).

We also note that

$$\int_{-\infty}^{\infty} \cos s(t-x)\, f(t)\, dt = \cos sx \int_{-\infty}^{\infty} \cos st\, f(t)\, dt + \sin sx \int_{-\infty}^{\infty} \sin st\, f(t)\, dt$$

whence it follows that if a locally piecewise smooth function $f \in L'(-\infty, \infty)$ is *even* then (see (4))

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos sx\, ds \int_{0}^{\infty} \cos st\, f(t)\, dt \tag{6}$$

and if it is *odd* then

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin sx\, ds \int_{0}^{\infty} \sin st\, f(t)\, dt \tag{7}$$

In formulas (6) and (7) we can regard the variable x as ranging within the semi-infinite interval $[0, \infty)$ and $f(t)$ as an arbitrary locally piecewise smooth function belonging to $L'(0, \infty)$ because these formulas only involve the values of f assumed on the positive x-axis. Let us discuss this question in more detail.

Suppose we are given a locally piecewise smooth function $f \in L'(0, \infty)$ such that $f(0) = f(0+0)$. On extending it to the whole real axis as an even function we obtain an even locally piecewise smooth function $f \in L'(-\infty, \infty)$ to which formula (6) applies; in particular, it holds for $x \geqslant 0$.

Let us assume now that the given locally piecewise smooth function $f \in L'(0, \infty)$ satisfies the condition $f(0) = 0$ (generally speaking, $f(0+0) \neq f(0)$). On extending f in an odd fashion to $(-\infty, \infty)$ we obtain an odd locally piecewise smooth function $f \in L'(-\infty, \infty)$ for which formula (7) holds; in particular, it holds for $x \geqslant 0$.

It should be stressed that in formula (7) we have $f(0) = 0$ while in formula (6) the value $f(0) = f(0+0)$ can be quite arbitrary.

The integrals

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos st\,dt \quad \text{and} \quad \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin st\,dt$$

are called, respectively, *Fourier's cosine transform* and *Fourier's sine transform of the function f*. It follows immediately from formulas (6) and (7) that if *Fourier's cosine or sine transformation* (i.e. the operation of forming the first or the second integral written above) is performed twice in a consecutive order on a given function f, the result is again the original function f. In this sense Fourier's sine and cosine transformations are inverse to themselves.

Exercises. Prove the following formulas for the locally piecewise smooth functions $f \in L'(-\infty, \infty)$:

$$1.\ \hat f(-t) = \tilde f(t).\quad 2.\ \tilde f(-x) = \hat f(x).\quad 3.\ \widetilde{f(at)} = \frac{1}{|a|}\tilde f\left(\frac{x}{a}\right).$$

$$4.\ \hat f(at) = \frac{1}{|a|}\hat f\left(\frac{x}{a}\right)\ (a \neq 0).\quad 5.\ \widehat{e^{i\mu t}\tilde f} = \widetilde{e^{-i\mu t}\hat f} = f(x+\mu)\ (\mu \text{ is real}).$$

For instance, Fourier's transform of $e^{i\mu t}\tilde f$ (see Exercise 5) is obtained as follows:

$$\widehat{e^{i\mu t}\tilde f} = \frac{1}{\sqrt{2\pi}}\int e^{i\mu t}\, e^{ixt}\,dt\,\frac{1}{\sqrt{2\pi}}\int f(u)\,e^{-iut}\,du =$$

$$= \frac{1}{\sqrt{2\pi}}\int e^{i(\mu+x)t}\,dt\,\frac{1}{\sqrt{2\pi}}\int f(u)\,e^{-iut}\,du = \hat{\tilde f}(\mu+x) = f(\mu+x)$$

§ 16.5. Differentiation and Fourier Transformation

Theorem. *Let f be continuous locally piecewise smooth function and let f and $t\tilde f(t)$ belong to $L' = L'(-\infty, \infty)$ (or to L). Then f possesses the continuous derivative $f'(x)$ (i.e. f is in fact a smooth function) expressed by the* *formula*

$$f'(x) = \widehat{it\tilde{f}}(x) \tag{1}$$

(*in the abbreviated form* $f' = \widehat{it\tilde{f}}$).

Proof. Since $f \in L'$, the function $\tilde{f}$ is everywhere continuous. Further, the inclusion relation $t\tilde{f} \in L'$ implies that $\tilde{f} \in L'$ ($|t| \geqslant 1$), and therefore $\tilde{f} \in L' = L'(-\infty, \infty)$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(u)\, e^{ixu}\, du \tag{2}$$

and

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} iu\tilde{f}(u)\, e^{ixu}\, du \tag{3}$$

The differentiation under the integral sign is legitimate here because, by virtue of the inequality,

$$|\, iu\tilde{f}(u)\, e^{-ixu}\,| \leqslant |u\tilde{f}(u)| \in L'$$

integral (3) is uniformly convergent with respect to x and the integrand in (3) is continuous in x, u. The theorem has been proved.

§ 16.6. Space S

Let us state the following definition.

By (L. Schwartz'[*]) *space S is meant the totality of all complex-valued functions* $\varphi = \varphi(x)$ (i.e. $\varphi = \varphi_1 + i\varphi_2$ where φ_1 and φ_2 are real; it may of course happen that $\varphi_1 \equiv 0$ or $\varphi_2 \equiv 0$ or both) *defined and infinitely differentiable throughout the real axis and satisfying the condition*

$$\sup_x\, (1+|x|^l)\, |\varphi^{(k)}(x)| = \varkappa(l, k, \varphi) < \infty \tag{1}$$

for any pair of nonnegative numbers l and k (where k is an integer).

It follows from this definition that for any k the derivative $\varphi^{(k)}(x)$ is bounded, tends to zero as $x \to \infty$ and belongs to L_p' ($1 \leqslant p < \infty$) because $|\varphi^{(k)}(x)| \leqslant \frac{\varkappa(2, k, \varphi)}{1+|x|^2}$. We also note that every finite infinitely differentiable function obviously belongs to S.

If φ and φ_m ($m = 1, 2, \ldots$) are functions belonging to S and the condition

$$\varkappa(l, k, \varphi_m - \varphi) \to 0 \qquad (m \to \infty)$$

holds for any of the indicated pairs (l, k) we shall write $\varphi_m \to \varphi\,(S)$ and say that φ_m *tends to φ in the sense of S* (or that φ_m *converges to φ with respect to the topology of the space S* or, simply, that *the sequence $\{\varphi_m\}$ is (S)-convergent to φ*).

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[*] L. Schwartz, a modern French mathematician.

It is obvious that S is a *linear space*.

We shall have to deal with *operators* A on the space S: $A\varphi = \psi\ (\varphi, \psi \in S)$. Such an operator A assigns to each function $\varphi \in S$ a function $\psi \in S$.

An operator A is called *linear* if

$$A(\alpha\varphi_1+\beta\varphi_2) = \alpha A\varphi_1+\beta A\varphi_2$$

for any complex numbers α and β and any functions $\varphi_1, \varphi_2 \in S$.

An operator A is said to be *continuous* if, given any sequence of functions $\varphi_k \in S$ convergent to a function $\varphi \in S$ with respect to the topology of S, there holds the condition

$$A\varphi_k \to A\varphi\ (S)$$

The following assertion provides a sufficient* test for continuity of a linear operator: *if for any pair* (l, k) *of nonnegative integers there is a system of pairs* $(l_1, k_1), \ldots, (l_m, k_m)$ *dependent on the former such that*

$$\varkappa(l, k, A\varphi) \leqslant C_{l,k} \sum_{j=1}^{m} \varkappa(l_j, k_j, \varphi)$$

for all $\varphi \in S$ *where* $C_{l,k}$ *are constants independent of* φ, *then the linear operator* A *is continuous.*

Indeed, if $\varphi_\nu \to \varphi(S)$ and an operator A satisfies the indicated condition, then, for any pair (l, k), we have

$$\varkappa(l, k, A(\varphi_\nu-\varphi)) \leqslant C_{l,k} \sum_{j=1}^{m} \varkappa(l_j, k_j; \varphi_\nu-\varphi) \to 0 \qquad (\nu \to \infty)$$

The operation of differentiating a function $\varphi \in S$ *an arbitrary number of times* μ *which results in the derivative* $\varphi^{(\mu)}$ *specifies a linear operator from* S *into itself. This operator is continuous* because

$$\varkappa(l, k, \varphi^{(\mu)}) = \varkappa(l, k+\mu, \varphi)$$

for any pair (l, k).

A function $\lambda = \lambda(x)$ infinitely differentiable on $(-\infty, \infty)$ is said to be of *polynomial growth* together with its derivatives if for any nonnegative integer k there is a nonnegative number $l(k) = l$ and a constant C such that

$$|\lambda^{(k)}(x)| \leqslant C(1+|x|^l)$$

For instance, the function $(ix)^s$ where s is a nonnegative integer is obviously an infinitely differentiable function of polynomial growth.

For any fixed $\lambda(x)$ of the indicated type the operation $\lambda\varphi = \lambda(x)\varphi(x)$ of multiplication of $\varphi \in S$ by λ specifies a *continuous linear operator under which* S *is mapped into* S $(S \ni \varphi \to \lambda\varphi \in S)$. The fact that this is an operator from S into S and that it is continuous follows from the inequalities

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* It can be shown that the property indicated in this assertion is in fact not only sufficient for the continuity of A but also necessary; here we shall not present the proof of the necessity.

$$\begin{aligned}(1+|x|^l)\,|(\lambda\varphi)^{(k)}| &= (1+|x|^l)\left|\sum_{j=0}^{k} C_k^j\lambda^{(j)}\varphi^{(k-j)}\right| \leqslant \\ &\leqslant C\sum_0^k \frac{|\lambda^{(j)}(x)|}{1+|x|^{l(j)}}(1+|x|^{l(j)})\,(1+|x|^l)\,|\varphi^{(k-j)}(x)| \leqslant \\ &\leqslant C_1\sum_0^k (1+|x|^{l+l(j)})\,|\varphi^{(k-j)}|\end{aligned}$$

which imply

$$\varkappa(l, k, \lambda\varphi) \leqslant C_1\sum_0^k \varkappa(l+l(j), k-j, \varphi)$$

The linearity of the operation $\lambda\varphi$ is quite evident.

Let us show that *Fourier's transformation* $\varphi \to \tilde{\varphi}$ *specified by the formula*

$$\tilde{\varphi}(x) = \frac{1}{\sqrt{2\pi}}\int \varphi(t)\,e^{-ixt}\,dt \tag{2}$$

$\left(\text{where } \int = \int\limits_{-\infty}^{\infty}\right)$ *determines a continuous linear operator under which* S *is mapped onto* S*, the mapping being one-to-one.*

Indeed, if $\varphi \in S$ then $\varphi \in L'$, and therefore Fourier's transform $\tilde{\varphi}$ is sure to be a continuous function. Further, we have

$$\left.\begin{aligned}\tilde{\varphi}^{(k)}(x) &= \int \psi(t)\,e^{-ixt}\,dt \\ \psi(t) &= \frac{1}{\sqrt{2\pi}}\varphi(t)\,(-it)^k = A\varphi\end{aligned}\right\} \tag{3}$$

The function $\psi(t)$ belongs to S as the product of $\varphi \in S$ by a continuously differentiable function of polynomial growth. Since $\psi \in L'$ integral (3) is uniformly convergent for any k, and therefore the differentiation of (2) with respect to x under the integral sign is legitimate. On integrating by parts we obtain

$$\tilde{\varphi}^{(k)}(x) = \int \psi(t)\,e^{-ixt}\,dt = \frac{1}{ix}\int \psi'(t)\,e^{-ixt}\,dt = \ldots = \frac{1}{(ix)^l}\int \psi^{(l)}(t)\,e^{-ixt}\,dt$$

because $\psi^{(s)}(t) \to 0$ $(t \to \pm\infty)$.

It follows that

$$|x|^l\,|\tilde{\varphi}^{(k)}(x)| \leqslant \int \frac{|\psi^{(l)}(t)|\,(1+t^2)}{1+t^2}\,dt \leqslant \varkappa(2, l, \psi)\int \frac{dt}{1+t^2} = c\varkappa(2, l, \psi)$$

In particular, $|\tilde{\varphi}^{(k)}(x)| \leqslant c\varkappa(2, 0, \psi)$, and therefore

$$\varkappa(l, k, \tilde{\varphi}) = \sup\,(1+|x|^l)\,|\tilde{\varphi}^{(k)}(x)| \leqslant c\big(\varkappa(2, l, A\varphi)+\varkappa(2, 0, A\varphi)\big)$$

Consequently, $\tilde{\varphi} \in S$ and $\tilde{\varphi}$ depends on $A\varphi$ continuously. Further, $A\varphi$ depends continuously on φ and thus $\tilde{\varphi}$ also depends continuously on $\varphi \in S$. The linearity of the operator in question specified by the transformation $\varphi \to \tilde{\varphi}$ is evident. We have shown that under this operator S is mapped onto S. It now remains to prove that S is in fact mapped onto S and that the mapping is one-to-one. To this end we first note that if χ is an arbitrary function belonging to S, it can be regarded as Fourier's transform of the function $\hat{\chi} \in S$ because χ is smooth and belongs to L'. Hence, under the transformation $\varphi \to \tilde{\varphi}$ the space S is mapped *onto* itself. Finally, the equality $\tilde{\varphi}_1 = \tilde{\varphi}_2$ $(\varphi_1, \varphi_2 \in S)$ implies $\varphi_1 - \varphi_2 = 0$, and hence $\varphi_1 - \varphi_2 = \hat{0} = 0$, that is $\varphi_1 = \varphi_2$, which shows that the mapping of S onto itself under Fourier's transformation $\varphi \to \tilde{\varphi}$ is one-to-one.

Now for any two functions $\varphi, \psi \in S$ we introduce the expression

$$(\varphi, \psi) = \int \varphi(x)\,\psi(x)\,dx$$

(*which involves the function ψ itself, not the complex conjugate of ψ!*).

By virtue of Lemma 1, § 16.2, we can write

$$(\varphi, \tilde{\psi}) = \int \varphi(x)\,\tilde{\psi}(x)\,dx = \int \varphi(x)\frac{dx}{\sqrt{2\pi}}\int \psi(t)\,e^{-ixt}\,dt =$$

$$= \int \psi(t)\,dt\frac{1}{\sqrt{2\pi}}\int \varphi(x)\,e^{-ixt}\,dx = (\psi, \tilde{\varphi}) = (\tilde{\varphi}, \psi)$$

and we thus arrive at the equality $(\varphi, \tilde{\psi}) = (\tilde{\varphi}, \psi)$ serving as an analogue of Parseval's relation established in the theory of Fourier's series*. In a similar way we can prove the equality $(\varphi, \hat{\psi}) = (\hat{\varphi}, \psi)$ which is also an analogue of Parseval's relation. *Thus, we have the two equalities*

$$(\varphi, \tilde{\psi}) = (\tilde{\varphi}, \psi) \quad \textit{and} \quad (\varphi, \hat{\psi}) = (\hat{\varphi}, \psi) \tag{4}$$

serving as analogues of Parseval's relation.

Here we also write the equalities

$$(\varphi', \psi) = \int \varphi'(t)\,\psi(t)\,dt = \varphi(t)\,\psi(t)\Big|_{-\infty}^{\infty} - \int \varphi(t)\,\psi'(t)\,dt = -(\varphi, \psi') \tag{5}$$

$$(\varphi, \psi \in S)$$

which follow from the fact that $\varphi(t), \psi(t) \to 0$ as $t \to \pm\infty$. Finally, we have the important equalities

$$\varphi'(x) = \widehat{it\tilde{\varphi}} = \widetilde{(-it)\hat{\varphi}} \qquad (\varphi \in S) \tag{6}$$

To prove the first of them one should take into account that the inclusion relation $\varphi \in S$ implies $\tilde{\varphi} \in S$ and, since ix is an infinitely differentiable function of polynomial growth, there must be $ix\tilde{\varphi} \in S$. Hence, φ and $ix\tilde{\varphi}$ belong to L', and therefore the theorem proved in § 16.5 applies to this case. The second equality (6) is proved analogously.

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* If we had put $(\varphi, \psi) = \int \varphi(x)\,\overline{\psi(x)}\,dx$ then there would be $(\varphi, \psi) = (\tilde{\varphi}, \tilde{\psi})$.

The functions $\varphi \in S$ obviously possess the properties (i)-(v) established in Exercises at the end of § 16.4.

Let $K \in L'$ (or $K \in L$) and $\varphi \in S$. The expression

$$K*\varphi = \frac{1}{\sqrt{2\pi}}\int K(x-t)\,\varphi(t)\,dt = \frac{1}{\sqrt{2\pi}}\int \varphi(x-t)\,K(t)\,dt = \varphi*K$$

is called the *convolution* (also *faltung*) *of the functions K and* φ (or *of* φ *and K*).

There hold the important relations

$$\widehat{\breve{K}\tilde{\varphi}} = \breve{\breve{K}\hat{\varphi}} = K*\varphi \qquad (K \in L',\ \varphi \in S) \tag{7}$$

The relation $\widehat{\breve{K}\tilde{\varphi}} = K*\varphi$ is proved as follows (see the explanations below):

$$\begin{aligned}\widehat{\breve{K}\tilde{\varphi}} &= \frac{1}{(2\pi)^{3/2}}\int e^{ixs}\,ds\int K(u)\,e^{-isu}\,du\int \varphi(v)\,e^{-isv}\,dv = \\ &= \frac{1}{(2\pi)^{3/2}}\int e^{ixs}\,ds\int K(u)\,du\int \varphi(v)\,e^{-is(u+v)}\,dv = \\ &= \frac{1}{(2\pi)^{3/2}}\int e^{ixs}\,ds\int K(u)\,du\int \varphi(\xi-u)\,e^{-s\xi i}\,d\xi = \\ &= \frac{1}{(2\pi)^{3/2}}\int e^{ixs}\,ds\int e^{-is\xi}\,d\xi\int K(u)\,\varphi(\xi-u)\,du = \\ &= \frac{1}{\sqrt{2\pi}}\int K(u)\,\varphi(x-u)\,du = K*\varphi\end{aligned}$$

The first member in these equalities makes sense because $\varphi \in S$, $\tilde{\varphi} \in S \in L'$, $K \in L'$, and $\breve{K}$ is a bounded continuous function, and therefore $\breve{K}\tilde{\varphi} \in L'$ is a continuous function. In the third equality we have replaced v by $\xi = u+v$, and in the fourth equality we have interchanged the integrations with respect to u and to ξ (see Lemma 1, § 16.2 or Fubini's theorem in Chapter 19). The fifth equality holds since the integral

$$\varkappa(\xi) = \int K(u)\,\varphi(\xi-u)\,du$$

is a function belonging to L' because

$$\int |\varkappa(\xi)|\,d\xi \leqslant \int d\xi\int |K(u)|\,|\varphi(\xi-u)|\,du = \int |\varphi(t)|\,dt\int |K(u)|\,du < \infty$$

and since the function $\varkappa(\xi)$ possesses the continuous derivative

$$\varkappa'(\xi) = \int K(u)\,\varphi'(\xi-u)\,du$$

(the last integral is uniformly convergent).

To complete the proof of (7) we must show that $\widetilde{\tilde{K}\hat{\varphi}} = \widehat{\tilde{K}\tilde{\varphi}}$:

$$\widetilde{\tilde{K}\hat{\varphi}} = \widetilde{\tilde{K}(-t)\,\tilde{\varphi}(-t)} = \frac{1}{\sqrt{2\pi}}\int \tilde{K}(-t)\,\tilde{\varphi}(-t)\,e^{-ixt}\,dt =$$

$$= \frac{1}{\sqrt{2\pi}}\int \tilde{K}(u)\,\tilde{\varphi}(u)\,e^{ixu}\,du = \widehat{\tilde{K}\tilde{\varphi}}$$

Exercises. Prove that the functions below belong to S.

1. e^{-x^2}.

2. $\psi(x)\begin{cases} e^{\frac{1}{x^2-1}} & \text{for} \quad |x| < 1, \\ 0 & \text{for} \quad |x| \geq 1 \end{cases}$

(see § 5.11 (8)).

§ 16.7. Space S' of Generalized Functions

If to each function $\varphi \in S$ there corresponds, according to a certain law, a number (F, φ) dependent on φ linearly and continuously (with respect to the topology of S), we say that there is a *continuous linear functional F defined on S*. A continuous linear functional F is also called a *generalized function* or a *distribution*.

A functional F of this kind possesses the following two properties:

(i) for any pair α, β of complex numbers and any pair φ, ψ of functions belonging to S there holds the relation

$$(F, \alpha\varphi + \beta\psi) = \alpha(F, \varphi) + \beta(F, \psi)$$

expressing the *linearity* of F,

(ii) F satisfies the condition

$$(F, \varphi_N) \to (F, \varphi) \qquad (\text{when } \varphi_N \to \varphi)\ (S)$$

expressing the *continuity* of F.

The collection of all the generalized functions (i.e. continuous linear functionals) F is denoted as S'.

Usually it is not difficult to find whether a given concrete functional (F, φ) defined on S is linear. As to the continuity of F, it can be checked with the aid of the following important sufficient* test:

If there is a constant C and a finite system of pairs $(k_1, l_1), \ldots, (k_m, l_m)$ such that the inequality

$$|(F, \varphi)| \leq C \sum_{j=1}^{m} \varkappa(l_j, k_j, \varphi) \tag{1}$$

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* It can be shown that the condition of this test is in fact not only sufficient but also necessary for a functional to be continuous; here we shall not present the proof of the necessity.

holds for all the functions φ belonging to S, then the functional (F, φ) is continuous. Indeed, (1) implies that if $\varphi_N \to \varphi \ (S)$ then

$$|(F,\varphi_N)-(F,\varphi)| = |(F,\varphi_N-\varphi)| \leq C \sum_{j=1}^{m} \varkappa(l_j, k_j, \varphi_N-\varphi) \to 0 \qquad (N \to \infty)$$

Let us consider an example. Let $F(x)$ be a locally integrable complex-valued function, defined throughout the real axis, such that there is a number $l \geq 0$ for which $|F(x)| \leq C(1+|x|^l)$ where C is independent of x. The integral

$$(F,\varphi) = \int F(x)\,\varphi(x)\,dx \tag{2}$$

$\left(\text{where } \varphi \in S \text{ and } \int = \int\limits_{-\infty}^{\infty}\right)$ is a continuous linear functional F on S, that is a generalized function $F \in S'$. Indeed, we have

$$|(F, \varphi)| \leq \int |F(x)\,\varphi(x)|\,dx \leq C \int \frac{1+|x|^l}{1+|x|^{l+2}}\,\varkappa(l+2, 0, \varphi)\,dx \leq C_1\varkappa(l+2, 0, \varphi)$$

whence it is seen that functional (2) is defined for all $\varphi \in S$ and is continuous. Its linearity is quite evident.

Equality (2) also specifies a functional $F \in S'$ in the case when the function $F(x)$ belongs to L'_p (or to L_p), $1 \leq p < \infty$. If $F(x) \in L'$, then the continuity of (F, φ) follows from the inequalities

$$|(F,\varphi)| \leq \int |F(x)\,\varphi(x)|\,dx \leq \int |F(x)|\,\varkappa(0, 0, \varphi)\,dx \leq C\varkappa(0, 0, \varphi)$$

and if $F(x) \in L'_p \ (1 < p < \infty)$ it is implied by the inequalities

$$|(F, \varphi)| \leq \int |F(x)\,\varphi(x)| \leq \int |F(x)|\,\frac{1}{1+|x|}\,\varkappa(1, 0, \varphi)\,dx \leq$$

$$\leq \left(\int |F|^p\,dx\right)^{1/p} \left(\int \frac{dx}{(1+|x|)^q}\right)^{1/q} \varkappa(1, 0, \varphi) = C\varkappa(1, 0, \varphi) \qquad \left(\frac{1}{p}+\frac{1}{q} = 1\right)$$

It is important to stress that *for two locally integrable (in Riemann's sense) functions* $F_1(x)$ *and* $F_2(x)$ *to specify equal generalized functions* $F_1 = F_2 \in S'$ *with the aid of equalities of type* (2) *it is necessary and sufficient that the equality* $F_1(x) = F_2(x)$ *should hold for all the points of continuity of* $F_1(x)$ *and* $F_2(x)$.

The sufficiency of the indicated condition is obvious. Its necessity can be proved as follows. Suppose that the equality

$$\int F_1(x)\,\varphi(x)\,dx = \int F_2(x)\,\varphi(x)\,dx \quad \text{for all } \varphi \in S \tag{3}$$

holds and that, simultaneously, there is a point of continuity x_0 of the functions F_1 and F_2 such that, for instance,

$$\psi(x_0) = F_1(x_0)-F_2(x_0) > 0$$

Then there exists an interval $(x_0-\delta, x_0+\delta)$ on which

$$\psi(x) > \eta > 0$$

Let us take a nonnegative function $\varphi \in S$ with the interval $[x_0-\delta, x_0+\delta]$ as its support (such a function exists; for instance, we can put $\varphi = \psi((x-x_0)/\delta)$ where $\psi(x)$ is the function considered in Exercise 2 at the end of § 16.6). For such a function φ we obtain

$$(F_1, \varphi)-(F_2, \varphi) = \int \psi(x)\,\varphi(x)\,dx > \eta \int\limits_{x_0-\delta}^{x_0+\delta} \varphi(x)\,dx > 0$$

which contradicts (3).

A more general proposition (proved in § 19.6, Theorem 1) reads: *two locally Lebesgue integrable functions specify one and the same linear functional with the aid of formula* (2) *if and only if they are equal to each other on every finite closed interval* $[a, b]$ *except possibly at the points of that interval forming a set of Lebesgue measure zero.*

A generalized function specified with the aid of integral (2) by an ordinary locally integrable function $F(x)$* *is identified* with the latter. For instance, $\sin x$, $(\sin x)/x$, e^{-x^2}, $\ln|x|$ and $\sum\limits_0^n a_k x^k$ are ordinary functions belonging to S and at the same time they are generalized functions belonging to S'.

The function e^{x^2} is an example of a function which cannot be regarded as belonging to S' (that is it does not specify a continuous linear functional on S with the aid of formula (2)) because, for instance, integral (2) does not exist for this function if we take $\varphi(x) = e^{-x^2} \in S$.

Example 1. The functional δ determined by the relation

$$\delta = (\delta, \varphi) = \varphi(0) \qquad (\varphi \in S) \tag{4}$$

is a generalized function called the *Dirac*** *delta-function* (or, simply, the *δ-function*). It is obvious that $\delta \in S'$ because

$$|\varphi(0)| \leq \sup_x |\varphi(x)| = \varkappa(0, 0, \varphi)$$

whence follows the continuity of δ (its linearity is quite evident).

We shall prove that there is no locally integrable function $F(x)$ *which generates the δ-function with the aid of* (2). In this sense δ is a "proper" generalized function distinct from those generalized functions which can be identified with ordinary functions.

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* A generalized function specified with the aid of formula (2) by a locally integrable function is usually called *regular* while all the other generalized functions are called *singular*. — *Tr.*

** P. A. M. Dirac (born 1902), an English physicist.

We shall prove this assertion by contradiction. Let us suppose that there is a locally integrable function $F(x)$ generating the δ-function with the aid of formula (2). Let $x_0 \neq 0$ be its point of continuity such that $F(x_0) > 0$. Then there is an interval $(x_0-\delta, x_0+\delta)$ not containing the point $x = 0$ on which an inequality $F(x) > \eta > 0$ is fulfilled; in this case it is also possible to choose a nonnegative function $\varphi \in S$ with support $[x_0-\delta, x_0+\delta]$ to obtain

$$(F, \varphi) = \int_{x_0-\delta}^{x_0+\delta} F(x)\,\varphi(x)\,dx > \eta \int_{x_0-\delta}^{x_0+\delta} \varphi(x)\,dx > 0$$

But this is impossible because for the functions φ with support not containing the point $x = 0$ the functional δ assumes the zero value: $(\delta, \varphi) = \varphi(0) = 0$. Thus, the function $F(x)$ cannot take a positive value at its any point of continuity distinct from $x = 0$.

It can similarly be proved that $F(x)$ cannot assume a negative value at a point of continuity $x_0 \neq 0$. Consequently, there must be

$$\int F(x)\,\varphi(x)\,dx = 0 \quad \text{for all} \quad \varphi \in S$$

which contradicts the existence of functions $\varphi \in S$ such that $\varphi(0) \neq 0$.

It is also possible to prove the more general assertion that *functional* (4) *cannot be represented by integral* (2) *in which* $F(x)$ *is any function locally integrable in Lebesgue's sense.*

However, the functional (δ, φ) can be written in the form of the *Stieltjes** *integral*

$$(\delta, \varphi) = \varphi(0) = \int_{-\infty}^{\infty} \varphi(x)\,d\theta(x) = \lim_{\substack{a\to-\infty\\ b\to+\infty}} \int_a^b \varphi(x)\,d\theta(x)$$

where

$$\theta(x) = \begin{cases} 0 & \text{for } x \leqslant 0\\ 1 & \text{for } x > 0 \end{cases}$$

is the so-called *Heaviside*** *(unit) function.*

The Stieltjes integral written above is defined by the relation

$$\int_a^b \varphi(x)\,d\theta(x) = \lim_{\max(x_j-x_{j-1})\to 0} \sum_1^N \varphi(\xi_j)\,[\theta(x_j)-\theta(x_{j-1})]$$

where the closed interval $[a, b]$ is split into parts with the aid of points of division $a = x_0 < x_1 < \ldots < x_N = b$ and $\xi_j \in [x_{j-1}, x_j]$ $(j = 1, \ldots, N)$.

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* T. J. Stieltjes (1856-1894), a Dutch mathematician.

** O. Heaviside (1850-1925), an English physicist.

It is obvious that if the point $x = 0$ belongs to an interval $[x_{l-1}, x_l]$ and does not coincide with its right end point then

$$\sum_1^N \varphi(\xi_j)\,[\theta(x_j)-\theta(x_{j-1})] = \varphi(\xi_l)[\theta(x_l)-\theta(x_{l-1})] = \varphi(\xi_l) \to \varphi(0)$$

$$(\max |x_j - x_{j-1}| \to 0)$$

If the point $x = 0$ coincides with the right end point of $[x_{l-1}, x_l]$ the above sum is equal to $\varphi(\xi_{l+1}) \to \varphi(0)$, that is we obtain the same result.

Example 2. The generalized function $\mathrm{P}.\frac{1}{x}$ is defined as the limit

$$\left(\mathrm{P}.\frac{1}{x}, \varphi\right) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left\{\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\,dx\right\} = \text{v.p.} \int_{-\infty}^{\infty} \frac{\varphi(x)}{x}\,dx \qquad (\varphi \in S) \quad (5)$$

The integral on the right-hand side of (5) is understood in the sense of the principal value ("v.p." means French "valeur principale" principal value). If $\varphi(0) \neq 0$, this integral does not exist in the ordinary Riemann (or Lebesgue) sense. On the other hand, limit (5) can be written in the form of the improper Riemann integral

$$\lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}} \left\{\int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x}\,dx + \int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x}\,dx\right\} = \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx =$$

$$= \int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx$$

(here we have $\varphi(x)-\varphi(-x) = 0(x)$, $x \to 0$).

Clearly, this functional is linear. Its continuity follows from the inequality

$$\left|\int_0^{\infty} \frac{\varphi(x)-\varphi(-x)}{x}\,dx\right| = \left|\int_0^1 \frac{\int_{-x}^{x} \varphi'(t)\,dt}{x}\,dx + \int_1^{\infty} \frac{x[\varphi(x)-\varphi(-x)]}{x^2}\,dx\right| \leqslant$$

$$\leqslant \int_0^1 \frac{2x\varkappa(0,1,\varphi)}{x}\,dx + \int_1^{\infty} \frac{2\varkappa(1,0,\varphi)}{x^2}\,dx < C[\varkappa(0,1,\varphi)+\varkappa(1,0,\varphi)]$$

Now we proceed to a number of important operations on generalized functions.

If $\lambda = \lambda(x)$ is an infinitely differentiable function of polynomial growth its product by a generalized function $F \in S'$ is written as $\lambda F = \lambda(x)\,F(x)$ and is defined by means of the equality

$$(\lambda F, \varphi) = (F, \lambda\varphi) \qquad (6)$$

This definition is correct because $\lambda\varphi$ is continuous with respect to $\varphi \in S$ and $(F, \lambda\varphi)$ is a functional continuous with respect to $\lambda\varphi$ and, consequently, with respect to φ as well. The linearity of $(F, \lambda\varphi)$ with respect to φ is apparent.

This definition also appears natural from the point of view of definition (2) because if a functional $F \in S'$ is generated by a locally integrable function $F(x)$, the function $\lambda(x)F(x)$ in its turn generates the functional $\lambda F \in S'$ with the aid of formula (2), and

$$(\lambda F, \varphi) = \int \lambda(x)\, F(x)\, \varphi(x)\, dx = \int F(x)\, \lambda(x)\, \varphi(x)\, dx = (F, \lambda\varphi)$$

The *derivative of a generalized function* $F \in S'$ is defined as the generalized function F' determined by the relation

$$(F', \varphi) = -(F, \varphi') \qquad (\varphi \in S) \tag{7}$$

Since the derivative φ' belongs to S and the operation of differentiation is continuous with respect to φ and, besides, (F, φ') is a continuous functional with respect to φ', we see that (F', φ) is a linear functional continuous with respect to φ. Its linearity is obvious.

Definition (7) is quite natural because if, for instance, the function $F(x)$ is continuous together with its derivative $F'(x)$ and $F, F' \in L'$ then

$$(F', \varphi) = \int F'(x)\, \varphi(x)\, dx = F(x)\, \varphi(x)\Big|_{-\infty}^{\infty} - \int F(x)\, \varphi'(x)\, dx = -(F, \varphi')$$

Indeed, we have $\varphi(x) \to 0\ (x \to \infty)$ and also $F(x) \to 0\ (x \to \infty)$; the latter can be, for instance, shown by means of the relation

$$F(x') - F(x) = \int_x^{x'} F'(t)\, dt \to 0 \qquad (x, x' \to \infty)$$

because the limit $\lim\limits_{x\to\infty} F(x)$ exists and cannot be different from zero since $F \in L'$.

It is obvious that any generalized function $F \in S'$ possesses the (generalized) derivative $F^{(k)}$ of an arbitrary order k defined by induction with the aid of the relation $F^{(k)} = (F^{(k-1)})'$. We thus have

$$(F^{(k)}, \varphi) = (-1)^k (F, \varphi^{(k)})$$

For example,

$$(\delta^{(k)}, \varphi) = (-1)^k (\delta, \varphi^{(k)}) = (-1)^k \varphi^{(k)}(0) \qquad (k = 1, 2, \dots)$$

and

$$(\theta', \varphi) = -(\theta, \varphi') = -\int_0^{\infty} \varphi'(t)\, dt = \varphi(0)$$

that is $\theta' = \delta$.

We say that *a sequence of generalized functions* $F_N \in S'$ $(N = 1, 2, \ldots)$ *converges to a generalized function* $F \in S'$ (and write $F_N \to F\,(S')$) if*

$$\lim_{N\to\infty} (F_N, \varphi) = (F, \varphi) \quad \text{for all} \quad \varphi \in S \tag{8}$$

It follows automatically that the sequence of the derivatives F_N' is convergent to the derivative F' because

$$(F_N', \varphi) = -(F_N, \varphi') \to -(F, \varphi') = (F', \varphi), \qquad N \to \infty$$

We can also consider *a series*

$$F = u_1 + u_2 + u_3 + \ldots \tag{9}$$

of generalized functions $u_k \in S'$ whose *sum* $F \in S'$ is a generalized function understood in the sense that

$$\sum_1^N u_k \to F(S'), \qquad N \to \infty$$

It obviously follows that series (9) can be differentiated termwise:

$$F' = u_1' + u_2' + u_3' + \ldots$$

Example 3. Let us consider the ordinary function

$$f_\varepsilon(x) = \begin{cases} \dfrac{1}{\varepsilon} & \text{for} \quad 0 < x < \varepsilon \\ 0 & \text{for} \quad x \notin (0, \varepsilon) \end{cases}$$

dependent on the parameter $\varepsilon > 0$. It can obviously be identified with the generalized function f_ε determined by formula (2).

Evidently,

$$(f_\varepsilon, \varphi) = \frac{1}{\varepsilon} \int_0^\varepsilon \varphi(x)\,dx \to \varphi(0), \qquad \varepsilon \to 0 \quad (\text{for all } \varphi \in S)$$

whence it follows that

$$\lim_{\varepsilon\to 0} f_\varepsilon = \delta\,(S')$$

Fourier's transform and *Fourier's inverse transform of a generalized function* $F \in S'$ are defined as the generalized functions $\tilde{F}$ and $\hat{F}$ determined, respectively, by the equalities

$$(\tilde{F}, \varphi) = (F, \tilde{\varphi}) \quad \text{and} \quad (\hat{F}, \varphi) = (F, \hat{\varphi}) \qquad (\varphi \in S) \tag{10}$$

This definition is correct: $\tilde{\varphi} \in S$ and depends continuously on φ while $(F, \tilde{\varphi})$ depends continuously on $\tilde{\varphi}$, and therefore $(F, \tilde{\varphi})$ is also continuous with respect to φ; the linearity of $(F, \tilde{\varphi})$ with respect to φ is evident. This definition is natural because, for instance, it is coherent with the equality $(\tilde{\varphi}, \psi) = (\varphi, \tilde{\psi})$ holding for $\varphi, \psi \in S$. Further, we have $\hat{\tilde{F}} = F$ because $(\hat{\tilde{F}}, \varphi) = (\tilde{F}, \hat{\varphi}) = (F, \tilde{\hat{\varphi}}) = (F, \varphi)$.

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* It can be proved that if a sequence of generalized functions $F_N \in S'$ is such that the sequence of the numbers (F_N, φ) satisfies the condition of Cauchy's criterion for any $\varphi \in S$, then there exists a (uniquely determined) generalized function $F \in S'$ for which relation (8) holds.

The transforms $\tilde{F}$ *and* $\hat{F}$ *depend continuously on* $F \in S'$. This assertion means that if a sequence of generalized functions $F_N \in S'$ is convergent to $F \in S'$ $(F_N \to F\,(S'))$ then $\tilde{F}_N$ is convergent to $\tilde{F}$ $(\tilde{F}_N \to \tilde{F}\,(S'))$. Indeed,

$$(\tilde{F}_N, \varphi) = (F_N, \tilde{\varphi}) \to (F, \tilde{\varphi}) = (\tilde{F}, \varphi), \qquad N \to \infty$$

It should also be noted that *Fourier's transformation* $F \to \tilde{F}$ *of generalized functions* $F \in S'$ specifies a one-to-one mapping of S' onto S'. *Fourier's inverse transformation* $F \to \hat{F}$ $(F \in S')$ possesses the same property. Let us, for instance, prove this assertion for $\tilde{F}$. We already know that $\tilde{F}$ determines a mapping of S' onto S'; if $\Phi \in S'$ is an arbitrary generalized function it can be represented in the form $\Phi = \tilde{\hat{\Phi}}$, which shows that Fourier's transformation does in fact specify a mapping of S' onto itself. Finally, if $F_1, F_2 \in S'$ and $\tilde{F}_1 = \tilde{F}_2$ then $\widetilde{F_1 - F_2} = 0$, $F_1 - F_2 = \hat{0} = 0$ and hence $F_1 = F_2$, which shows that the mapping in question is one-to-one.

It follows from (10) that

$$F' = \widehat{ix\tilde{F}} = \widetilde{-ix\hat{F}} \tag{11}$$

because, for instance, we can write (see § 16.6 (6))

$$\left(\widetilde{-ix\hat{F}}, \varphi\right) = (-ix\hat{F}, \tilde{\varphi}) = (\hat{F}, -ix\tilde{\varphi}) = \left(F, \widehat{-ix\tilde{\varphi}}\right) = -(F, \varphi') = (F', \varphi)$$

As readily follows by induction from (11), there holds the following general formula for the derivative of the kth order:

$$F^{(k)} = \widehat{(ix)^k\tilde{F}} = \widetilde{(-ix)^k\hat{F}} \qquad (k = 0, 1, 2, \ldots) \tag{12}$$

If $K \in S'$ is a generalized function whose Fourier transform $\tilde{K}$ is an ordinary infinitely differentiable function of polynomial growth, it is possible to state a correct definition of the convolution of K with an arbitrary generalized function $F \in S'$ by putting

$$K * F = \widehat{\tilde{K}\tilde{F}} = \widetilde{\hat{K}\hat{F}} \tag{13}$$

This definition is obviously coherent with the definition of the convolution of $K \in L'$ (or $K \in L$) with $\varphi \in S$ stated in the foregoing section in the sense that these definitions provide the same result when the conditions of both are fulfilled simultaneously.

Example 4. We have $\tilde{\delta} = \hat{\delta} = \frac{1}{\sqrt{2\pi}}$ because, for instance,

$$(\tilde{\delta}, \varphi) = (\delta, \tilde{\varphi}) = \frac{1}{\sqrt{2\pi}} \int \varphi(t)\, dt$$

Consequently,

$$\delta^{(k)} = \widetilde{(-ix)^k\hat{\delta}} = \frac{1}{\sqrt{2\pi}}\widetilde{(-ix)^k 1} = \frac{1}{\sqrt{2\pi}}\widetilde{(-ix)^k} \qquad (k = 0, 1, 2, \ldots)$$

and thus

$$F^{(k)} = \widetilde{(-ix)^k\hat{F}} = \sqrt{2\pi}\,\widetilde{\widehat{\delta^{(k)}}\hat{F}} = \sqrt{2\pi}\,(\delta^{(k)} * F)$$

Example 5. We have (see the explanations below)

$$\begin{aligned}
\left(\widehat{\operatorname{sgn} x}, \varphi\right) &= (\operatorname{sgn} x, \hat{\varphi}) = \frac{1}{\sqrt{2\pi}}\int \operatorname{sgn} u\, du \int e^{iut}\varphi(t)\, dt = \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty du \int \varphi(t)\,(e^{iut}-e^{-iut})\, dt = \\
&= \sqrt{\frac{2}{\pi}}\, i \lim_{N\to\infty} \int \varphi(t)\, dt \int_0^N \sin ut\, du = \\
&= \sqrt{\frac{2}{\pi}}\, i \lim_{N\to\infty} \int \varphi(t)\frac{1-\cos Nt}{t}\, dt = \\
&= \sqrt{\frac{2}{\pi}}\, i \lim_{N\to\infty} \int_0^\infty [\varphi(t)-\varphi(-t)]\,\frac{1-\cos Nt}{t}\, dt = \\
&= \sqrt{\frac{2}{\pi}}\, i \int_0^\infty \frac{\varphi(t)-\varphi(-t)}{t}\, dt = \sqrt{\frac{2}{\pi}}\, i \text{ v.p.} \int_{-\infty}^\infty \frac{\varphi(t)}{t}\, dt
\end{aligned} \tag{14}$$

which results in the formula

$$\widehat{\operatorname{sgn} x} = \sqrt{\frac{2}{\pi}}\, i \text{ v.p. } \frac{1}{t} \tag{15}$$

The function $\operatorname{sgn} x$ is locally integrable and bounded; consequently it belongs to S, and therefore the first member in (14) makes sense and, together with it, the second and the third members as well. In the passage to the fifth member we have changed the order of integration (for finite N; see § 13.15). In the passage to the sixth member we take into account that

$$\lim_{N\to\infty} \int_0^\infty \frac{\varphi(t)-\varphi(-t)}{t} \cos Nt\, dt = 0$$

because the smooth function $(\varphi(t)-\varphi(-t))/t$ belongs to $L'(0, \infty)$. The last member is written as a singular integral in the sense of the principal value.

We also note that if $F(x) \in S'$ and $a \neq 0$ is a real number, the generalized functions $F(a-x)$ and $F(ax)$ are defined with the aid of the equalities

$$(F(a-x), \varphi(x)) = (F(x), \varphi(a-x))$$

and

$$(F(ax), \varphi(x)) = |a|^{-1}(F(x), \varphi(x/a))$$

The correctness of these definitions follows from the fact that the operation of passing from $\varphi(x) \in S$ to $\varphi(x-a)$ or to $\varphi(x/a)$ is continuous in the sense (S). On applying the definitions stated to an ordinary locally integrable function $F(x)$ which simultaneously is a generalized function we readily see that these definitions look natural from the point of view of the theory of ordinary functions.

Exercises. Prove the following relations.

1. $\dfrac{1}{\pi}\dfrac{\sin Nx}{x} \xrightarrow[N\to\infty]{} \delta(x)\ \ (S)$.

2. $\dfrac{1}{\varepsilon\sqrt{\pi}} e^{-x^2/\varepsilon^2} \xrightarrow[\varepsilon\to 0]{} \delta(x)\ \ (S)$.

Hint: the integral $\dfrac{1}{\varepsilon\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} e^{-x^2/\varepsilon^2}\varphi(x)\,dx$ can be represented as the sum I_1+I_2 of the integrals I_1 and I_2 taken, respectively, over the domains $|x|<1$ and $|x|>1$. In the first of these integrals one should put $\varphi(x)=\varphi(0)+\psi(x)$ and take into account that $|\psi(x)| \leqslant c|x|$; in the second integral it should be taken into account that the function $\varphi(x)$ is bounded $\big(|\varphi(x)| \leqslant M\big)$.

3. $\dfrac{1}{x\pm i\varepsilon} \to \mp i\pi\delta(x)+P\cdot\dfrac{1}{x}, \qquad \varepsilon\to 0\ \ (S)$.

Hint: $\left(P\cdot\dfrac{1}{x}, \varphi(x)\right) = \text{v.p.}\displaystyle\int_{-\infty}^{\infty}\dfrac{\varphi(x)}{x}\,dx \quad$ (see (5)).

Below it is supposed that $F \in S'$.

4. $\tilde{F}(-x) = \hat{F}(x)$.

5. $\tilde{F}(-x) = \hat{F}(x)$.

6. $\widetilde{F(ax)} = \dfrac{1}{|a|}\tilde{F}\left(\dfrac{x}{a}\right)$.

7. $\widehat{F(ax)} = \dfrac{1}{|a|}\hat{F}\left(\dfrac{x}{a}\right) \qquad (a \neq 0)$.

8. $\widehat{e^{i\mu t}\tilde{F}} = \widetilde{e^{-i\mu t}\hat{F}} = F(x+\mu)$ (where μ is real; take into account that $e^{i\mu t}$ is an infinitely differentiable function of polynomial growth).

9. Prove that the generalized function $P(x)f(x)$ belongs to S' when $P(x) = \sum_{0}^{n} a_k x^k$ is an arbitrary polynomial of degree n and $f(x) \in L'_p$ (or $f \in L_p$), $1 \leqslant p < \infty$, or if $f(x)$ is a bounded locally integrable function.

§ 16.8. Many-dimensional Fourier Integrals and Generalized Functions

Let $R_n = R$ be the n-dimensional space of the points $x = (x_1, \ldots, x_n)$ and let $\Delta_N = \Delta_N^{(n)} = \{|x_j| \leq N;\ j = 1, \ldots, n\}$ $(N > 0)$ be a cube belonging to R.

Let us consider a locally integrable function $f(x)$ defined in R, that is $f \in L'(\Delta_N)$ or $f \in L(\Delta_N)$ for any N (in the general case $f(x)$ can be complex-valued). For such a function make sense the integrals

$$\left.\begin{aligned} \tilde{f}^N(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u) e^{-ixu}\, du \\ \hat{f}^N(x) &= \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u) e^{ixu}\, du \quad \left(xu = \sum_1^n x_j u_j\right) \end{aligned}\right\} \tag{1}$$

which are continuous functions of x (see the lemma in § 16.2 and the remark to that lemma) tending to zero when $|x|^2 = \sum_1^n x_j^2 \to \infty$ (see Theorem 1 below).

By *Fourier's transform of f* and by *Fourier's inverse transform of f* we shall mean, respectively, the functions

$$\tilde{f}(x) = \lim_{N\to\infty} \tilde{f}^N(x) \quad \text{and} \quad \hat{f}(x) = \lim_{N\to\infty} \hat{f}^N(x) \tag{2}$$

These limits are sometimes understood in the sense of the convergence in the mean square (i.e. $\|\tilde{f}-\tilde{f}^N\|_{L_2} \to 0,\ N \to \infty$).

Theorem 1. *If $f \in L' = L'(R)$ $(or\ f \in L = L(R))$, then limits* (2) *exist in the ordinary sense and determine $\tilde{f}(x)$ and $\hat{f}(x)$ as bounded continuous functions possessing the property*

$$\lim_{|x|\to\infty} \tilde{f}(x) = \lim_{|x|\to\infty} \hat{f}(x) = 0$$

Proof. The continuity and the boundedness of $\tilde{f}$ and $\hat{f}$ follow from the lemma proved in § 16.2 and from Remark 1 after that lemma. Further, if e_k is the unit vector of the x_k-axis, then (see § 14.4, Theorem 6)

$$\begin{aligned} |(2\pi)^{n/2}\tilde{f}(x)| &= \frac{1}{2}\left|\int f(u) e^{-ixu}\, du + \int f\left(u+\frac{\pi e_k}{x_k}\right) e^{-i(xu+\pi)}\, du\right| \leq \\ &\leq \frac{1}{2}\int \left|f\left(u+\frac{\pi e_k}{x_k}\right) - f(u)\right| du \to 0 \qquad (x_k \to \infty) \end{aligned}$$

If $|x|^2 = \sum_1^n x_k^2 \to \infty$ then $\max_k |x_k| \to \infty$ and $\tilde{f}(x) \to 0$, and therefore $\hat{f}(x) = \tilde{f}(-x) \to 0$.

The expression

$$S_N(x) = \hat{\tilde{f}}^N = \frac{1}{(2\pi)^n} \int\limits_{\Delta_N} e^{ixv}\, dv \int f(u) e^{-ivu}\, du =$$

$$= \frac{1}{(2\pi)^n} \int f(u)\, du \int\limits_{\Delta_N} e^{-iv(x-u)}\, dv =$$

$$= \frac{1}{\pi^n} \int \prod_{j=1}^{n} \frac{\sin N(x_j-u_j)}{x_j-u_j} f(u)\, du =$$

$$= \frac{1}{\pi^n} \int \prod_{j=1}^{n} \frac{\sin Nu_j}{u_j} f(x+u)\, du \tag{3}$$

where the function $f(x)$ belongs to L' (or to L), is a multi-dimensional analogue of Fourier's single integral studied for functions of one variable in the foregoing sections.

The legitimacy of the change of the order of integration in (3) follows from the lemma in § 16.2 and from Remark 1 to that lemma.

Under some rather general conditions (e.g. see §§ 16.9 and 16.10) imposed on the function f it can be shown that

$$f(x) = \lim S_N(x) = \lim_{N\to\infty} \frac{1}{(2\pi)^n} \int\limits_{\Delta_N} e^{ixv}\, dv \int f(u) e^{-ivu}\, du =$$

$$= \frac{1}{(2\pi)^n} \int e^{ixv}\, dv \int f(u) e^{-ivu}\, du \tag{4}$$

where the outer integral is understood in the singular sense (that is the existence of the limits in (3) as $N \to \infty$ is guaranteed for the domains Δ_N of that very type indicated above and not for arbitrary Δ_N's). Formula (4) is a multi-dimensional generalization of Fourier's ordinary integral formula.

Thus, if the function f satisfies certain conditions there hold the equalities

$$f(x) = \tilde{\hat{f}}(x) = \hat{\tilde{f}}(x) \tag{5}$$

In our further discussion we shall be mainly concerned with the two-dimensional case although all the results obtained can readily be extended to the general case of an arbitrary dimension n.

Let $f(x)$ and $\varphi(y)$ be two locally integrable functions of one variable, that is belonging to $L'(\Delta_N^{(1)})\,(L(\Delta_N^{(1)}))$ for any $N\,(\Delta_N^{(1)} = (-N, N))$. Then obviously

$$f(x)\,\varphi(y) \in L'(\Delta_N^2)$$

$$\Delta_N^{(2)} = \{|x|, |y| < N\} = \Delta_N^{(1)} \times \Delta_N^{(1)}$$

and

$$\widetilde{f\varphi}^N(x, y) = \frac{1}{2\pi} \iint\limits_{\Delta_N^{(2)}} f(u)\,\varphi(v)\, e^{-i(xu+yv)}\, du\, dv =$$

$$= \frac{1}{\sqrt{2\pi}} \int\limits_{\Delta_N^{(1)}} f(u)\, e^{-ixu}\, du \frac{1}{\sqrt{2\pi}} \int\limits_{\Delta_N^{(1)}} \varphi(v)\, e^{-iyv}\, dv =$$

$$= \tilde{f}^N(x)\, \tilde{\varphi}^N(y) \tag{6}$$

Therefore, if

$$\tilde f^N(x) \to \tilde f(x) \quad \text{and} \quad \tilde\varphi^N(y) \to \tilde\varphi(y) \qquad (N \to \infty) \tag{7}$$

then

$$\tilde f(x)\,\tilde\varphi(y) = \widetilde{f\varphi}(x, y) \tag{8}$$

If relations (7) hold in the sense of the metric of L_2, relation (8) also holds in the same sense:

$$\|\tilde f(x)\,\tilde\varphi(y)-\widetilde{f\varphi}^N\|_{L_2(R_2)} = \|\tilde f(x)\,\tilde\varphi(y)-\tilde f^N(x)\,\tilde\varphi^N(y)\|_{L_2(R_2)} \leqslant$$
$$\leqslant \|\tilde f(x)\,(\tilde\varphi(y)-\tilde\varphi^N(y))\|_{L_2(R_2)}+\|(\tilde f(x)-\tilde f^N(x))\,\tilde\varphi^N(y)\|_{L_2(R_2)} =$$
$$= \|\tilde f\|_{L_2(R_1)}\|\tilde\varphi-\tilde\varphi^N\|_{L_2(R_1)}+\|\tilde f-\tilde f^N\|_{L_2(R_1)}\|\tilde\varphi^N\|_{L_2(R_1)} \to 0 \qquad (N \to \infty)$$

It follows that if the functions $f(x)$ and $\varphi(y)$ are such that

$$f(x) = \hat{\tilde f}(x) = \tilde{\hat f}(x) \quad \text{and} \quad \varphi(y) = \hat{\tilde\varphi}(y) = \tilde{\hat\varphi}(y)$$

in the sense of the ordinary convergence or convergence in the mean then, in the same sense, there hold the relations

$$\widehat{\widetilde{f\varphi}}(x, y) = \hat{\tilde f}(x)\,\hat{\tilde\varphi}(y) = f(x)\,\varphi(y) = \widetilde{\widehat{f\varphi}}(x, y)$$

Let us come back to the general n-dimensional case. We shall say that a function $\varphi(\boldsymbol{x})$ belongs to the class (space) S if it is complex-valued, infinitely differentiable on $R = R_n$ and satisfies the condition that for any pair formed by an integer $l \geqslant 0$ and an integral vector $\boldsymbol{k} = (k_1, \ldots, k_n) \geqslant 0$ (with nonnegative integral components $k_j \geqslant 0$) there holds the condition

$$\sup_{\boldsymbol{x}} (1+|\boldsymbol{x}|^l)\,|\varphi^{(\boldsymbol{k})}(\boldsymbol{x})| = \varkappa(l, \boldsymbol{k}, \varphi) < \infty \qquad \left(|\boldsymbol{x}| = \left(\sum_1^n x_j^2\right)^{1/2}\right)$$

Since for any such $\boldsymbol{k}$ we have

$$|\varphi^{(\boldsymbol{k})}(\boldsymbol{x})| \leqslant \frac{\varkappa(n+1, \boldsymbol{k}, \varphi)}{1+|\boldsymbol{x}|^{n+1}}$$

it obviously follows that the partial derivative $\varphi^{(\boldsymbol{k})}$ is a bounded function belonging to $L_p(R)$ $(1 \leqslant p < \infty)$.

If functions φ and φ_m $(m = 1, 2, \ldots)$ belong to S, and for any pair $(l, \boldsymbol{k})$ of the indicated type the limiting relation

$$\varkappa(l, \boldsymbol{k}; \varphi_m-\varphi) \to 0 \qquad (m \to \infty)$$

holds, we shall write $\varphi_m \to \varphi$ (S) and say that φ_m *tends to* φ *with respect to the topology of* S *(in the sense of* S*)*.

An operator A $(\varphi, A\varphi \in S)$ under which S is mapped into itself is said to be *linear* if $A(\alpha\varphi+\beta\psi) = \alpha A\varphi+\beta A\psi$ where $\varphi, \psi \in S$ and α and β are arbitrary complex numbers; it is said to be *continuous* if $\varphi_N \to \varphi$ (S) implies $A\varphi_N \to A\varphi$ (S).

For a linear operator A *to be continuous it is sufficient** *that for any pair*

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* This condition is in fact not only sufficient but necessary too.

(l, k) there should exist a constant $C_{l,k}$ and a system of pairs (l_j, k^j) $(j = 1, \ldots, m)$ dependent on (l, k) such that

$$\varkappa(l, k, A\varphi) \leqslant C_{l,k} \sum_{j=1}^{m} \varkappa(l_j, k^j, \varphi) \qquad (\text{for all } \varphi \in S)$$

because $\varphi_\nu \to \varphi$ (S) implies

$$\varkappa(l, k, A(\varphi_\nu - \varphi)) \leqslant C_{l,k} \sum_{j=1}^{m} \varkappa(l_j, k^j, \varphi_\nu - \varphi) \to 0 \qquad (\nu \to \infty)$$

Below are examples of some important continuous linear operators A on S $(\varphi, A\varphi \in S)$.

(i) The operator corresponding to the differentiation of φ:

$$A\varphi = \varphi^{(\mu)} = \frac{\partial^{|\mu|}\varphi}{\partial x_1^{\mu_1} \ldots \partial x_n^{\mu_n}} \qquad \left(|\mu| = \sum_1^n \mu_j\right)$$

The linearity is quite obvious and the continuity of this operator follows from the relation

$$\varkappa(l, k, \varphi^{(\mu)}) = \varkappa(l, k+\mu, \varphi)$$

(ii) The operator corresponding to the multiplication of φ by an infinitely differentiable function $\lambda(x)$ of polynomial growth $(\lambda\varphi = \lambda(x)\varphi(x))$, that is such that for any nonnegative integral vector k there is a number $l(k)$ for which

$$|\lambda^{(k)}(x)| \leqslant C(1+|x|^{l(k)})$$

where C is independent of x. The linearity of this operator is obvious and its continuity is implied by the inequalities

$$\begin{aligned} |(1+|x|^l)(\lambda\varphi)^{(k)}| &= (1+|x|^l)\left|\sum_{0 \leqslant s \leqslant k} \lambda^{(s)}\varphi^{(k-s)}\right| \leqslant \\ &\leqslant C' \sum_{0 \leqslant s \leqslant k} (1+|x|^l)(1+|x|^{l(s)})\,|\varphi^{(k-s)}| \leqslant \\ &\leqslant C_1 \sum_{0 \leqslant s \leqslant k} \varkappa(l+l(s), k-s, \varphi) \end{aligned}$$

where

$$\frac{k!\,(k-s)!}{s!} = \frac{k_1! \ldots k_n!\,(k_1-s_1)! \ldots (k_n-s_n)!}{s_1! \ldots s_n!}$$

(iii) The operators generated by Fourier's transformation $\varphi \to \tilde{\varphi}$ and Fourier's inverse transformation $\varphi \to \hat{\varphi}$.

The linearity of these operators is quite clear because for any $\varphi \in S$ we have

$$\begin{aligned} \tilde{\varphi} = \frac{1}{(2\pi)^{n/2}} \int \varphi(u) e^{-ixu}\, du = \frac{1}{(2\pi)^{n/2}} \int e^{-ix_1u_1}\, du_1 \ldots \\ \ldots \int e^{-ix_nu_n}\varphi(u)\, du_n = {}^{1\sim \ldots n}\tilde{\varphi} \end{aligned} \tag{9}$$

where $\varphi \in S$ and $j \sim$ is the sign of the operation of forming Fourier's transform only with respect to the variable x_j (in the analogous sense we shall use the symbols $x \sim$, $j \frown$, $x \frown$ and the like). This reduces the investigation of the continuity of the operation $\sim$ to that of the operation $j \sim$ $(j = 1, \ldots, n)$.

Let us show that *the transformation* $\varphi \rightarrow {}^j\tilde{\varphi}$ *generates a continuous and one-to-one mapping of S onto itself.*

We shall confine ourselves to the two-dimensional case. For $\varphi(x, y) \in S$ we have

$$\psi(x, y) = \frac{1}{\sqrt{2\pi}} \int \varphi(t, y) e^{-ixt}\, dt$$

On putting (see the explanations below)

$$g(t, y) = \frac{1}{\sqrt{2\pi}} (-it)^{k_1} \frac{\partial \varphi^{k_2}}{\partial y^{k_2}} (t, y)$$

we obtain (with the aid of the integration by parts) the equalities

$$\frac{\partial^{k_1+k_2}\psi}{\partial x^{k_1}\, \partial y^{k_2}} = \int g(t, y) e^{-ixt}\, dt = \frac{1}{ix} \int \frac{\partial g}{\partial x} (t, y) e^{-ixt}\, dt =$$

$$= \ldots = \frac{1}{(ix)^l} \int \frac{\partial^l g}{\partial x^l} (t, y) e^{-ixt}\, dt \tag{10}$$

Since t^{k_1} is an infinitely differentiable function of polynomial growth ($k_1 \geqslant 0$ is an integer), the function $g(t, y)$ belongs to S and is continuous (in the sense (S)) with respect to φ. Therefore, in particular, $g(t, y) \rightarrow 0$ $(t \rightarrow \pm \infty)$ whence follow equalities (10).

From (10) it follows that

$$\leqslant \left| x^l \frac{\partial^{k_1+k_2}\psi}{\partial x^{k_1}\, \partial y^{k_2}} \right| \int \frac{dt}{1+t^2} \varkappa(2; l, 0; g) \leqslant C\varkappa(2; l, 0; g)$$

and therefore, on applying (10) once again, for $l = 0$, we obtain

$$(1+|x|^l) \left| \frac{\partial^{k_1+k_2}\psi}{\partial x^{k_1}\, \partial y^{k_2}} \right| \leqslant C_1(\varkappa(2; 0, 0; g) + \varkappa(2; l, 0; g))$$

Consequently, ψ depends continuously (relative to the topology of S) on g, and hence, together with g, on φ as well.

If $\psi(x, y) \in S$ is an arbitrary function, then ${}^x\hat{\psi} \in S$ and $\psi = {}^{x\sim}({}^x\hat{\psi})$, whence we see that the mapping of S *onto* S corresponding to the transformation $\varphi \rightarrow {}^x\tilde{\varphi}$ is in fact a mapping of S *onto* itself.

Finally, the equality ${}^x\tilde{\varphi}_1 = {}^x\tilde{\varphi}_2$ implies $\widetilde{{}^x(\varphi_1 - \varphi_2)} = 0$ whence $\varphi_1 - \varphi_2 = = {}^x\hat{0} = 0$, which proves that the mapping generated by the transformation $\varphi \rightarrow {}^x\tilde{\varphi}$ is one-to-one.

It now follows from (9) that *the mapping of S onto S generated by Fourier's transformation* $\varphi \rightarrow \tilde{\varphi}$ *is one-to-one and continuous.*

Any two of the operations

$$1 \sim, \ldots, n \sim, 1 \frown, \ldots, n \frown$$

commute with each other. Consequently, *for any function* $\varphi \in S$ *the equalities*

$$\varphi = \hat{\tilde{\varphi}} = \tilde{\hat{\varphi}} \tag{11}$$

hold at every point $x \in R$. For example, in the two-dimensional case we have for $\varphi(x, y)$ the relations

$$\tilde{\tilde{\varphi}} = {}^{x\frown y\frown y\sim x}\tilde{\varphi} = {}^{x\frown x}\tilde{\varphi} = \varphi$$

Confining ourselves to the case of a function $\varphi(x, y)$ of two variables we also note the following fact:

$$\widehat{\lambda(x)\, \tilde{\varphi}(x, y)} = \widehat{{}^{x}\lambda(x)\, {}^{x}\tilde{\varphi}(x, y)} \tag{12}$$

where $\lambda(x)$ is any infinitely differentiable function of polynomial growth. Indeed, we have

$$\widehat{\lambda(x)\tilde{\varphi}} = \widehat{\lambda(x)^{y\sim x}\tilde{\varphi}} = \widehat{{}^{y\sim}\big(\lambda(x)^{x}\tilde{\varphi}\big)} = \widehat{{}^{x}\lambda(x)^{x}\tilde{\varphi}}$$

because $\frown y \sim\, = x\frown y\frown y\sim\, = x\frown$.

For functions $\varphi, \psi \in S$ we also have the relations*

$$(\tilde{\varphi}, \psi) = (\varphi, \tilde{\psi}),\ (\varphi, \psi) = \int_R \varphi(t)\, \psi(t)\, dt \tag{13}$$

$$(\varphi^{(k)}, \psi) = (-1)^{|k|}\, (\varphi, \psi^{(k)}) \tag{14}$$

$$\varphi^{(k)}(x) = \widehat{(ix)^k\tilde{\varphi}} = \widetilde{(-ix)^k\hat{\varphi}} \qquad \left((ix)^k = |i|^{|k|}\, x_1^{k_1} \ldots x_n^{k_n}\right) \tag{15}$$

and

$$\widehat{K\tilde{\varphi}} = \widetilde{\hat{K}\hat{\varphi}} = K * \varphi = \frac{1}{(2\pi)^{n/2}} \int K(u)\, \varphi(x-u)\, du \qquad (K \in L' \text{ or } K \in L) \tag{16}$$

They generalize, respectively, formulas (4), (5), (6) and (7) in § 16.6 and are proved analogously.

By analogy with the one-dimensional case, for the space S of the functions $\varphi(x)$ $\big(x = (x_1, \ldots, x_n)\big)$ we consider the space S' of all generalized functions (distributions) F which are the continuous linear functionals (F, φ) $(\varphi \in S)$ on S.

For a linear functional (F, φ) *to be continuous it is sufficient that there exist a system of pairs* $(l_1, k^1), \ldots, (l_m, k^m)$ *dependent on it and a constant* C *such that* (see § 16.7 (1))

$$|(F, \varphi)| \le C \sum_{j=1}^{m} \varkappa(l_j, k^j, \varphi) \qquad (\text{for all } \varphi \in S)$$

A function $F(x) \in L_p'$ (or $F(x) \in L_p$), $1 \le p < \infty$, or a locally integrable function $F(x)$ of polynomial growth (i.e. $|F(x)| \le C(1+|x|^l)$ for some l)

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* Note that the integral in (13) involves the function ψ itself and not its complex conjugate.

generates the generalized function

$$(F, \varphi) = \int F(x)\,\varphi(x)\,dx$$

where $\varphi \in S$ and $\int = \int\limits_{R_n}$.

Any two such functions differing at least at one of their points of continuity (or, in the case of the Lebesgue integrability, differing on a set of positive Lebesgue's measure) generate different (regular) generalized functions, which is proved like in the case of one variable using the function $\psi(|(x-x^0)/\delta|)$ dependent on n variables (see Exercise 1 below).

The functional $(\delta, \varphi) = \varphi(0)$ is the n-dimensional delta-function dependent on $x = (x_1, \ldots, x_n)$; this is a generalized function which cannot be generated as a regular generalized function by any locally integrable function. The operations λF (where λ is an infinitely differentiable function of polynomial growth together with its derivatives), $F^{(k)}, \tilde{F}, \hat{F}, {}^j\tilde{F}$ and ${}^j\hat{F}$ are defined with the aid of the equalities

$$(\lambda F, \varphi) = (F, \lambda\varphi),\ (F^{(k)}, \varphi) = (-1)^{|k|}\,(F, \varphi^{(k)}),$$

$$(\tilde{F}, \varphi) = (F, \tilde{\varphi}),\ (\hat{F}, \varphi) = (F, \hat{\varphi}),\ ({}^j\tilde{F}, \varphi) = F, {}^j\tilde{\varphi} \quad \text{and} \quad ({}^j\hat{F}, \varphi) = (F, {}^j\hat{\varphi})$$

whence, in particular, follows

$$F^{(k)} = \widehat{(ix)^k\tilde{F}} = \widetilde{(-ix)^{(k)}\hat{F}}$$

The convergence $F_N \to F(S')$ is understood in the sense that $(F_N, \varphi) \to (F, \varphi)\,(N \to \infty)$ for all $\varphi \in S$. Finally, by analogy with the one-dimensional case, the convolution is defined by the formula

$$K * F = \widehat{\tilde{K}\tilde{F}} = \widetilde{\hat{K}\hat{F}} \qquad (F \in S')$$

where $K \in S'$ is such that $\tilde{K}$ is an infinitely differentiable function of polynomial growth (see also § 18.3).

In the two-dimensional case, for a function $f(x, y) \in L' = L'(R_2)$ (or $f(x, y) \in L(R_2)$) and any $\eta > 0$ we can write

$$S_N(x, y) = \frac{1}{\pi^2}\iint\limits_{K_\eta} \frac{\sin Nu}{u}\,\frac{\sin Nv}{v} f(x+u, y+v)\,du\,dv + o(1) \quad (N \to \infty) \qquad (17)$$

where K_η is the cross-shaped region whose points (u, v) satisfy one of the inequalities

$$|u| < \eta \quad \text{and} \quad |v| < \eta$$

This follows from the relation

$$\int\limits_{\eta}^{\infty}\int\limits_{\eta}^{\infty} \frac{\sin Nu}{u}\,\frac{\sin Nv}{v} f(x+u, y+v)\,du\,dv \to 0 \qquad (N \to \infty)$$

which is a simple generalization of Lemma 1, § 15.4 to the two-dimensional case and from similar relations for the integrals taken over the domains symmetric to $\{\eta \leqslant x, y\}$ with respect to the coordinate axes and with respect to the origin.

It should be stressed that the cross-shaped region K_η in (17) cannot be replaced by the square

$$\Delta_\eta = \{|u|, |v| \leqslant \eta\}$$

Here lies an essential distinction between the many- and one-dimensional cases. In the one-dimensional case the convergence of Fourier's single integral of a function $f \in L'$ (or $f \in L_1$) at a point x or the convergence of Fourier's partial sum of a periodic function $f \in L'^*$ ($f \in L^*$) depends solely on the behaviour of f in an arbitrarily small neighbourhood of x. In the two-dimensional (many-dimensional) case this is no longer so: a function $f \in L'$ (L'^*) equal to zero in a neighbourhood of a point (x, y) may have Fourier's partial sum (or the expression of the type $S_N(x)$) which is not convergent to $f(x, y)$ as $N \to \infty$.

To elucidate what has been said let us consider the set $C_0(\Delta')$ of all continuous finite functions $f(u, v)$ with supports belonging to the rectangle

$$\Delta' = \{0 < u < \eta, \eta < v < 2\eta; \eta > 0\}$$

equipped with the norm

$$||f||_{C_0(\Delta')} = \max_{(u, v) \in \Delta'} |f(u, v)|$$

To each function $f \in C_0(\Delta')$ there corresponds the value

$$S_N(f) = S_N(f, 0, 0) = \frac{1}{\pi^2} \iint_{\Delta'} \frac{\sin Nu}{u} \frac{\sin Nv}{v} f(u, v)\, du\, dv$$

of S_N at the point (0, 0).

The supremum of this integral extended over all the functions $f = C_0(\Delta')$ with $||f|| \leqslant 1$ is equal to

$$\Lambda_N = \sup S_N(f, 0, 0) = \frac{1}{\pi^2} \iint_{\Delta'} \left| \frac{\sin Nu}{u} \frac{\sin Nv}{u} \right| du\, dv =$$

$$= \frac{1}{\pi^2} \int_0^{N\eta} \left| \frac{\sin u}{u} \right| du \int_{\eta N}^{2\eta N} \left| \frac{\sin v}{v} \right| dv$$

It can readily be proved that

$$\Lambda_N \to \infty, \qquad N \to \infty$$

(to this end one should use the argument of Theorem 3 in § 9.15).

From the point of view of functional analysis, Λ_N is the norm of the functional $S_N(f, 0, 0)$ defined in the space $C_0(\Delta')$. As is proved in functional analysis, the fact that $\Lambda_N \to \infty$ implies the existence of a function $f \in C_0(\Delta')$ such that the functional $S_N(f, 0, 0)$ is unbounded for it (on the set $N = 1, 2, \ldots$).

Exercises

1. Prove that $e^{-|x|^2} \in S$ and $\psi\left(\left|\frac{x-x^0}{\delta}\right|\right) \in S\left(|x|^2 = \sum_1^n x_j^2\right)$. Here $\delta > 0$; see Exercise 2 in § 16.6.

2. Show that

$$f(x+\mu) = \widehat{e^{i\mu t}\tilde{f}} = \widetilde{e^{-i\mu t}\hat{f}}\left(\mu = (\mu_1, \ldots, \mu_n),\ t = (t_1, \ldots, t_n),\ \mu t = \sum_1^n \mu_j t_j\right)$$

3. Show that the supremum of the Fourier sums $S_n(f, 0)$ at the point $x = 0$ extended over all the functions $f \in C_*$ of one variable with norm $\|f\|_{C_*} \leqslant 1$ is expressed as

$$\sup_{\|f\|_{C_*} \leqslant 1} |S_n(f, 0)| = \frac{2}{\pi} \int_0^\pi |D_n(t)|\, dt \to \infty, \qquad n \to \infty$$

where $D_n(t)$ is Dirichlet's kernel. It follows, by virtue of the theorem of functional analysis mentioned above, that there is a function $f \in C_*$ whose Fourier series is divergent at the point $x = 0$.

§ 16.9. Finite Step Functions. Approximation in the Mean Square

Let us begin with the simplest finite *step function*

$$\varphi_{a,\delta}(x) = \begin{cases} 1 & \text{for } x \in (a-\delta, a+\delta) \\ 0 & \text{for } x \notin [a-\delta, a+\delta] \\ \frac{1}{2} & \text{for } x = a-\delta, a+\delta \end{cases} \tag{1}$$

dependent on one variable x ($\delta > 0$).

Its Fourier's transform is

$$\tilde{\varphi}_{a,\delta}(x) = \frac{1}{\sqrt{2\pi}} \int_{a-\delta}^{a+\delta} e^{-ixt}\, dt = \sqrt{\frac{2}{\pi}}\, e^{-ixa} \frac{e^{ix\delta} - e^{-ix\delta}}{2ix} = \sqrt{\frac{2}{\pi}}\, e^{-ixa} \frac{\sin x\delta}{x} \tag{2}$$

and therefore, taking into account that $\varphi_{a,\delta} \in L'$ is a locally piecewise smooth function, we obtain

$$\varphi_{a,\delta}(x) = \widehat{\sqrt{\frac{2}{\pi}}\, e^{-ixa} \frac{\sin x\delta}{x}} \tag{3}$$

The function $\varphi_{a,\delta}$ obviously belongs to L_2'. Formula (2) also indicates that $\tilde{\varphi}_{a,\delta}$ does not belong to L' but belongs to L_2' (generally, to L_p', $1 < p < \infty$), and the relations

$$\|\varphi_{a,\delta}\|_{L_2}^2 = 2\delta$$

and

$$\|\hat{\varphi}_{a,\delta}\|^2_{L_2} = \|\tilde{\varphi}_{a,\delta}\|^2_{L_2} = \frac{2}{\pi}\int\limits_{-\infty}^{\infty}\left(\frac{\sin x\delta}{x}\right)^2 dx = 2\delta$$

hold (see § 15.9 (9)). Thus, we have the equalities

$$\|\varphi_{a,\delta}\|_{L_2} = \|\tilde{\varphi}_{a,\delta}\|_{L_2} = \|\hat{\varphi}_{a,\delta}\|_{L_2} \tag{4}$$

We shall show that they make sense for all the functions $\varphi \in L_2$.

According to the definitions of the operations $\sim$, $\frown$, $\sim N$ and $\frown N$ (see § 16.4) and by virtue of the fact that $\varphi_{a,\delta} \in L'$ is a locally piecewise smooth function, we have the relations

$$\left.\begin{aligned} \tilde{\varphi}^N_{a,\delta}(x) &\to \tilde{\varphi}_{a,\delta}(x), \\ \hat{\tilde{\varphi}}^N_{a,\delta}(x) &\to \varphi_{a,\delta}(x) \end{aligned} \qquad (N \to \infty)\right\} \tag{5}$$

which hold for any real x.

It is important to note that here we have not only the pointwise convergence but the convergence with respect to the metric of $L_2 = L_2(-\infty, \infty)$ as well, that is

$$\|\tilde{\varphi}^N_{a,\delta} - \tilde{\varphi}_{a,\delta}\| L_2 \to 0 \qquad (N \to \infty) \tag{6}$$

and

$$\|\hat{\tilde{\varphi}}^N_{a,\delta} - \varphi_{a,\delta}\| L_2 \to 0 \qquad (N \to \infty) \tag{7}$$

Relation (6) is trivial because the interval $[a-\delta, a+\delta]$ belongs to $[-N, N]$ for sufficiently large N, whence

$$\tilde{\varphi}_{a,\delta}(x) = \frac{1}{\sqrt{2\pi}}\int\limits_{-N}^{N} \varphi_{a,\delta}(t)\, e^{-ixt}\, dt = \tilde{\varphi}^N_{a,\delta}(x)$$

The proof of (7) is more complicated. We have

$$\begin{aligned} \hat{\tilde{\varphi}}^N_{a,\delta}(x) &= \frac{1}{\pi}\int\limits_{a-\delta}^{a+\delta} \frac{\sin N(t-x)}{t-x}\, dt = \frac{1}{\pi}\int\limits_{N(a-\delta-x)}^{N(a+\delta-x)} \frac{\sin z}{z}\, dz = \\ &= 1 - \frac{1}{\pi}\int\limits_{N(a+\delta-x)}^{\infty} \frac{\sin z}{z}\, dz - \frac{1}{\pi}\int\limits_{-\infty}^{N(a-\delta-x)} \frac{\sin z}{z}\, dz \end{aligned}$$

since

$$\frac{1}{\pi}\int\limits_{-\infty}^{\infty} \frac{\sin z}{z}\, dz = 1$$

Therefore, taking into account (1), we arrive at

$$\sqrt{\int\limits_{-\infty}^{\infty} |\varphi_{a,\delta}(x) - \hat{\tilde{\varphi}}^N_{a,\delta}(x)|^2\, dx} \leqslant I_1 + I_2 + I_3 + I_4 \tag{8}$$

where

$$I_1 = \left(\int\limits_{a-\delta}^{a+\delta} \left(\frac{1}{\pi} \int\limits_{N(a+\delta-x)}^{\infty} \frac{\sin z}{z}\,dz\right)^2 dx\right)^{1/2}$$

$$I_2 = \left(\int\limits_{a-\delta}^{a+\delta} \left(\frac{1}{\pi} \int\limits_{-\infty}^{N(a-\delta-x)} \frac{\sin z}{z}\,dz\right)^2 dx\right)^{1/2}$$

$$I_3 = \left(\int\limits_{-\infty}^{a-\delta} \left(\frac{1}{\pi} \int\limits_{N(a-\delta-x)}^{N(a+\delta-x)} \frac{\sin z}{z}\,dz\right)^2 dx\right)^{1/2}$$

and

$$I_4 = \left(\int\limits_{a+\delta}^{\infty} \left(\frac{1}{\pi} \int\limits_{N(a-\delta-x)}^{N(a+\delta-x)} \frac{\sin z}{z}\,dz\right)^2 dx\right)^{1/2}$$

Now, putting $u = N(a+\delta-x)$, we receive

$$(\pi I_1)^2 = \frac{1}{N} \int\limits_0^{2\delta N} du \left(\int\limits_u^{\infty} \frac{\sin z}{z}\,dz\right)^2 \tag{9}$$

But

$$\int\limits_u^{\infty} \frac{\sin z}{z}\,dz = -\frac{\cos z}{z}\Bigg|_u^{\infty} - \int\limits_u^{\infty} \frac{\cos z}{z^2}\,dz$$

and, consequently,

$$\left|\int\limits_u^{\infty} \frac{\sin z}{z}\,dz\right| \leqslant \frac{1}{u} + \frac{1}{u} = \frac{2}{u}$$

We shall use this estimate for $u > 1$. For $u < 1$ it is sufficient to take into account that

$$\left|\int\limits_u^{\infty} \frac{\sin z}{z}\,dz\right| < K$$

where K is a constant independent of u. It thus follows from (9) that

$$(\pi I_1)^2 \leqslant \frac{1}{N}\left(\int\limits_0^1 K^2\,du + \int\limits_1^{\infty} \frac{2du}{u^2}\right) \leqslant \frac{C}{N} \to 0 \qquad (N \to \infty)$$

By analogy it is proved that

$$I_2 \to 0 \qquad (N \to \infty)$$

Next, making the substitutions $u = N(x-a-\delta)$ and $z = -z'$, we obtain

$$(\pi I_4)^2 = \frac{1}{N}\int_0^\infty du\left(\int_{-u-2N\delta}^{-u} \frac{\sin z}{z}\,dz\right)^2 = \frac{1}{N}\int_0^\infty du\left(\int_u^{u+2N\delta} \frac{\sin z}{z}\,dz\right)^2$$

Consequently,

$$(\pi I_4)^2 \leqslant \frac{1}{N}\left(\int_0^1 (2K)^2\,dz + \int_1^\infty \frac{4}{z^2}\,dz\right) < \frac{C_1}{N} \qquad (N \to \infty)$$

The relation

$$I_3 \to 0 \qquad (N \to \infty)$$

is proved similarly, which completes the proof of relation (7).

It is quite evident that in (5), (6) and (7) it is allowable to interchange the symbols $\sim$ and $\frown$.

It should be noted that if the intervals $(a-\delta, a+\delta)$ and $(b-\sigma, b+\sigma)$ do not intersect then, together with the obvious equality

$$\int \varphi_{a,\delta}(x)\,\varphi_{b,\sigma}(x)\,dx = 0$$

there hold the following important relations:

$$\int \tilde{\varphi}_{a,\delta}(x)\,\bar{\tilde{\varphi}}_{b,\sigma}(x)\,dx = \int \hat{\varphi}_{a,\delta}\,\bar{\hat{\varphi}}_{b,\sigma}\,dx = 0$$

Indeed, $\tilde{\varphi}_{a,\delta}, \tilde{\varphi}_{b,\sigma} \in L_2$, and therefore the integral

$$\int \tilde{\varphi}_{a,\delta}\bar{\tilde{\varphi}}_{b,\sigma}\,dx = \lim_{N\to\infty}\int_{-N}^{N} \tilde{\varphi}_{a,\delta}\bar{\tilde{\varphi}}_{b,\sigma}\,dx =$$

$$= \lim_{N\to\infty}\frac{1}{2\pi}\int_{-N}^{N}\int e^{-ixu}\varphi_{a,\delta}(u)\,du\int e^{ixv}\varphi_{b,\sigma}(v)\,dv\,dx =$$

$$= \lim_{N\to\infty}\int \varphi_{a,\delta}(u)\left(\frac{1}{\pi}\int \varphi_{b,\sigma}(v)\frac{\sin N(u-v)}{u-v}\,dv\right)du =$$

$$= \lim_{N\to\infty}\int \varphi_{a,\delta}(u)\,\hat{\tilde{\varphi}}^{N}_{b,\sigma}(u)\,du = \int \varphi_{a,\delta}\varphi_{b,\sigma}\,du$$

exists and is absolutely convergent.

All the three integrals in the third member have finite limits of integration and therefore the change of order of integration is legitimate, which leads, after the integration with respect to x is performed, to the fourth member. The passage from the fifth member to the sixth one is justified by the fact that $\hat{\tilde{\varphi}}^{N}_{b,\sigma} \to \varphi_{b,\sigma}$ in the sense of L_2.

An arbitrary finite step function of one variable can be written in the form

$$f(x) = \sum_1^m c_k\varphi_{a_k,\delta_k}(x)$$

where c_k are constant coefficients (complex, in the general case) and the intervals $(a_k-\delta_k, a_k+\delta_k)$ and $(a_l-\delta_l, a_l+\delta_l)$ do not intersect for $k \neq l$.

In the further presentation we limit ourselves to the two-dimensional case although all the statements and proofs can readily be extended to the n-dimensional case.

In the case of dimension 2 the simplest finite step function is of the form

$$\varphi_\Delta(x, y) = \begin{cases} 1 & \text{for} \quad (x, y) \overline{\in} \Delta \\ 0 & \text{for} \quad (x, y) \overline{\in} \Delta \end{cases}$$

where

$$\Delta = \{a-\delta \leqslant x < a+\delta, \quad b-\sigma \leqslant y < b+\sigma\}$$

is a rectangle. At all the points of continuity of this function we have

$$\varphi_\Delta(x, y) = \varphi_{a,\delta}(x)\,\varphi_{b,\sigma}(y) \tag{10}$$

An arbitrary finite step function f of the two variables (x, y) can be written as a finite sum of the form

$$f(x, y) = \sum_1^m c_k \varphi_{\Delta^k}(x, y) = \sum_1^m c_k \varphi_{a_k, \delta_k}(x)\,\varphi_{b_k, \sigma_k}(y) \tag{11}$$

where c_k are, generally speaking, some complex constants and the rectangles

$$\Delta^1, \ldots, \Delta^m \tag{12}$$

are pairwise disjoint.

It follows that

$$\begin{aligned} &\tilde{f}^N(x, y) \to \tilde{f}(x, y) = \sum_1^m c_k \tilde{\varphi}_{a_k, \delta_k}(x)\,\tilde{\varphi}_{b_k, \sigma_k}(y) \qquad (N \to \infty) \\ &\hat{f}^N = \sum_1^m c_k \hat{\tilde{\varphi}}^N_{a_k, \delta_k}(x)\,\hat{\tilde{\varphi}}^N_{b_k, \sigma_k}(y) \to \sum_1^m c_k \varphi_{a_k, \delta_k}(x)\,\varphi_{b_k, \sigma_k}(y) = f(x, y) \qquad (N \to \infty) \end{aligned} \tag{13}$$

where the convergence is understood in the pointwise sense (in the second relation at the points of continuity of the functions $\varphi_{a_k, \delta_k}(x)\,\varphi_{b_k, \sigma_k}(y)$) and also in the sense of the mean square.

The set of all step functions will be denoted $\mathfrak{M}$.

For any two step functions $f, \psi \in \mathfrak{M}$ we can write

$$\psi(x, y) = \sum_1^m c_k' \varphi_{\Delta^k}(x, y) \tag{14}$$

where the system of rectangles Δ^k is the same as in (12) because the union of the two systems of rectangles specifying, respectively, f and ψ can obviously be represented as a sum of a finite number of rectangles which are only allowed to intersect along some parts of their boundaries.

It is important to note that

$$(f, \varphi) \iint f(x, y)\overline{\varphi(x,y)}\, dx\, dy =$$

$$= \sum_{1}^{m} c_k \bar{c}'_k \iint \varphi_{a_k, \delta_k}(x)\, \varphi_{b_k, \sigma_k}(y)\, \overline{\varphi_{a_k, \delta_k}(x)\, \varphi_{b_k, \sigma_k}(y)}\, dx\, dy =$$

$$= \sum_{1}^{m} c_k \bar{c}'_k \int |\varphi_{a_k, \delta_k}(x)|^2\, dx \int |\varphi_{b_k, \sigma_k}(y)|^2\, dy =$$

$$= \sum_{1}^{m} c_k \bar{c}'_k \int |\tilde{\varphi}_{a_k, \delta_k}(x)|^2\, dx \int |\tilde{\varphi}_{b_k, \sigma_k}(y)|^2\, dy = (\tilde{f}, \tilde{\varphi}) \tag{15}$$

because the functions $\varphi_{\Delta^k}(x, y)$ and $\varphi_{\Delta^l}(x, y)$ are obviously orthogonal for $k \neq l$ in the two-dimensional sense, and in the one-dimensional case there hold equalities (4).*

The last equality in (15) is proved like the equality of the first and fourth members by replacing in the latter f and φ by $\tilde{f}$ and $\tilde{\varphi}$ respectively and taking into account the orthogonality of $\tilde{\varphi}_{a_k, \delta_k}$ and $\tilde{\varphi}_{a_i, \delta_i}$ for $k \neq i$. In particular, from (15) it follows that

$$(f, f) = (\tilde{f}, \tilde{f}) \qquad \text{for all } f \in \mathfrak{M}$$

§ 16.10. Plancherel's Theorem. Estimating Speed of Convergence of Fourier's Integral

The complete statement of Plancherel's theorem involves the space L_2 whose definition is based on the notion of the Lebesgue integral; the advantage of the space L_2 over L'_2 is that the former is complete (see property 20, § 19.3). Here we shall state this theorem and present its proof based on the fact (see property 18 in § 19.3) that *the set of all finite step functions is dense in* L_2.

Theorem 1 (M. Plancherel). *For every function* $f \in L_2 = L_2(R_n)$ *there exist its Fourier transforms* $\tilde{f}, \hat{f} \in L_2$:

$$\tilde{f}(x) = \lim_{N \to \infty} \tilde{f}^N(x) = \lim \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u) e^{-ixu}\, du \tag{1}$$

and

$$\hat{f}(x) = \lim_{N \to \infty} \hat{f}^N(x) = \lim \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u) e^{ixu}\, du \tag{1'}$$

where $u = (u_1, \ldots, u_n)$, $x = (x_1, \ldots, x_n)$, $ux = \sum_{j=1}^{n} u_j x_j$

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* In this argument we consider the scalar product of f and φ (involving the complex conjugate of φ) in order to use formula (15) in the next section (see the footnote concerning (4) in § 16.6).

and $$\Delta_N = \{|x_j| \leqslant N; j = 1, \ldots, n\}$$

and the convergence is understood in the sense of the space L_2.

Fourier's transformations $f \to \tilde{f}$ *and* $f \to \hat{f}$ *possess the following properties:*

(i) *they are linear;*

(ii) *they produce one-to-one mappings of* L_2 *onto* L_2*;*

(iii) *they are mutually inverse, that is* $f = \hat{\tilde{f}} = \tilde{\hat{f}}$, $f \in L_2$*;*

(iv) *under these mappings the scalar product*

$$(f, \varphi) = \int f(u)\overline{\varphi(u)}\, du \qquad \left(\int_{R_n} = \int\right)$$

is preserved $((f, \varphi) = (\tilde{f}, \tilde{\varphi}) = (\hat{f}, \hat{\varphi});\ f, \varphi \in L_2)$, *and thus the mappings are isometric* $(||f|| = ||\tilde{f}|| = ||\hat{f}||)$.

Proof. In the foregoing section we proved that the finite step functions $f\ (f \in \mathfrak{M})$ not only belong to L_2 but also are mapped under Fourier's transformations $f \to \tilde{f}$ and $f \to \hat{f}$ into L_2 and that property (iv) holds for them.

Now let us consider an arbitrary function $f \in L_2$ and suppose, at present, that it is finite. Then, for sufficiently large N, its support belongs to Δ_N and

$$\tilde{f}(x) = \tilde{f}^N(x) \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} f(u) e^{-ixu}\, du \tag{2}$$

In the case under consideration $f \in L$, and therefore (see § 16.2) $\tilde{f}$ is a continuous function on $R = R_n$. Let us show that it belongs to L_2 and that

$$||\tilde{f}|| = ||f|| \tag{3}$$

(here $||\ || = ||\ ||_{L_2}$). Indeed, there is a sequence of finite step functions $f_\nu\ (f_\nu \in \mathfrak{M})$ with supports belonging to Δ_N such that

$$||f - f_\nu|| \to 0 \qquad (\nu \to \infty) \tag{4}$$

It follows that

$$|\tilde{f}(x) - \tilde{f}_\nu(x)| = \frac{1}{(2\pi)^{n/2}} \left| \int_{\Delta_N} [f(t) - f_\nu(t)] e^{-ixt}\, dt \right| \leqslant$$
$$\leqslant \frac{1}{(2\pi)^{n/2}} ||f - f_\nu|| (2N)^{n/2} \to 0 \qquad (\nu \to \infty)$$

Consequently, $\tilde{f}_\nu(x)$ uniformly converges to $\tilde{f}(x)$ on R_n. By virtue of (4) and property (iv) (the latter has already been checked for the functions $f_\nu \in \mathfrak{M}$), we can write, for an arbitrary $\varepsilon > 0$ and sufficiently large s, the inequalities

$$\varepsilon > ||f_p - f_q|| = ||\tilde{f}_p - \tilde{f}_q|| \geqslant ||\tilde{f}_p - \tilde{f}_q||_{L_2(\Delta_\lambda)}, \qquad p, q > s \tag{5}$$

where $\lambda > 0$ is an arbitrary number. Now, since $\tilde{f}_q$ converges uniformly to $\tilde{f}$ and the rectangle Δ_λ has finite dimensions, we can pass to the limit as $q \to \infty$ under the sign of the norm (i.e. under the integral sign), which yields

$$\varepsilon \geqslant ||\tilde{f}_p - \tilde{f}||_{L_2(\Delta_\lambda)} \qquad (p > s)$$

On fixing p and making λ increase indefinitely we arrive at the limiting relation

$$\varepsilon \geqslant \|\tilde{f}_p-\tilde{f}\| \qquad (p > s)$$

Since we also have

$$\|\tilde{f}_p-\tilde{f}\| \geqslant \big|\|\tilde{f}_p\|-\|\tilde{f}\|\big| \qquad (p > s)$$

it follows that

$$\|\tilde{f}\| = \lim_{p\to\infty} \|\tilde{f}_p\| = \lim_{p\to\infty} \|f_p\| = \|f\|$$

which proves that $\tilde{f} \in L_2$ and that equality (3) holds.

The validity of relation (1) for the function $f \in L_2$ (with compact support) readily follows from equality (2) for sufficiently large N.

Now let us take an arbitrary function $f \in L_2$ (not necessarily finite). In this case for positive N and N' $(N < N')$ we have (see the explanations below)

$$\|f^{N'}-f^N\| = \left\| \frac{1}{(2\pi)^{n/2}} \int\limits_{\Delta_{N'}-\Delta_N} f(u)e^{-ixu}\,du \right\| =$$

$$= \|f\|_{L_2(\Delta_{N'}-\Delta_N)} \to 0, \qquad N, N' \to \infty \tag{6}$$

and, by the completeness of L_2, there is a function belonging to L_2 which we denote by $\tilde{f}$ such that

$$\|\tilde{f}-\tilde{f}^N\| \to 0 \qquad (N \to \infty) \tag{7}$$

that is relation (1) holds.

The function $\tilde{f}^{N'}-\tilde{f}^N$ can be regarded as Fourier's transform of a function with compact support which is equal to $f(x)$ on $\Delta_{N'}-\Delta_N$ and equal to zero at the other points. For a function of this type we have already proved that its norm is preserved under Fourier's transformation, which accounts for (6).

From (7) it follows that

$$\|\tilde{f}\| = \lim_{N\to\infty} \|\tilde{f}^N\| = \lim_{N\to\infty} \|f\|_{L_2(\Delta_N)} = \|f\|$$

The linearity of the operation $\sim$ is obvious. Its continuity can also be readily shown:

$$\|\tilde{f}_k-\tilde{f}\| = \|\widetilde{f_k-f}\| = \|f_k-f\| \to 0 \qquad (k \to \infty)$$

Since $\tilde{f}(-x) = \hat{f}(x)$, the results obtained apply to the operation $\wedge$ as well.

Now we can easily prove the equalities

$$f = \hat{\tilde{f}} = \tilde{\hat{f}}, \qquad f \in L_2 \tag{8}$$

Indeed, these equalities hold for the functions $f \in \mathfrak{M}$, and, since for $f \in L_2$ there is a sequence of functions $f_\nu \in \mathfrak{M}$ such that $\|f_\nu-f\| \to 0$, the continuity of the operations $\sim$ and $\wedge$ allows us to obtain equality (8) by passing to the limit as $\nu \to \infty$ in the equalities

$$f_\nu = \hat{\tilde{f}}_\nu = \tilde{\hat{f}}_\nu$$

If $\psi \in L_2$ then $\hat{\psi} \in L_2$ and $\tilde{\hat{\psi}} = \psi$. This shows that under the operation $\sim$ the space L_2 is mapped onto itself, the mapping being one-to-one since $\tilde{\psi}_1 = \tilde{\psi}_2$ implies $\psi_1 - \psi_2 = \widehat{\widetilde{\psi_1 - \psi_2}} = \hat{0} = 0$.

In these considerations the symbols $\sim$ and $\frown$ can be interchanged. We have thus proved property (ii).

Finally, if

$$\|f - f_\nu\|,\ \|\varphi - \varphi_\nu\| \to 0 \qquad (\nu \to \infty, f_\nu, \varphi_\nu \in \mathfrak{M})$$

then on passing to the limit in the equalities

$$(f_\nu, \varphi_\nu) = (\tilde{f}_\nu, \tilde{\varphi}_\nu)$$

we arrive at

$$(f, \varphi) = (\tilde{f}, \tilde{\varphi})$$

Indeed, we have

$$|(f, \varphi) - (f_\nu, \varphi_\nu)| = |((f - f_\nu), \varphi) - (f_\nu, \varphi_\nu - \varphi)| \le$$
$$\le \|f - f_\nu\|\ \|\varphi\| + \|f_\nu\|\ \|\varphi_\nu - \varphi\| \to 0 \qquad (\nu \to \infty)$$

The theorem has been proved.

Theorem 2 (Analogue of Theorem 6, § 15.11). *Let* $\boldsymbol{\lambda} = (\lambda, \ldots, \lambda)$ *be a vector whose all components are equal to a natural number* λ *and let, for any nonnegative integral vector* $\boldsymbol{k} \le \boldsymbol{\lambda}$, *the partial derivative* $f^{(\boldsymbol{k})}$ *(in particular, the function* f *itself) be continuous. Also let the inequality*

$$\frac{1}{(2\pi)^n} \int |f^{(\boldsymbol{l})}(x)|^2\, dx \le M^2$$

hold for any vector $\boldsymbol{l} = (l_1, \ldots, l_n)$ *with components* l_j *equal to* 0 *or to* λ. *Then the deviation of the integral* $S_N(x)$ *corresponding to the function* $f(x)$ *from that function satisfies the inequality*

$$|f(x) - S_N(x)| \le \frac{CM}{N^{\lambda - 1/2}} \tag{9}$$

where C *depends on* λ *but does not depend on* M *and* N.

Proof. Let us estimate the expression

$$\varrho_N(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta'_N} \tilde{f}(u) e^{ixu}\, du$$

where

$$\Delta'_N = R_n - \Delta_N$$

The set Δ'_N can be described as the collection of the points $u = (u_1, \ldots, u_n)$ such that $\max\limits_j |u_j| \ge N$.

Since $f \in L_2$, Plancherel's theorem implies that $\tilde{f} \in L_2$, and hence $\varrho_N(x)$ can be regarded as the limit (in the mean) of the form

$$\varrho_N(x) = \lim_{N_1 \to \infty} \frac{1}{(2\pi)^{n/2}} \int_{\Delta_{N_1} - \Delta_N} \tilde{f}(u) e^{ixu}\, du$$

Now, taking an arbitrary integer m satisfying the inequalities $0 \leqslant m < n$, we define the set Ω_m of the points $u = (u_1, \dots, u_n)$ such that

$$\left.\begin{array}{ll} |u_j| \leqslant 1 \qquad (j = 1, \dots, m \quad \text{if} \quad m > 0) & \\ |u_{m+1}| \geqslant N,\ |u_j| \geqslant 1 \qquad (j = m+2, \dots, n \quad \text{if } n > m+1) & \end{array}\right\} \tag{10}$$

The symbol Ω_m will also denote any set reducible to the one defined above by means of the corresponding renumbering of the coordinates.

We obviously have

$$\Delta'_N = \sum_{m=0}^{n-1} \sum \Omega_m$$

where the inner sum extends over all the possible Ω_m's corresponding to m. On arranging the summands entering into this sum into a sequence and renumbering them as $(\Delta_1, \Delta_2, \dots, \Delta_p)$, we put

$$E_1 = \Delta_1, \quad E_2 = \Delta_2 - E_1, \dots, E_p = \Delta_p - \sum_1^{p-1} E_k$$

and obtain

$$\Delta'_N = \sum_1^p E_k, \quad E_k E_s = 0 \qquad (k \neq s)$$

Every set E_k thus constructed belongs to some Ω_m.

Now we can write (at present, purely formally; see the explanations below)

$$\varrho_N(x) = \frac{1}{(2\pi)^{n/2}} \sum_{k=1}^{p} \int_{E_k} \tilde{f}(u) e^{ixu}\, du \tag{11}$$

Let us estimate one of the integrals $\int\limits_{E_k}$ $(k = 1, \dots, p)$. For definiteness, let $E_k \subset \Omega_m$ where Ω_m is that very set which was defined by inequalities (10).

By virtue of the equality

$$\widetilde{\frac{\partial^{(n-m)\lambda} f}{\partial x_{m+1}^{\lambda} \dots \partial x_n^{\lambda}}} = i^{(n-m)\lambda}\, u_{m+1}^{\lambda} \dots u_n^{\lambda} \tilde{f} \tag{12}$$

which can be rewritten as

$$\tilde{f}(u) = \frac{1}{i^{(n-m)\lambda} u_{m+1}^{\lambda} \dots u_n^{\lambda}}\, \widetilde{\frac{\partial^{(n-m)\lambda} f}{\partial x_{m+1}^{\lambda} \dots \partial x_n^{\lambda}}}(u)$$

we obtain

$$\int_{E_k} |\tilde{f}(u) e^{ixu}|\, du \leqslant \int_{E_k} \left| \frac{1}{u_{m+1}^{\lambda} \dots u_n^{\lambda}}\, \frac{\widetilde{\partial^{(n-m)\lambda} f}(u)}{\partial x_{m+1}^{\lambda} \dots \partial x_n^{\lambda}} \right| du \leqslant$$

$$\leqslant \left(\int_{E_k} \frac{du}{u_{m+1}^{2\lambda} \dots u_n^{2\lambda}} \right)^{1/2} \left(\int_{R_n} \left| \frac{\partial^{(n-m)\lambda} f}{\partial x_{m+1}^{\lambda} \dots \partial x_n^{\lambda}} \right|^2 du \right)^{1/2} \leqslant \frac{C'M}{N^{\lambda - 1/2}} \tag{13}$$

It follows that

$$|\varrho_N(x)| \leqslant \frac{CM}{N^{\lambda-1/2}} \tag{14}$$

where C, as well as C', is independent of M and N.

Estimates (13) show that integrals (11) are in fact convergent for any x, the convergence being absolute. Moreover, for $N \to \infty$, these integrals, and, together with them, $\varrho_N(x)$ converge uniformly to zero. Thus, the sequence of the functions

$$S_N(x) = \frac{1}{(2\pi)^{n/2}} \int_{\Delta_N} \tilde{f}(u) e^{ixu}\, du$$

converges uniformly for $N \to \infty$. At the same time, since $f \in L_2$, Plancherel's theorem shows that $S_N(x)$ tends to $f(x)$ as $N \to \infty$ in the sense of the mean square. Therefore (see Lemma 1, § 15.11) $S_N(x)$ converges uniformly to that very function $f(x)$, and consequently

$$\varrho_N(x) = f(x) - S_N(x) \tag{15}$$

Relation (9) follows from (14) and (15).

§ 16.11. Generalized Periodic Functions

Let S^* be the set of all infinitely differentiable periodic functions φ with period 2π dependent on one variable x.

Every function $\varphi \in S^*$ can be written as the sum of its Fourier series uniformly convergent to that function (see Theorem 2, § 16.10):

$$\varphi(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}, \qquad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{-ikt}\, dt \tag{1}$$

$$(k = 0, \pm 1, \pm 2, \ldots)$$

The series can be differentiated termwise any number of times (see § 15.7), that is

$$\varphi^{(s)}(x) = \sum_{-\infty}^{\infty} (ik)^s c_k e^{ikx} \qquad (s = 1, 2, \ldots) \tag{2}$$

and the series on the right-hand side of (2) is uniformly convergent to $\varphi^{(s)}(x)$ for any natural s.

If $\varphi_n, \varphi \in S^*$ $(n = 1, 2, \ldots)$ and $\varphi_n^{(s)}(x)$ uniformly converges to $\varphi^{(s)}(x)$ as $n \to \infty$ for any integer $s \geqslant 0$, we shall write

$$\varphi_n \to \varphi\ (S^*)$$

In particular, it is evident that if a function φ belongs to S^* and $S_N(x) = \sum_{-N}^{N} c_k e^{ikx}$ is its Nth Fourier sum then $S_N \to \varphi\ (S^*)$.

Let us denote by S'^{*} the totality of all continuous linear functionals f (generalized functions) on S^{*}.

An ordinary function $f \in L'^{*}$ (or $f \in L^{*}$) determines with the aid of the equality

$$(f, \varphi) = \int_{0}^{2\pi} f(t)\,\varphi(t)\,dt \qquad (\varphi \in S^{*}) \tag{3}$$

a (regular) generalized function belonging to S'^{*} which is denoted by the same letter f. Here we shall not repeat the argument of § 16.7 concerning the case of the space S; it can obviously be applied to the space S^{*} as well to show that two functionals (3) determined by two functions $f_1, f_2 \in L'^{*}$ coincide identically on S^{*} if and only if $f_1(x) = f_2(x)$ at the points of continuity of f_1 and f_2 (in the case of the space L^{*}, if and only if $f_1(x) = f_2(x)$ almost everywhere).

Since the functions e^{ikx} ($k = 0, \pm 1, \pm 2$) belong to S^{*}, to any functional $f \in S'^{*}$ there correspond the numbers

$$c_k = \frac{1}{2\pi}(f, e^{-ikt}) \qquad (k = 0, \pm 1, \pm 2, \ldots)$$

called the *Fourier coefficients* of f.

We shall prove the equality

$$\frac{1}{2\pi}(f, \varphi) = \lim_{N\to\infty} \frac{1}{2\pi}\left(\sum_{-N}^{N} c_k e^{ikx}, \varphi\right) = \sum_{-\infty}^{\infty} c_k c'_{-k},\ 2\pi c'_{-k} = (e^{ikx}, \varphi) \tag{4}$$

showing that the series

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx} \tag{5}$$

called the *Fourier series of the generalized function* f is convergent to f in the sense of the space S'^{*}.

Indeed, we have

$$\left(\sum_{-N}^{N} c_k e^{ikx}, \varphi\right) = \sum_{-N}^{N} = c_k(e^{ikx}, \varphi) = \frac{1}{2\pi}\sum_{-N}^{N} (f, e^{-ikx})(e^{ikx}, \varphi) =$$

$$= \frac{1}{2\pi}\sum_{-N}^{N} (f, (e^{ikx}, \varphi)e^{-ikx}) = \left(f, \frac{1}{2\pi}\sum_{-N}^{N} (e^{ikx}, \varphi)e^{-ikx}\right) =$$

$$= (f, S_N(x)) \to (f, \varphi) \qquad (N \to \infty)$$

because $S_N(x) \to \varphi(x)\,(S^{*})$.

It should be noted that, by virtue of the orthogonality of the functions e^{ikx}, there hold the relations

$$(f, e^{ilx}) = \lim_{N\to\infty} \sum_{-N}^{N} c_k(e^{ikx}, e^{ilx}) = 2\pi c_{-l}$$

It follows that the representation of a generalized function $f \in S'^*$ in the form of the sum of a convergent (in the sense of S'^*) series (5) with constant coefficients is unique.

Let us show that for every function $f \in S'^*$ there is a natural number λ dependent on f (but independent of k) and a constant A such that

$$|c_k| \leqslant A|k|^\lambda \qquad (k = \pm 1, \pm 2, \ldots) \tag{6}$$

Indeed, let us suppose the contrary. Then for each $\lambda = 1, 2, \ldots$ there must exist k_λ such that

$$|c_{k_\lambda}| \geqslant |k_\lambda|^\lambda \quad \text{and} \quad |k_1| < |k_2| < \ldots$$

Let us put

$$\varphi_\lambda(x) = \frac{e^{-ik_\lambda x}}{|k_\lambda|^\lambda} \qquad (\lambda = 1, 2, \ldots)$$

It is obvious that $\varphi_\lambda \in S^*$ and that the limiting relation

$$\varphi_\lambda^{(s)}(x) = \frac{(-ik_\lambda)^s e^{-ik_\lambda x}}{|k_\lambda|^\lambda} \to 0 \qquad (\lambda \to \infty)$$

holds uniformly for any nonnegative integer s; therefore

$$\varphi_\lambda \to 0 \qquad (S^*)$$

and

$$(f, \varphi_\lambda) \to 0 \qquad (\lambda \to \infty)$$

On the other hand, by virtue of (4) and by the orthogonality of the functions e^{ikx}, we have

$$\frac{1}{2\pi}|(f, \varphi_\lambda)| = |c_{k_\lambda} c' - k_\lambda| \geqslant |k_\lambda|^\lambda \frac{1}{|k_\lambda|^\lambda} = 1 \qquad (\lambda = 1, 2, \ldots)$$

which leads to a contradiction.

Now let us show that, conversely, if numbers c_k $(k = \pm 1, \pm 2, \ldots)$ satisfy inequality (6) for some A and $\lambda > 0$ for all k, then the functional

$$\frac{1}{2\pi}(f, \varphi) = \sum_{-\infty}^{\infty} c_k c'_{-k}, \quad 2\pi c'_{-k} = (\varphi, e^{ikx}) \tag{7}$$

is a generalized function $f \in S'^*$. Indeed, for $k \neq 0$ we have

$$c'_k = \frac{1}{2\pi(ik)^s} \int_0^{2\pi} e^{-ikt} \varphi^{(s)}(t)\, dt$$

whence

$$|c'_k| \leqslant \frac{\|\varphi^{(s)}\|C}{k^s} \quad \text{and} \quad \|\varphi^{(s)}\|_C = \max_t |\varphi^{(s)}(t)|$$

and for $s = \lambda + 2$ we obtain

$$\left|\frac{1}{2\pi} \sum_{-\infty}^{\infty}{}' c_k c'_k\right| \leqslant \frac{1}{2\pi} \sum_{-\infty}^{\infty}{}' A k^{\lambda - s} \|\varphi^{(s)}\|_C = \frac{A}{2\pi} \sum_{-\infty}^{\infty}{}' k^{-2} \|\varphi^{(s)}\|_C$$

(the prime in the symbol $\sum'$ indicates that the corresponding sum does not include the term with $k = 0$). On the other hand,

$$\frac{1}{2\pi}|c_0c_0'| \leqslant \frac{|c_0|}{2\pi}\|\varphi\|_C$$

Thus,

$$|(f, \varphi)| \leqslant B(\|\varphi^{(s)}\|_C+\|\varphi\|_C)$$

where B is a constant independent of φ; consequently, if $\varphi_n \to \varphi$ (S^*) then

$$|(f,\varphi)-(f,\varphi_n)| = |f,\varphi-\varphi_n)| \leqslant B(\|\varphi^{(s)}-\varphi_n^{(s)}\|_C+\|\varphi-\varphi_n\|_C)\to 0 \qquad (n\to\infty)$$

whence it follows that $f \in S'^*$.

As was already mentioned, functional (7) can also be written in form (5). It follows that the space S'^* can be defined as the totality of the formal series

$$f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

with coefficients satisfying inequalities (6) where A, $\lambda > 0$ are some constants dependent on the series in question, the values of f on $\varphi \in S^*$ being defined with the aid of equality (4). The delta-function $\delta(x)$ belonging to S'^* is determined by the series

$$\delta(x) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} e^{ikx} = \frac{1}{\pi}\left(\frac{1}{2}+\cos x+\cos 2x+ \ldots\right)$$

For each function $\varphi \in S^*$ we have

$$(\delta, \varphi) = \lim_{N\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2}+\sum_{1}^{N}\cos kt\right)\varphi(t)\,dt = \lim_{N\to\infty} S_N(\varphi, 0) = \varphi(0)$$

where $S_N(\varphi, 0)$ is the value of the Nth partial Fourier sum of the function φ assumed at the point $x = 0$.

By analogy with S' (see § 16.7 (7)), the derivative F' of a functional $F \in S'^*$ will be understood as the functional determined by the formula

$$(F', \varphi) = -(F, \varphi')$$

Let us show that if

$$F(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

then

$$F'(x) = \sum_{-\infty}^{\infty} (ik)\, c_k e^{ikx} \tag{8}$$

Note that, by virtue of the inequalities $|c_k| \leqslant A|k|^{\lambda}$ $(k = \pm 1, \pm 2, \ldots)$ holding for some A, $\lambda > 0$, there also holds the inequality

$$|ikc_k| \leqslant A|k|^{\lambda+1}$$

which, as we know, implies that series (8) is convergent in the sense of S'^* to a generalized function $\Phi \in S'^*$. Equality $\Phi = F'$ follows from the following calculations:

$$(\Phi, \varphi) = \lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{-N}^{N} (ik) c_k e^{ikx} \varphi(x)\, dx =$$

$$= - \lim_{N\to\infty} \int_{-\pi}^{\pi} \sum_{-N}^{N} c_k e^{ikx} \varphi'(x)\, dx = -(F, \varphi')$$

$\left(\text{here one should take into account that } \int_{-\pi}^{\pi} \varphi'(x)\, dx = 0\right)$.

The result we have established is a generalization of Theorem 1, § 15.7 on the termwise differentiation of the ordinary Fourier series.

If a function f belongs to L'^* (or $f \in L^*$) it also belongs to S'^* and its Fourier coefficients understood in the ordinary sense and in the sense of S'^* coincide. The Fourier series of a function $f \in L^*$ is not necessarily convergent to it; there is an example of a function $f \in L^*$ whose Fourier series is divergent everywhere on the real line (this is Kolmogorov's example, see § 15.5). On the other hand, what has been established above in the present section shows that the Fourier series of a function $f \in L'^*$ ($f \in L^*$) is always convergent to f in the sense of S'^*.

Here we shall also dwell on the representation of the convolution $\Phi * f$ of two functions $\Phi, f \in L^*$ in terms of their Fourier coefficients. Let

$$\Phi(x) = \sum_{-\infty}^{\infty} \lambda_k e^{ikx} \quad \text{and} \quad f(x) = \sum_{-\infty}^{\infty} c_k e^{ikx}$$

Then (see § 18.3)

$$\psi(x) = \Phi * f = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x-t) f(t)\, dt \qquad (\psi \in L^*)$$

The Fourier coefficients of the function ψ can be transformed with the aid of Fubini's theorem as follows:

$$\mu_\nu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x)\, e^{-i\nu x}\, dx =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \Phi(x-t)\, e^{-i\nu(x-t)} f(t)\, e^{-i\nu t}\, dt \right\} dx =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(t)\, e^{-i\nu t}\, dt \left\{ \frac{1}{2\pi} \int_0^{2\pi} \Phi(x-t)\, e^{-i\nu(x-t)}\, dx \right\} = c_\nu \lambda_\nu$$

Consequently,

$$\Phi * f = \sum_{-\infty}^{\infty} c_k \lambda_k e^{ikx} \tag{9}$$

It appears natural to define the convolution of two arbitrary generalized functions Φ, $f \in S'^*$ by means of equality (9) because, together with the coefficients c_k and λ_k, the products $c_k\lambda_k$ also satisfy inequalities of type (6).

CHAPTER 17

Differentiable Manifolds and Differential Forms

§ 17.1. Differentiable Manifolds

By a *k-dimensional* $(0 < k < n)$ *differentiable manifold* in the n-dimensional space $R = R_n$ is meant a set $S \subset R$ for whose any point x^0 there are a permutation $j_1, \ldots, j_n$ of the natural numbers $1, \ldots, n$ and a rectangle

$$\Delta = \{|x_{j_s} - x_{j_s}^0| < \delta;\ s = 1, \ldots, k;\ |x_{j_l} - x_{j_l}^0| < \sigma;\ l = k+1, \ldots, n\}$$

which cuts from S a part $S\Delta \subset S$ described by equations

$$x_{j_l} = f_l(x_{j_1}, \ldots, x_{j_k});\ l = k+1, \ldots, n;\ (x_{j_1}, \ldots, x_{j_k}) \in \Delta'$$
$$\Delta' = \{|x_{j_s} - x_{j_s}^0| < \delta,\ s = 1, \ldots, k\} \tag{1}$$

where f_l are continuously differentiable functions on Δ'.

An *n-dimensional (differentiable) manifold* in $R = R_n$ is an arbitrary open set $S \subset R_n$.

If a number $\delta > 0$ mentioned in the above definition has been found it can obviously be arbitrarily decreased with the retention of the property stated. For instance, we can take as Δ a rectangle

$$\Delta = \{|x_{j_s} - x_{j_s}^0| < \delta_s,\ s = 1, \ldots, k,\ |x_{j_l} - x_{j_l}^0| < \sigma,\ l = k+1, \ldots, n\}$$

where $\delta_s \leqslant \delta$. We can also set an arbitrary number $\sigma_0 < \sigma$ and, using the continuity of f_l, choose $\delta_0 \leqslant \delta$ so that the functions f_l differ, respectively, from $x_{j_l}^0$ by less than σ_0 for $|x_{j_s} - x_{j_s}^0| < \delta_0$, $s = 1, \ldots, k$; then, obviously, the rectangle

$$\Delta^0 = \{|x_{j_s} - x_{j_s}^0| < \delta_0,\ s = 1, \ldots, k,\ |x_{j_l} - x_{j_l}^0| < \sigma_0,\ l = k+1, \ldots, n\}$$

will also satisfy the requirements of the definition.

Thus, the expression "there is a rectangle Δ" entering into this definition can be replaced by "for any $\varepsilon > 0$ there is a rectangle Δ with diameter less than ε", the new definition being equivalent to the former.

System of equations (1) is a special case of a system

$$x_i = F_i(u) = F_i(u_1, \ldots, u_k),\ u \in \omega;\ i = 1, \ldots, n \tag{2}$$

where ω is a domain in the k-dimensional space of the points $u = (u_1, \ldots, u_k)$ and the functions F_i are continuously differentiable on ω, the matrix

$\left\|\frac{\partial F_i}{\partial u_j}\right\|$ having rank k. When the point $\boldsymbol{u}$ runs over ω the point $\boldsymbol{x}$ describes a set $S \subset R$ which can naturally be called a *k-dimensional surface*. The correspondence between the points $\boldsymbol{u}$ and $\boldsymbol{x}$ established by (2) is supposed to be one-to-one: $\omega \ni \boldsymbol{u} \rightleftarrows \boldsymbol{x} \in S$ (in the contracted notation we shall also write $\boldsymbol{u} \rightleftarrows \boldsymbol{x}$ or $\omega \rightleftarrows S$).

For the sake of brevity we shall simply speak of a surface S *represented (described) by functions* (or *equations*) (2) without enumerating all the indicated properties.

Let $\boldsymbol{u} = (u_1, \dots, u_k)$ be connected with $\boldsymbol{u}' = (u_1', \dots, u_k')$ with the aid of a continuously differentiable one-to-one mapping

$$u_j' = \varphi_j(u_1, \dots, u_k), \qquad \boldsymbol{u} \in \omega \rightleftarrows \omega' \ni \boldsymbol{u}' \tag{3}$$

whose Jacobian is different from zero on ω. In such a case we shall say that mapping (3) is *continuously differentiable in both directions* meaning that the mapping is continuously differentiable and one-to-one and that its Jacobian is different from zero (note that the condition that the mapping in question is one-to-one implies that the Jacobian of the transformation from $\boldsymbol{u}$ to $\boldsymbol{u}$ which is equal to 1 is expressible as the product of the Jacobians of the transformations from $\boldsymbol{u}$ to $\boldsymbol{u}'$ and from $\boldsymbol{u}'$ to $\boldsymbol{u}$, and hence both Jacobians are different from zero).

The solution of equations (3) with respect to u_j yields

$$u_j = \psi_j(u_1', \dots, u_k'), \qquad \boldsymbol{u}' \in \omega'$$

On substituting u_j in (2) we obtain a new (parametric) representation of S:

$$x_i = \Phi_i(\boldsymbol{u}') = \Phi_i(u_1', \dots, u_k') = F_i(\psi_1, \dots, \psi_k), \qquad \boldsymbol{u}' \in \omega', i = 1, \dots, n$$

where the functions Φ_i possess the same properties as F_i.

A surface S determined by equations of type (2) is not necessarily a k-dimensional manifold in the sense of the above definition (e.g. see Fig. 6.1 in § 6.5). In this connection the lemma below provides a simple sufficient test for a surface S described by equations (2) to be a k-dimensional differentiable manifold.

Lemma 1. *Let S be a surface representable by equations* (2) *with functions F_i possessing the above-indicated properties. Let ω in* (2) *be a bounded domain and let the functions F_i be defined not only on ω but also on $\bar{\omega}$ and specify a homeomorphism $\bar{\omega} \rightleftarrows \bar{S}$, that is a one-to-one mapping of $\bar{\omega}$ onto $\bar{S}$ continuous in both directions.*

Then S is a k-dimensional differentiable manifold.

Proof. Let $\boldsymbol{x}^0 \in S$ be the point corresponding to a value $\boldsymbol{u} = \boldsymbol{u}^0$ of the vector parameter $\boldsymbol{u}$; then ω contains a sufficiently small open cube λ with centre at $\boldsymbol{u}^0$ such that one of the minors of the kth order of the matrix $\left\|\frac{\partial F_i}{\partial u_j}\right\|$, say $\frac{D(x_1, \dots, x_k)}{D(u_1, \dots, u_k)}$, is different from zero in that cube, and the first k equations of system (2) can be resolved with respect to $u_1, \dots, u_k$:

$$u_j = \psi_j(x_1, \dots, x_k),\ (x_1, \dots, x_k) \in \mu \rightleftarrows \lambda \qquad (j = 1, \dots, k)$$

Consequently, a certain part S' of S ($S' \subset S$) is described by the continuously differentiable functions

$$x_i = \Phi_i(x_1, \dots, x_k), \quad (x_1, \dots, x_k) \in \mu, \quad i = k+1, \dots, n \tag{4}$$

resulting from the substitution of $u_j = \psi_j$ ($j = 1, \dots, k$) into the last $n-k$ equations (2).

The image of the *compact* $\bar{\omega}-\lambda$ (by a compact in R_n is meant a bounded and closed set) under the mapping $S \rightleftarrows \bar{\omega}$ is a compact F not containing the point x^0 and therefore not containing some rectangle Δ with centre at x^0 and edges parallel to the coordinate axes. By the continuity of the functions Φ_i, the dimensions of the projection Δ' of the rectangle Δ into the coordinate subspace $x_1, \dots, x_k$ can be diminished (we retain the notation Δ for the new rectangle) so that the points $(x_1, \dots, x_k, \Phi_{k+1}, \dots, \Phi_n)$ belong to Δ for all $(x_1, \dots, x_k) \in \Delta' \subset \mu$. Then the part $S\Delta$ is described by the equations

$$x_i = \Phi_i(x_1, \dots, x_k), \quad (x_1, \dots, x_k) \in \Delta', \quad i = k+1, \dots, n$$

Indeed, the points x described by these equations belong to $S\Delta$ and there are no other points of S belonging to Δ because these are either points of form (4) where $(x_1, \dots, x_k) \in \mu-\Delta'$ or points belonging to F.

Lemma 2. *Let a surface S be represented by equations* (2) *where the functions F_i possess all the indicated properties and let S contain a surface σ described by functions*

$$x_i = \Phi_i(v), \quad v = (v_1, \dots, v_k) \in G, \quad i = 1, \dots, n \tag{5}$$

where G is a domain and Φ_i possess the same properties as F_i.

Then the obvious one-to-one correspondence

$$G \ni v \rightleftarrows u \in \omega' \subset \omega \tag{6}$$

determined by systems (2) *and* (5) *is continuously differentiable in both directions and hence σ is described by functions* (2):

$$x_i = F_i(u), \quad u \in \omega', \quad i = 1, \dots, n$$

where ω' is a domain.

Besides, if S is a k-dimensional differentiable manifold then so is σ.

In particular, if σ is described by functions (2) *in which $u \in G \subset \omega$ then both σ and S are k-dimensional differentiable manifolds.*

The proof of the lemma is based on the following proposition (see § 7.18). Suppose we are given two continuously differentiable functions

$$v = \alpha(u), \quad u \in g \subset R_k \quad \text{and} \quad w = \beta(v), \quad v \in g' \in R_k$$

which specify, respectively, some mappings of the domains g and g' belonging to R_k into R_k. Let $u^0 \in g$ and $v^0 = \alpha(u^0) \in g'$. Then the composite function (mapping) $w = \beta(\alpha(u))$ exists in a sufficiently small neighbourhood $\omega \subset g$ of the point u^0 and is continuously differentiable in ω, its Jacobian (i.e. the Jacobian of the transformation from u to w) being equal to the product of the Jacobians of the transformations α and β.

In particular, if the Jacobian of the transformation from u to w is different from zero on ω, so is the Jacobian of the transformation from u to v, and the image of the domain ω under the mapping specified by α is also a domain.

Relations (5) assign to an arbitrary point $v \in G$ a definite point $x = (x_1, \ldots, x_n) \in \sigma \subset S$, to which, by virtue of (2), there corresponds a uniquely determined point $u \in \omega'$. At the latter point the rank of the matrix $\left\| \frac{\partial F_i}{\partial u_j} \right\|$ is equal to k, and consequently, for some pairwise distinct values $i = i_1, \ldots, i_k$ of the index i, equations (2) can be resolved with respect to $u_1, \ldots, u_k$. This results in the two consecutive continuously differentiable mappings

$$G \ni v \to (x_{i_1}, \ldots, x_{i_k}) \to u \in \omega'$$

making sense not only at the original point v but in some neighbourhood of v as well.

A similar argument shows that to the point $u \in \omega'$ thus obtained there corresponds, under two consecutive continuously differentiable mappings, the original point $v \in G$.

We have proved that correspondence (6) is continuously differentiable in both directions, and, since by the hypothesis G is a domain, so is ω'.

Now let S be a k-dimensional differentiable manifold, and $x^0 \in \sigma$. According to the definition of S, it is possible to renumber the coordinates and to find an open rectangle Δ with centre at x^0 so that $S\Delta$ is described by some continuously differentiable functions

$$x_i = f_i(x_1, \ldots, x_k), \quad i = k+1, \ldots, n,. \quad (x_i, \ldots, x_k) \in \Delta' \tag{7}$$

where Δ' is the projection of Δ into the coordinate subspace $(x_1, \ldots., x_k)$. By what has already been proved, there is a correspondence

$$\Delta' \ni (x_1, \ldots, x_k) \rightleftarrows u \in \omega'' \subset \omega$$

continuously differentiable in both directions, ω'' being a domain. The intersection $\omega'\omega''$ of ω' and ω'' is an open set containing the point u^0 corresponding, under (2), to the point x^0. Consequently, the cube Δ' can obviously be replaced by a smaller cube with faces parallel to those of the former and centre at the same point (for the diminished cube we retain the notation Δ') so that the new domain ω'' belongs to ω'. We see that under these assumptions functions (7) now describe $\sigma\Delta = S\Delta$, which shows that σ is a manifold.

Lemma 3. *The intersection of two k-dimensional (differentiable) manifolds*

$$\sigma_1 = \{x: x_i = \varphi_i(u), \quad i = 1, \ldots, n, \quad u \in U\}$$

and

$$\sigma_2 = \{x: x_i = \psi_i(v), \quad i = 1, \ldots, n, \quad v \in V\}$$

belonging to a k-dimensional manifold

$$S = \{x\colon x_i = f_i(w),\quad i = 1, \ldots, n,\quad w \in W\}$$

is a k-dimensional manifold representable as

$$\sigma_1\sigma_2 = \{x\colon x_i = f_i(w),\quad i = 1, \ldots, n,\quad w \in W_*\}\qquad W_* \subset W$$

where generally speaking, W_* is an open set (and not necessarily a domain even when U, V and W are domains).

Proof. Since $\sigma_1, \sigma_2 \subset S$, Lemma 2 implies that

$$\sigma_1 = \{x\colon x_i = f_i(w),\quad i = 1, \ldots, n,\quad w \in W_1\},\qquad W_1 \subset W$$

and

$$\sigma_2 = \{x\colon x_i = f_i(w),\quad i = 1, \ldots, n,\quad w \in W_2\},\qquad W_2 \subset W$$

where W_1 and W_2 are domains together with U, V and W. It follows that the assertion of the lemma holds with $W_* = W_1W_2$.

Now let us show that the following definition of a k-dimensional differentiable manifold S is equivalent to the one stated at the beginning of the section: *a k-dimensional differentiable manifold is a set $S \subset R_n$ for whose any point x^0 and any $\varepsilon > 0$ there is an n-dimensional neighbourhood Ω of that point with diameter $d(\Omega) < \varepsilon$ such that $S\Omega$ is representable by relations* (2) (*with functions possessing the properties indicated for* (2)) *where S should be replaced by $S\Omega$.*

Indeed, if we put $\Delta = \Omega$, $\Delta' = \omega$

$$x_{j_s} = u_s = F_{j_s}(u_1, \ldots, u_k)\qquad s = 1, 2, \ldots, k$$

and

$$x_{j_s} = f_{j_s}(u_1, \ldots, u_k) = F_{j_s}(u_1, \ldots, u_k)\qquad s = k+1, \ldots, n$$

in the former definition where, as we know, it is allowable to assume that $d(\Delta) < \varepsilon$ for any $\varepsilon > 0$, this will result in a system of functions F_i satisfying the requirements of the latter definition.

Now, let S satisfy the conditions of the latter definition and let x^0 be a point belonging to S. Then there is a neighbourhood Ω of that point with diameter $d(\Omega) < 1$ such that $S\Omega$ is described by functions (2) (possessing the properties indicated for (2)). Let us suppose that the point x^0 corresponds to a point $u^0 \in \omega$ and let us take $\delta > 0$ so small that the ball $|u-u^0| \leqslant \delta$ belongs to ω. The spherical surface $|u-u^0| = \delta$ is mapped, under the transformation specified by functions (2), onto a set $\Gamma \subset S$ which is bounded and closed. Since it does not contain the point x^0, we have

$$\min_{x \in \Gamma} |x-x^0| = m > 0$$

Now, proceeding from the latter definition, let us find a second neighbourhood Ω_1 of the point x^0 with diameter $d(\Omega_1) < m/2$ such that $S\Omega_1$ is described by functions

$$x_i = \Phi_i(v),\quad v \in G,\quad i = 1, \ldots, k$$

where G is a domain. By Lemma 2, the set $S\Omega_1$ can also be described by the equations

$$x_i = F_i(\boldsymbol{u}), \quad \boldsymbol{u} \in \omega' \subset \omega \tag{8}$$

where ω' is a domain.

It is important to note that we have in fact

$$\omega' \subset \bar{\omega}' \subset \omega \tag{9}$$

because ω' belongs to the closed ball $|\boldsymbol{u}-\boldsymbol{u}^0| \leqslant \delta$ which belongs to ω according to the definition. It follows that equalities (8) not only set up a one-to-one correspondence $S\Omega_1 \rightleftharpoons \omega'$ but also $\overline{S\Omega_1} \rightleftharpoons \bar{\omega}'$, which indicates, by virtue of Lemma 1, that $S\Omega_1$ is a k-dimensional differentiable manifold (in the sense of the former definition). Hence, there is a rectangle $\Delta \subset \Omega_1$ with centre at $\boldsymbol{x}^0$ for which there hold the properties indicated in the former definition for $S\Omega_1$; consequently they hold for S as well since $S\Delta = S\Omega_1\Delta$.

It should be noted that the second definition of a k-dimensional differentiable manifold (in contrast to the first one) cannot be restated as "for any point $\boldsymbol{x}^0 \in S$ there is its n-dimensional neighbourhood Ω such that ..." without indicating that there should exist a neighbourhood of this kind with arbitrarily small diameter. For if, for instance, the functions $F_i(\boldsymbol{u})$ specifying the surface S by means of (2) are bounded on ω, the set S can be embedded in a cube Δ, and then $S\Delta = S$ while S must not necessarily be a k-dimensional differentiable manifold.

Finally, we note that the equivalence of the two definitions implies that if a set S satisfies the conditions of the first definition relative to one coordinate system it satisfies them relative to any other system as well.

Lemma 4.* *If the intersection $\sigma_1\sigma_2$ of two k-dimensional (differentiable) manifolds*

$$\sigma_1 = \{\boldsymbol{x}: x_i = \varphi_i(\boldsymbol{u}), \boldsymbol{u} \subset \omega\}$$

and

$$\sigma_2 = \{\boldsymbol{x}: x_i = \psi_i(\boldsymbol{v}), \boldsymbol{v} \in g\}$$

*($i = 1, \ldots, n$) belonging to a k-dimensional manifold S (of arbitrary nature) is nonempty, this intersection is a k-dimensional manifold. Between the parameters **u** and **v** with the aid of which $\sigma_1\sigma_2$ is representable there is a one-to-one correspondence*

$$U^0 \ni \boldsymbol{u} \rightleftharpoons \boldsymbol{v} \in V^0 \tag{10}$$

where U^0 and V^0 are open sets.

Proof. Let $\boldsymbol{x}^0 \in \sigma_1\sigma_2$. Then $x_i^0 = \varphi_i(\boldsymbol{u}^0) = \psi_i(\boldsymbol{v}^0)$ $(i = 1, \ldots, n)$. There exists an n-dimensional neighbourhood Δ of the point $\boldsymbol{x}^0$ such that

$$S\Delta = \{\boldsymbol{x}: x_i = f_i(w), w \in W\} \tag{11}$$

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* Lemma 4 generalizes Lemma 3 while Lemma 5, in its turn, generalizes Lemma 4.

There also exist two n-dimensional rectangles $\Omega_1, \Omega_2 \subset \Delta$ with centres at x^0 of sufficiently small diameters such that

$$\sigma_1\Omega_1 = \{x: x_i = \varphi_i(u), u \subset U_1 \subset \omega\} \tag{12}$$

and

$$\sigma_2\Omega_2 = \{x: x_i = \psi_i(v), v \in V_1 \subset g\} \tag{13}$$

It follows that $\sigma_1\Omega_1, \sigma_2\Omega_2 \subset S\Delta$, and consequently, by Lemma 2, there are one-to-one correspondences $U_1 \rightleftarrows U_1' \subset W$ and $V_1 \rightleftarrows V_1' \subset W$ continuously differentiable in both directions.

Let $W' = U_1'V_1'$ (it is an open set). The indicated correspondences induce the correspondences

$$U_1 \supset U_* \rightleftarrows W' \rightleftarrows V_* \subset V_1 \tag{14}$$

where U_* and V_* are open sets.

Note that the equations

$$x_i = \varphi_i(u), u \in U_*$$

as well as the equations

$$x_i = \psi_i(v), v \in V_*$$

describe the part $\sigma_1\sigma_2\Omega_1\Omega_2$ of $\sigma_1\sigma_2$, whence, by the arbitrariness of the point $x^0 \in \sigma_1\sigma_2$, it follows that $\sigma_1\sigma_2$ is a k-dimensional manifold. We have also shown that for any point $x^0 \in \sigma_1\sigma_2$ ($x_i^0 = \varphi_i(u^0) = \psi_i(v^0)$; $i = 1, \ldots, n$) there are open sets $U_* \ni u^0$ and $V_* \ni v^0$ between whose points u and v there is a one-to-one correspondence continuously differentiable in both directions. Therefore it becomes evident that all the values of the parameters u and v to which the points of $\sigma_1\sigma_2$ correspond and between which there is therefore a trivial one-to-one correspondence are in fact in a correspondence continuously differentiable in both directions and, besides, the sets over which these parameters u and v range are open.

Lemma 5. *If σ_1, σ_2 and S are arbitrary k-dimensional manifolds and $\sigma_1, \sigma_2 \subset \subset S$ then the intersection $\sigma_1\sigma_2$, provided it is nonempty, is also a k-dimensional manifold.*

Proof. Let $x^0 \in \sigma_1\sigma_2$. Arguing as in the proof of the foregoing lemma we can construct two manifolds of the type of $\sigma_1\Omega_1$ and $\sigma_2\Omega_2$ belonging to $S\Delta$ (see (11), (12) and (13)) and then, as is explained in that proof, it turns out that $\sigma_1\sigma_2\Omega_1\Omega_2$ is a k-dimensional manifold, and, together with it, so is $\sigma_1\sigma_2$.

If a k-dimensional manifold σ is described by equations

$$x_i = f_i(u) = f_i(u_1, \ldots, u_k), u \in \omega \qquad (i = 1, \ldots, n) \tag{15}$$

we say that *it is represented parametrically (with the aid of the parameter u).* By definition, we say in this case that *the manifold σ is oriented.*

The replacement of u by u' with the aid of a mapping

$$u_j = \psi_j(u_1', \ldots, u_k') \qquad (u' \in \omega', j = 1, \ldots, k)$$

continuously differentiable in both directions on the domain ω' leads to another parametric representation

$$x_i = F_i(\boldsymbol{u}') = f_i(\psi_1, \ldots, \psi_k), \ \boldsymbol{u}' \in \omega'$$

of σ.

By definition, we say that the latter parametric representation specifies *the same orientation of* σ as that specified by (15) or *the reverse (opposite) orientation* depending on whether the Jacobian of the transformation from $\boldsymbol{u}$ to $\boldsymbol{u}'$ is positive or negative.

A k-dimensional manifold S is said to be orientable if all the possible descriptions of S (the parametric representations of its submanifolds) can be divided into two classes $\mathfrak{M}_+$ *and* $\mathfrak{M}_-$ *so that if two representations with parameters* $\boldsymbol{u}$ and $\boldsymbol{u}'$ *belong to one class and the manifolds* σ_1 *and* σ_2 *which are described by them intersect, the Jacobian of the transformation from* $\boldsymbol{u}$ to $\boldsymbol{u}'$ *is positive on* $\sigma_1\sigma_2$.

The parametric representations belonging to $\mathfrak{M}_+$ *specify one orientation of S and those belonging to* $\mathfrak{M}_-$ *specify another orientation reverse with respect to the former.* However, it should be taken into account that there exist nonorientable k-dimensional (differentiable) manifolds, that is such that their descriptions (parametric representations) cannot be split into two classes $\mathfrak{M}_+$ and $\mathfrak{M}_-$ with the indicated properties (for the case $n = 3$, $k = 2$ an example of such a nonorientable manifold is the Möbius strip).

A given representation (with the aid of a vector parameter $\boldsymbol{u} = (u_1, \ldots \ldots, u_k)$) belonging to one of the classes $\mathfrak{M}_+$ and $\mathfrak{M}_-$ can be transformed to the other class, that is to the reverse orientation, by changing the sign of u_1, that is by the substitution $u_1' = -u_1$, $u_2' = u_2$, $\ldots$, $u_k' = u_k$.

It should be noted that an oriented k-dimensional differentiable manifold L_k is obviously a smooth oriented curve (see § 6.5)* in the case $k = 1$ and that in the case $k = 2$, $n = 3$ the definition of L_k is completely coherent with the definition of an oriented smooth surface. Indeed, in the latter case all the parametric representations $\boldsymbol{r}(u, v) = \varphi\boldsymbol{i}+\psi\boldsymbol{j}+\chi\boldsymbol{k}$ of a smooth surface S (which is a two-dimensional differentiable manifold) belonging to one and the same class $\mathfrak{M}_+$ are such that if the unit normal vector

$$\boldsymbol{n}(x) = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_v}{|\boldsymbol{r}_u \times \boldsymbol{r}_v|}$$

at a point $\boldsymbol{x} = \boldsymbol{r}(u, v)$ is computed in terms of a parametric representation, with the aid of parameters (u, v), belonging to $\mathfrak{M}_+$ the result is independent of whether we compute $\boldsymbol{n}$ in terms of (u, v) or in terms of another parametric representation, with the aid of some other parameters (u', v'), belonging to the same class $\mathfrak{M}_+$ connected with the parameters (u, v) by a transformation continuously differentiable in both directions. This follows from the fact that the Jacobian $\frac{D(u, v)}{D(u', v')}$ is positive and, consequently, the unit normal

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* On the contrary, a smooth curve is not necessarily a differentiable manifold of the type of L_1 (see the considerations concerning Fig. 6.1).

$n(x)$ depends continuously on x, and the surface S is oriented in the sense of the definition stated in § 7.20. On applying the same argument to the representations forming the class $\mathfrak{M}_-$ we arrive at the opposite orientation of S in the sense of § 7.20.

Let G be a domain in the space R_n of the points $x = (x_1, \ldots, x_n)$. The equations

$$x_i = u_i, \quad i = 1, \ldots, n, \quad u = (u_1, \ldots, u_n) \in G$$

in which x_i's simultaneously play the role of the coordinates $x = (x_1, \ldots, x_n)$ and that of the parameters $u = (u_1, \ldots, u_n)$, describe a definitely oriented n-dimensional manifold S. If equations of the form

$$x_i = f_i(w_1, \ldots, w_n), \quad w = (w_1, \ldots, w_n) \in \omega \rightleftarrows G$$

specify a one-to-one correspondence $x \rightleftarrows w$ continuously differentiable in both directions, then they determine the same manifold S with the same or reverse orientation depending on whether the Jacobian of the transformation from x to w is positive or negative.

The following lemma is sometimes of use.

Lemma 6. *Let a k-dimensional manifold S possess a class $\mathfrak{M}$ of its representations*

$$x_i = f_i(u), \quad u \in w, \quad i = 1, \ldots, n \tag{16}$$

satisfying the following conditions.

(i) *The manifolds $\sigma \subset S$ determined by the representations belonging to the class $\mathfrak{M}$ cover S.*

(ii) *Any two representations belonging to $\mathfrak{M}$ and specifying some submanifolds $\sigma, \sigma' \subset S$ whose intersection $\sigma\sigma'$ is nonempty are such that the corresponding parameters u and u' are connected by a transformation with positive Jacobian on $\sigma\sigma'$* (see Lemma 4).

Then the manifold S is orientable and the classes $\mathfrak{M}_+$ and *$\mathfrak{M}_-$ into which split all the representations of S can be chosen so that $\mathfrak{M} \subset \mathfrak{M}_+$. In this way $\mathfrak{M}$ specifies a definite orientation of S.*

Proof. Let us consider a representation of type (16) (with a parameter u) of a manifold $\sigma \subset S$, and let x^0 be a point belonging to σ. By property (i) of the class $\mathfrak{M}$, there is a representation of a manifold $\sigma' \subset S$, with the aid of some parameter v, belonging to $\mathfrak{M}$, which covers the point x^0. Let us attribute the orientation of σ to the class $\mathfrak{M}_+$ or $\mathfrak{M}_-$ depending on whether the Jacobian of the transformation from u to v is positive or negative on $\sigma\sigma'$ in a small neighbourhood of x^0. This rule requires a justification, that is we have to show that it is consistent in the sense that it does not in fact depend on the point $x^0 \in \sigma$ and on the representation of the submanifold $\sigma' \in \mathfrak{M}$ covering x^0.

For definiteness, let the Jacobian of the transformation from u to v be positive in a sufficiently small neighbourhood of the point x^0, that is let, in accordance with the rule stated, the representation of σ belong to $\mathfrak{M}_+$. The point x^0 can also be covered by another submanifold σ'' with a representation belonging to $\mathfrak{M}$ described with the aid of some other parameter, say w. Since the transformation from v to w has a positive Jacobian on $\sigma'\sigma''$, the Jacobian of the transformation from u to w (which is equal to the product of the Jacobians corresponding to the transformations from u to v and from v to w) is positive, and hence the rule stated above leads to the same result, that is the representation of σ belongs to $\mathfrak{M}_+$.

We shall denote as σ_x a manifold covering a point x whose representation belongs to $\mathfrak{M}$, the symbol v_x designating the corresponding vector parameter. Let $y^0 \in \sigma$ and $y^0 \neq x^0$. Suppose that our rule is inconsistent, i.e. the Jacobian of the transformation from u to v_{y^0} is negative in a small neighbourhood of y^0. Let us join x^0 and y^0 with a continuous curve $\Gamma \subset \sigma$ ($x = x(t)$, $0 \leqslant t \leqslant 1$, $x(0) = x^0$, $x(1) = y^0$). There must exist a value t_0 $(0 < t_0 < 1)$ of the parameter t such that in its any neighbourhood, however small, there are some values t' and t'' such that the Jacobians of the transformations from u to $v_{x(t')}$ and from u to $v_{x(t'')}$ are of opposite signs in sufficiently small neighbourhoods of the points $x(t')$ and $x(t'')$. This conclusion contradicts the fact that there is a neighbourhood of the points $x(t_0)$ within which the Jacobian retains sign. The correctness of the rule has thus been proved.

Now we see that the transformation $u' \to u''$ also has a positive Jacobian. The same argument applies to $\mathfrak{M}_-$. It is also evident that $\mathfrak{M} \subset \mathfrak{M}_+$ and that every local representation of S belongs to only one of the classes $\mathfrak{M}_+$ and $\mathfrak{M}_-$.

We note that *an orientable manifold can obviously be also defined as a manifold S for which there exists a class $\mathfrak{M}$ of its representations satisfying conditions* (i) *and* (ii) *of Lemma 6.*

Remark. Any bounded and closed differentiable manifold S (e.g. the circumference of a circle, the spherical surface bounding a ball or the boundary of an ellipsoid and the like) cannot be represented as a whole with the aid of equations (2), that is in such a way that the correspondence $S \rightleftarrows \omega$ is one-to-one. Indeed, suppose that S is described by equations of form (2). Consider a sequence of points $\{u^\nu\} \subset \omega$ convergent to a point u^0 belonging to the boundary γ of ω or, if the set γ is empty (i.e. $\omega = R_k$), receding to infinity (i.e. $|u^\nu| \to \infty$). Since S is bounded and closed, this sequence contains a subsequence (let it be again denoted as $\{u^\nu\}$) such that for the points $x^\nu \in S$ corresponding to u^ν under the mapping specified by (2) there holds the relation $x^\nu \to x^0$ where x^0 is a point belonging to S and thus corresponding to a definite point $u' \in \omega$ under (2). Let $V \in \omega$ be a closed ball with centre at u'. Under (2) this ball is mapped continuously onto a closed set $\sigma \subset S$ and, consequently, the mapping is also continuous in the other direction (see § 12.20, Theorem 1), and $x^\nu \to x^0$ implies $u^\nu \to u'$, that is $u' = u^0$. But this is impossible because u^0 is a boundary point of ω or a point at infinity while u' is a (finite) interior point of ω.

§ 17.2. Boundary of a Differentiable Manifold and Its Orientation

Let us denote by $E^{(k)}$ the projection of a set E belonging to the space R_n into the coordinate subspace $(x_1, \ldots, x_k)$.

Theorem 1. *Let $\Gamma \subset S$ where Γ and S are k-dimensional and $(k+1)$-dimensional differentiable manifolds respectively and let $x^0 \in \Gamma$. Then it is possible to renumber the coordinates and to find a rectangle*

$$\begin{aligned} \Delta = \{|x_i - x_i^0| < \delta_1,\ i = 1, \ldots, k;\ |x_{k+1} - x_{k+1}^0| < \delta_2; \\ |x_j - x_j^0| < \sigma,\ j = k+2, \ldots, n\} \end{aligned} \tag{1}$$

so that $S\Delta$ is described by continuously differentiable functions

$$x_j = f_j(x_1, \ldots, x_{k+1}),\ j = k+2, \ldots, n,\ (x_1, \ldots, x_{k+1}) \in \Delta^{(k+1)} \tag{2}$$

and $\Gamma\Delta$ by continuously differentiable functions

$$\left.\begin{aligned} x_{k+1} &= \varphi(x_1, \ldots, x_k),\ (x_1, \ldots, x_k) \in \Delta^{(k)} \\ x_j &= f_j(x_1, \ldots, x_k, \varphi(x_1, \ldots, x_k)),\ j = k+2, \ldots n \end{aligned}\right\} \tag{3}$$

Proof. All the functions we shall deal with will be continuously differentiable and we shall simply call them functions.

Let us choose a point $x^0 \in \Gamma \subset S$. We shall assume that in a neighbourhood of the point x^0 the manifold S can be projected (in a one-to-one manner) into the subspace R_{k+1} of the points $(x_1, \ldots, x_{k+1}, 0, \ldots, 0)$, which can always be achieved by an appropriate renumbering of the coordinates. As will be shown below, since $\Gamma \subset S$, the manifold Γ must necessarily be projectable into a k-dimensional coordinate *subspace R_k of the subspace R_{k+1}*. The first $k+1$ coordinates can be renumbered once again so that R_k becomes the subspace of the points $(x_1, \ldots, x_k, 0, \ldots, 0)$.

Thus, there is a rectangle $\Delta_1 = \{|x_j - x_j^0| < \delta_1, j = 1, \ldots, k; |x_{k+1} - x_{k+1}^0| < \mu;\ |x_j - x_j^0| < \nu, j = k+1, \ldots, n\}$ such that the part $S\Delta_1$ of S is representable by the equations

$$x_j = f_j(x_1, \ldots, x_{k+1}),\ j = k+2, \ldots, n,\ (x_1, \ldots, x_{k+1}) \in \Delta_1^{(k+1)} \tag{4}$$

We shall make use of the remark made immediately after the definition of a k-dimensional manifold in § 17.1 according to which, given some numbers δ_1 and μ entering in the above relations and satisfying the necessary requirements, they can be arbitrarily diminished* while equations (4) will continue to describe $S\Delta_1$.

By virtue of the definition of Γ, there also exists a rectangle

$$\Delta_2 = \{|x_j - x_j^0| < \delta_1,\ j = 1, \ldots, k;\ |x_j - x_j^0| < \sigma,\ j = k+1, \ldots, n\}$$

belonging to Δ_1 (i.e. $\sigma \leqslant \mu, \nu$, and it may be necessary to diminish δ_1 in Δ_1

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* Diminishing, when necessary, the numbers δ_1 and μ we shall often retain the same letters δ_1 and μ for their notation.

appropriately) such that $\Gamma\Delta_2$ is representable by equations

$$\begin{gathered} x_{k+1} = \varphi(x_1, \dots, x_k), \quad x_j = F_j(x_1, \dots, x_k) \\ j = k+2, \dots, n, \quad (x_1, \dots, x_k) \in \Delta_2^{(k)} \end{gathered}$$

Since $\Gamma\Delta_2 \subset S\Delta_1$, we have

$$F_j(x_1, \dots, x_k) = f_j(x_1, \dots, x_k, \varphi), \ (x_1, \dots, x_k) \in \Delta_2^{(k)}, \ \varphi = \varphi(x_1, \dots, x_k)$$

and the equations describing $\Gamma\Delta_2$ can be written in the form

$$\left.\begin{aligned} x_{k+1} &= \varphi(x_1, \dots, x_k), \ (x_1, \dots, x_k) \in \Delta_2^{(k)} \\ x_j &= f_j(x_1, \dots, x_k, \varphi), \ j = k+2, \dots, n \end{aligned}\right\}$$

Now, using the remark mentioned above, we find a rectangle $\Delta \subset \Delta_2$ (see (1)) such that $S\Delta$ is described by equations (2) in which f_j are the functions already found (for Δ_1). This may be connected with a necessary decrease of the values of δ_1 in Δ_1 and Δ_2; besides, let us choose δ_1 so that

$$|\varphi(x_1, \dots, x_k) - x_{k+1}^0| < \delta_2, \quad (x_1, \dots, x_k) \in \Delta^{(k)} \tag{5}$$

Thus, $S\Delta$ can be described by equations of form (2). Now let $x = (x_1, \dots, x_n)$ be a point belonging to $\Gamma\Delta$; then $(x_1, \dots, x_k) \in \Delta^{(k)} \subset \Delta_2^{(k)}$, $x \in \Gamma\Delta_2$ and, consequently, x can be written in the form

$$\left.\begin{aligned} x_{k+1} &= \varphi(x_1, \dots, x_k) \\ x_j &= f_j(x_1, \dots, x_k, \varphi), \ (x_1, \dots, x_k) \in \Delta^{(k)}, \ j = k+2, \dots, n \end{aligned}\right\} \tag{6}$$

Conversely, if a point x is representable in form (6), then, by virtue of (5), we have $(x_1, \dots, x_{k+1}) \in \Delta^{(k+1)}$, and therefore $x \in S\Delta$. On the other hand, $(x_1, \dots, x_k) \in \Delta^{(k)} \subset \Delta_2^{(k)}$, and consequently $x \in \Gamma\Delta_2 \subset \Gamma$. Hence, $x \in \Gamma S\Delta = \Gamma\Delta$.

Now we proceed to the proof of the fact that if $\Gamma \subset S$ then, in a neighbourhood of any point $x^0 \in \Gamma$, the manifold Γ is sure to be projectable into a k-dimensional subspace of the $(k+1)$-dimensional space on which S is projectable. Let the manifold S be representable in a neighbourhood of x^0 by functions

$$x_i = f_i(x_1, \dots, x_{k+1}), \qquad i = k+2, \dots, n \tag{7}$$

and the manifold Γ by functions

$$x_j = \varphi_j(u_1, \dots, u_k), \qquad j = 1, \dots, n \tag{8}$$

the rank of the matrix $\left\|\dfrac{\partial\varphi_j}{\partial u_i}\right\|$ being equal to k. Since $\Gamma \subset S$, we have

$$\left.\begin{aligned} x_s &= \varphi_s(\boldsymbol{u}), \qquad s = 1, \dots, k+1 \\ x_j &= \varphi_j(\boldsymbol{u}) \equiv f_j(\varphi_1(\boldsymbol{u}), \dots, \varphi_{k+1}(\boldsymbol{u})), \quad j = k+2, \dots, n \end{aligned}\right\}$$

in a small neighbourhood of the point u^0. Therefore

$$\left\|\frac{\partial\varphi_j}{\partial u_l}\right\| = \left\|\begin{matrix} \dfrac{\partial\varphi_1}{\partial u_1} \cdots \dfrac{\partial\varphi_{k+1}}{\partial u_1} \displaystyle\sum_{s=1}^{k+1} \dfrac{\partial f_{k+2}}{\partial x_s}\dfrac{\partial\varphi_s}{\partial u_1} \cdots \sum_{s=1}^{k+1} \dfrac{\partial f_n}{\partial x_s}\dfrac{\partial\varphi_s}{\partial u_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{\partial\varphi_1}{\partial u_k} \cdots \dfrac{\partial\varphi_{k+1}}{\partial u_k} \displaystyle\sum_{s=1}^{k+1} \dfrac{\partial f_{k+2}}{\partial x_s}\dfrac{\partial\varphi_s}{\partial u_k} \cdots \sum_{s=1}^{k+1} \dfrac{\partial f_n}{\partial x_s}\dfrac{\partial\varphi_s}{\partial u_k} \end{matrix}\right\|$$

The rank of this matrix being equal to k, we can readily prove by contradiction that the rank of the matrix

$$\left\|\begin{matrix} \dfrac{\partial\varphi_1}{\partial u_1} \cdots \dfrac{\partial\varphi_{k+1}}{\partial u_1} \\ \cdots\cdots\cdots \\ \dfrac{\partial\varphi_1}{\partial u_k} \cdots \dfrac{\partial\varphi_{k+1}}{\partial u_k} \end{matrix}\right\|$$

is also equal to k, that is there is a minor of order k of this matrix different from zero. Let it be the determinant $\left|\frac{\partial\varphi_j}{\partial u_l}\right|$; $j, l = 1, \ldots, k$. It follows that in a small neighbourhood of u^0 the first k equations (8) can be resolved with respect to $u_1, \ldots, u_k$, which shows that in a neighbourhood of the point x^0 the manifold Γ can be projected in a one-to-one manner into the subspace R_k of the points $(x_1, \ldots, x_k, 0, \ldots, 0)$.

By a *boundary* of a k-dimensional manifold L_k is meant the set $\partial L_k = \bar{L}_k - L_k$. If L_k is closed, it obviously has no boundary.

Theorem 2. *Let L_{k+1} and L'_{k+1} be $(k+1)$-dimensional differentiable manifolds and $\bar{L}_{k+1} \subset L'_{k+1}$; let the manifold L'_{k+1} be connected and let L_{k+1} have a nonempty connected boundary $\partial L_{k+1} = L_k$ which is a k-dimensional manifold.*

Then for any point $x^0 \in L_k$ there is a permutation $j_1, j_2, \ldots, j_n$ of the natural numbers $1, \ldots, n$ and a rectangle

$$\Delta = \{|x_{j_s} - x^0_{j_s}| < \delta_1,\ s = 1, \ldots, k;$$
$$|x_{j_{k+1}} - x^0_{j_{k+1}}| < \delta_2,\ |x_{j_s} - x^0_{j_s}| < \delta,\ s = k+2, \ldots, n\}$$

such that the part $L'_{k+1}\Delta$ of L'_{k+1} can be described by functions

$$x_{j_s} = f_{j_s}(x_{j_1}, \ldots, x_{j_{k+1}}),\ (x_{j_1}, \ldots, x_{j_{k+1}}) \in \Delta^{(k+1)},\ s = k+2, \ldots, n \quad (9)$$

continuously differentiable on $\overline{\Delta^{(k+1)}}$ and the part $L_k\Delta$ of L_k is described by the same functions in which a substitution

$$x_{j_{k+1}} = \varphi(x_{j_1}, \ldots, x_{j_k}),\ (x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)} \quad (10)$$

is made where φ is a continuously differentiable function on $\overline{\Delta^{(k)}}$.

Under the transformation of the coordinates specified by equations (9) the points of $\Delta^{(k+1)}$ belonging to the domain

$$\Delta^{(k+1)}_+ = \{x_{j_{k+1}} > \varphi(x_{j_1}, \ldots, x_{j_k}),\ (x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)}\}$$

are mapped entirely onto one of the parts $L_{k+1}\Delta$ or $(L'_{k+1} - \bar{L}_{k+1})\Delta$ and *the points of $\Delta^{(k+1)}$ belonging to the domain*

$$\Delta_{-}^{(k+1)} = \{x_{j_{k+1}} < \varphi(x_{j_1}, \ldots, x_{j_k}), (x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)}\}$$

onto the other part.

Proof. Since L_k is a k-dimensional manifold belonging to the $(k+1)$-dimensional manifold L'_{k+1}, the first part of the theorem is simply a modified restatement of Theorem 1. It remains to prove the second part. To this end we make use of the fact that L_k is the boundary of L_{k+1} and that L_k and L'_{k+1} are connected manifolds.

Surface (10) cuts the rectangle $\Delta^{(k+1)}$ into two domains not containing those points which are mapped into L_k under transformation (10). Consequently, each of these domains is mapped entirely on one and only one of the parts $L_{k+1}\Delta$ and $(L'_{k+1}-L_{k+1})\Delta$. Further, it cannot happen that both domains are mapped entirely onto $(L'_{k+1}-\bar{L}_{k+1})\Delta$ because in such a case $L_{k+1}\Delta$ would be an empty set and Δ would not contain points of L_k. It is also impossible that both domains are mapped onto $L_{k+1}\Delta$. Indeed, let us say that a point $x^0 \in L_k$ is *regular* if, for this point, the different domains $\Delta_{+}^{(k+1)}$ and $\Delta_{-}^{(k+1)}$ are mapped onto different parts $L_{k+1}\Delta$ and $(L'_{k+1}-\bar{L}_{k+1})\Delta$. We have to prove that the set of all irregular points is empty. Let us begin with the remark that there cannot exist two points x^0 and y^0 belonging to L_k one of which is regular while the other is not. Indeed, if such points existed then, by the connectedness of L_k, there would exist a curve Γ belonging to L_k and joining the points x^0 and y^0, and therefore the application of the method of Dedekind's cuts would imply the existence of a point z^0 on Γ in whose arbitrarily small neighbourhood there would be both a regular point and an irregular point. But this is obviously impossible. Now let us suppose that all the points of L_k are irregular. On choosing two points $x^0 \in L'_{k+1}-\bar{L}_{k+1}$ and $y^0 \in L_{k+1}$ and using the connectedness of L'_{k+1} we can join them by a continuous curve

$$x = x(t),\ 0 \leqslant t \leqslant 1,\ x(0) = x^0,\ x(1) = y^0$$

belonging to L'_{k+1}. Further, using the method of Dedekind's cuts we can find the greatest value t_0 of the parameter t for which $x(t) \in L'_{k+1}-\bar{L}_{k+1}$, $t < t_0$. The point $z^0 = x(t_0)$ obviously belongs to L_k, is irregular and, by what was said above, for this point both domains $\Delta_{+}^{(k+1)}$ and $\Delta_{-}^{(k+1)}$ must be mapped onto $L_{k+1}\Delta$, which means that there cannot be points of $(L'_{k+1}-\bar{L}_{k+1})\Delta$ in a neighbourhood of z^0. Since such points do in fact exist for $t < t_0$ lying sufficiently close to t_0, we have arrived at a contradiction.

Hence, every point of $x^0 \in L_k$ is regular, that is under transformation (9) the domains $\Delta_{+}^{(k+1)}$ and $\Delta_{-}^{(k+1)}$ are mapped onto different parts $L_{k+1}\Delta$ and $(L'_{k+1}-L_{k+1})\Delta$.

Remark. Taking into account that φ is continuous and satisfies the equality $x_{j_{k+1}}^0 = \varphi(x_{j_1}^0, \ldots, x_{j_k}^0)$ and that the diameter of $\Delta^{(k)}$ can be arbitrarily decreased we conclude that there is $\lambda > 0$ such that the domain

$$\begin{aligned} \Delta = \{(x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)}, \varphi(x_{j_1}, \ldots, x_{j_k})-\lambda < x_{j_{k+1}} < \\ < \varphi(x_{j_1}, \ldots, x_{j_k})+\lambda,\ |x_{j_s}-x_{j_s}^0| < \sigma,\ s = k+2, \ldots, n\} \end{aligned} \qquad (11)$$

cuts from L'_{k+1} a portion $L'_{k+1}\Lambda$ described by the functions

$$x_{j_s} = f_{j_s}(x_{j_1}, \ldots, x_{j_{k+1}}), \ (x_{j_1}, \ldots, x_{j_{k+1}}) \in \Lambda^{(k+1)} \qquad (s = k+2, \ldots, n)$$

Besides, surface (10) cuts $\Lambda^{(k+1)}$ into the two parts

$$\left.\begin{aligned} \Lambda_{+}^{(k+1)} &= \{(x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)}, \varphi(x_{j_1}, \ldots, x_{j_k}) \leqslant \\ &\leqslant x_{j_{k+1}} < \varphi(x_{j_1}, \ldots, x_{j_k})+\lambda\} \\ \Lambda_{-}^{(k+1)} &= \{(x_{j_1}, \ldots, x_{j_k}) \in \Delta^{(k)}, \varphi(x_{j_1}, \ldots, x_{j_k})-\lambda \leqslant \\ &\leqslant x_{j_{k+1}} < \varphi(x_{j_1}, \ldots, x_{j_k})\} \end{aligned}\right\} \qquad (12)$$

one of which is mapped under transformation (9) onto $L_{k+1}\Lambda$ and the other onto $(L'_{k+1}-L_{k+1})\Lambda$.

If $\Lambda_{+}^{(k+1)}$ is mapped onto $(L'_{k+1}-L_{k+1})\Lambda$ we introduce the variables

$$u_1 = x_{j_1}, \ldots, u_k = x_{j_k}, u_{k+1} = x_{j_{k+1}}-\varphi(x_{j_1}, \ldots, x_{j_k}) \qquad (13)$$

(Correspondence (13) between $u_1, \ldots, u_{k+1}$ and $x_{j_1}, \ldots, x_{j_{k+1}}$ is continuously differentible in both directions.) Then the functions

$$\left.\begin{aligned} x_{j_s} &= F_{j_s}(u_1, \ldots, u_{k+1}) = u_s, \qquad s = 1, \ldots, k \\ x_{j_{k+1}} &= F_{j_{k+1}}(u_1, \ldots, u_{k+1}) = u_{k+1}+\varphi(u_1, \ldots, u_k) \\ x_{j_s} &= F_{j_s}(u_1, \ldots, u_{k+1}) = f_{j_s}(u_1, \ldots, u_k, u_{k+1}+\varphi(u_1, \ldots, u_k)) \\ s &= k+2, \ldots, n \\ |u_s&-u_s^0| \leqslant \delta_1, s = 1, \ldots, k, |u_{k+1}| \leqslant \lambda \end{aligned}\right\} \qquad (14)$$

represent $L'_{k+1}\Lambda$ and the points $(u_1, \ldots, u_{k+1})$ with coordinates $u_{k+1} > 0$, $u_{k+1} < 0$ and $u_{k+1} = 0$ go into $(L'_{k+1}-L_{k+1})\Lambda$, $L_{k+1}\Lambda$ and $L_k\Lambda$ respectively.

If $\Lambda_{+}^{(k+1)}$ is mapped onto $L_{k+1}\Lambda$, the same result follows if we replace u_{k+1} by $-u_{k+1}$ in equations (13) and (14).

Finally, note that the replacement of the parameter u_1 by $-u_1$ obviously does not alter these results, that is after the replacement the points $(u_1, \ldots, u_{k+1})$ with $u_{k+1} > 0$ will again go into $(L'_{k+1}-L_{k+1})\Lambda$. Our considerations thus result in the proof of the following theorem.

Theorem 3. *Under the conditions of Theorem 2, for any point $x^0 \in L_k$ there is its (n-dimensional) neighbourhood Λ such that $L'_{k+1}\Lambda$ can be described by continuously differentiable functions*

$$x_i = F_i(u) = F_i(u_1, \ldots, u_{k+1}), \ u \in \omega \qquad (i = 1, \ldots, n) \qquad (15)$$

where ω is a domain belonging to the space R_{k+1} of the points u and $L_k\Lambda$ is described by the equations

$$x_i = F_i(u_1, \ldots, u_k, 0), \quad (u_1, \ldots, u_k) \in \lambda \qquad (16)$$

where the set λ is the section of ω by the plane $u_{k+1} = 0$ in R_{k+1} cutting ω into two nonempty domains. Equations (15) specify a mapping of the points $u \in \omega$ with coordinates $u_{k+1} > 0$ or $u_{k+1} < 0$ onto $(L'_{k+1}-L_{k+1})\Lambda$ or $L_{k+1}\Lambda$ respectively.

Finally, if L'_{k+1} is an oriented manifold, the functions F_i possessing the indicated properties can be defined so that they determine the orientation of L'_{k+1}.

The last assertion should be understood in the sense that if the functions F_i with the indicated properties appearing in the above process determine the orientation reverse to that of L'_{k+1} then in order to obtain the desired orientation it suffices to replace u_1 by $-u_1$ in F_i.

Theorem 4. *If the conditions of Theorem 2 (or, which is the same, of Theorem 3) hold, then if L'_{k+1} is orientable, so is L_k, and there is a rule for making their orientations coherent.*

Proof. Let us assign to each point $x^0 \in L_k$ its neighbourhood Λ and functions F_i describing $L'_{k+1}\Lambda$ (and specifying its orientation) so that the conditions of Theorem 3 are fulfilled. Equations (16) provide a representation of the manifold L_k which is by now nonoriented.

Let $\mathfrak{M}$ denote the totality of all representations of form (16). Since to each point $x^0 \in L_k$ is assigned at least one representation of L_k of form (16) covering x^0 and specifying the manifold $L_k\Lambda$ the totality $\mathfrak{M}$ satisfies condition (i) of Lemma 6 proved in the foregoing section. To show that $\mathfrak{M}$ satisfies condition (ii) of that lemma as well, let us consider, besides (16), some other representation of L_k belonging to $\mathfrak{M}$. Thus, we assume that for a point $y^0 \in L_k$ there is its n-dimensional neighbourhood Ω such that functions

$$x_i = \Phi_i(v_1, \dots, v_{k+1}),\ (v_1, \dots, v_{k+1}) \in \varkappa \qquad (i = 1, \dots, n)$$

determine the oriented manifold $L'_{k+1}\Omega$ while functions

$$x_i = \Phi_i(v_1, \dots, v_k, 0)\ (v_1, \dots, v_k) \in \mu$$

determine $L_k\Omega$ where μ is the section of $\varkappa$ by the plane $v_{k+1} = 0$, and to positive v_{k+1} there correspond the points $x \in (L'_{k+1}-L_{k+1})\Lambda$. Let us suppose that the intersection $L_k\Lambda\Omega$ of the manifolds $L_k\Lambda$ and $L_k\Omega$ is nonempty. By virtue of Lemma 4 of the foregoing section, on that intersection we have a correspondence

$$U_k^0 \ni (u_1, \dots, u_k) \rightleftarrows (v_1, \dots, v_k) \in V_k^0$$

continuously differentiable in both directions where U_k^0 and V_k^0 are open sets. Then the intersection $L'_{k+1}\Lambda\Omega$ of the manifolds $L'_{k+1}\Lambda$ and $L'_{k+1}\Omega$ is also nonempty, and on $L'_{k+1}\Lambda\Omega$ we also have a correspondence

$$U_{k+1}^0 \ni (u_1, \dots, u_{k+1}) \rightleftarrows (v_1, \dots, v_{k+1}) \in V_{k+1}^0$$

continuously differentiable in both directions.

Note that in the case under consideration we have $\frac{\partial v_{k+1}}{\partial u_j} = 0\,(j = 1, \dots, k)$ for $v_{k+1} = 0$ or, which is the same, for $u_{k+1} = 0$, and therefore

$$\frac{D(v_1, \dots v_{k+1})}{D(u_1, \dots, u_{k+1})} = \frac{\partial v_{k+1}}{\partial u_{k+1}} \frac{D(v_1, \dots, v_k)}{D(u_1, \dots, u_k)} \tag{17}$$

for $u_{k+1} = 0$.

The Jacobian on the left-hand side is positive since the functions $F_i(u_1, \ldots, u_{k+1})$ and $\Phi_i(v_1, \ldots, v_{k+1})$ determine the oriented parts of L'_{k+1} and the Jacobian of the transformation from $(u_1, \ldots, u_{k+1})$ to $(v_1, \ldots, v_{k+1})$ has a positive Jacobian on the intersection of these parts. The factor $\frac{\partial v_{k+1}}{\partial u_{k+1}}$ is also positive (for $u_{k+1} = 0$) because the parameters u and v are chosen so that $x \in (L'_{k+1} - L_{k+1})\Lambda\Omega$ for $u_{k+1} > 0$ and therefore we also have $v_{k+1} > 0$. Hence, the Jacobian on the right-hand side of (17) is positive, and we have thus shown that the representations of form (16) satisfy condition (ii) of Lemma 6, § 17.1 (the parameters of these representations are connected by transformations with positive Jacobians on the intersections of the manifolds described by them).

Thus, according to the terminology of the foregoing section, the representations of form (16) form a class of the type of $\mathfrak{M}$. Consequently, L_k is an orientable manifold, and as a class specifying its orientation we can choose $\mathfrak{M}_+$. The construction method for $\mathfrak{M}_+$ was indicated in the lemma we referred to ($\mathfrak{M} \subset \mathfrak{M}_+$).

Theorem 4 asserts the possibility of setting a coherence rule for the orientations of L_{k+1} and L_k, and its proof does in fact provide such a rule. But for practical purposes it will be more convenient to use another rule stated below.

Coherence rule for the orientations of L_{k+1} *and* L_k*: if, under the conditions of Theorem 2, there is a neighbourhood* Ω *of any point* $x^0 \in L_k \subset L'_{k+1}$ *such that the part* ΩL_{k+1} *of* L_{k+1} *can be described by equations*

$$x_i = f_i(u_1, \ldots, u_{k+1}), \qquad i = 1, \ldots, n \tag{18}$$

which go, for $u_1 = 0$, *into the equations*

$$x_i = f_i(0, u_2, \ldots, u_{k+1}), \qquad i = 1, \ldots, n \tag{19}$$

describing the part ΩL_k, *and the points* $u = (u_1, \ldots, u_{k+1})$ *with* $u_1 > 0$ *are mapped under transformation* (18) *into the exterior of* L_{k+1}, *then, by definition, equations* (19) *determine an orientation of* L_k *coherent with the orientation of* L_{k+1}.

To show that the new rule is correct it is sufficient to make the substitution $u'_1 = u_{k+1}, u'_2 = u_1, \ldots, u'_{k+1} = (-1)^{k+1}u_k$ in the above-stated rule (Theorem 4) and then remove the primes.

The latter rule which we shall refer to as a *formal* rule has an advantage over the former since it is in a natural concordance with the visual rules for the orientation of geometric figures given in the elementary presentation of the Green, Gauss-Ostrogradsky and Stokes formulas.

Besides L_k, let us also consider L_k^- which is the same manifold but with the reverse orientation. It is evident that the manifold L_k^- is the boundary of the manifold $L'_{k+1} - L_{k+1}$ or, possibly, its part, the orientations of these two manifolds being coherent in accordance with the formal rule.

Remark. Let L_{k+1}, L'_{k+1} and L_k have the same sense as before and, besides, let $L_{1,k+1}, \ldots, L_{s,k+1}$ be $(k+1)$-dimensional manifolds whose closures belong to L_{k+1} and are pairwise disjoint. Let these manifolds have connected boundaries $L_{1,k}, \ldots, L_{s,k}$ oriented coherently with them. Then, obviously, the (s-connected) manifold

$$L_{k+1} - \sum_{j=1}^{s} L_{j,k+1} \tag{20}$$

has the boundary

$$L_k + \sum_{j=1}^{s} L_{\bar{j},k} \tag{21}$$

oriented coherently with it.

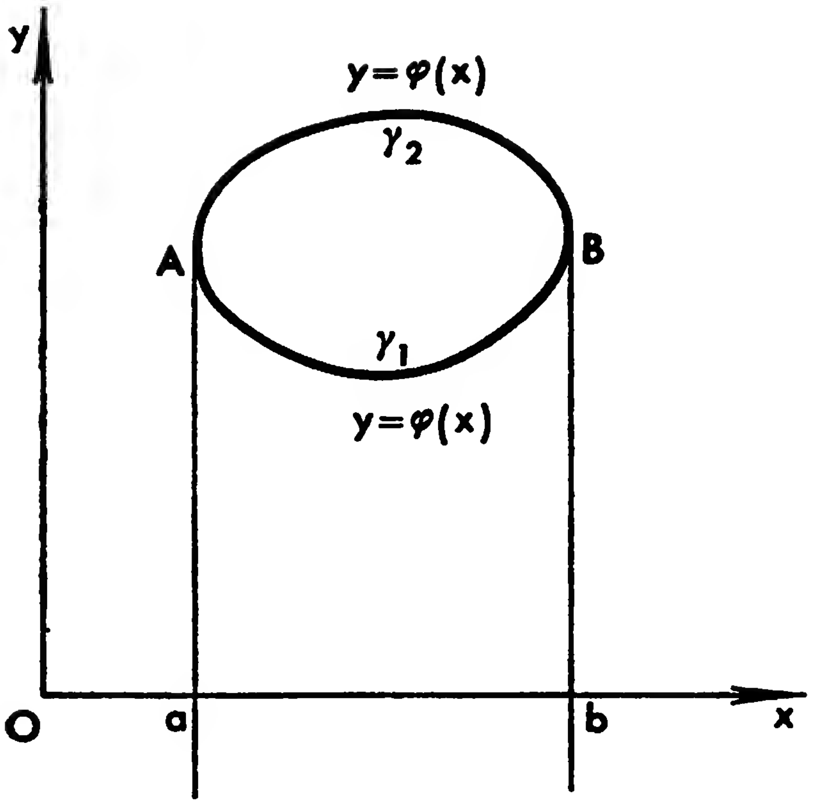

Fig. 17.1

Example 1. Consider, in the xy-plane, the convex domain G with smooth boundary Γ shown in Fig. 17.1. Let Π denote the strip $a < x < b$, $-\infty < y < \infty$ which we shall regard as a (positively) oriented two-dimensional manifold represented by the equations

$$x = x, y = y, (x, y) \in \Pi \tag{22}$$

G can be regarded as a manifold with the corresponding orientation. Further, Π can also be represented by the equations

$$x = x, y = -v + \varphi(x), a < x < b, -\infty < v < \infty \tag{23}$$

setting up a (continuously differentiable in both directions) correspondence $(x, y) \rightleftarrows (v, x)$ with the Jacobian

$$\frac{D(x, y)}{D(v, x)} = \begin{vmatrix} 0 & 1 \\ -1 & \varphi'(x) \end{vmatrix} = 1$$

or by the equations

$$x = -x', y = v + \psi(-x'), -b < x' < -a, -\infty < v < \infty \tag{24}$$

specifying a correspondence $(x, y) \rightleftarrows (v, x')$ with the Jacobian

$$\frac{D(x, y)}{D(v, x')} = \begin{vmatrix} 0 & -1 \\ 1 & -\psi'(-x') \end{vmatrix} = 1$$

Besides, in the vicinity of γ_1 and γ_2 the corresponding points (v, x) and (v, x') (with $v > 0$) are mapped into the exterior of G, and therefore, according to the coherence rule for the orientations, the equations

$$x = x, y = \varphi(x), \qquad a < x < b$$

and

$$x = -x', y = \psi(-x'), \qquad -b < x' < -a$$

of the parts γ_1 and γ_2 obtained from (23) and (24) for $v = 0$ specify the orientation of Γ coherent with that of G.

We see that as the parameters x and x' increase continuously the point (x, y) describes, respectively, the parts γ_1 and γ_2 of Γ so that the domain G remains on the left, that is Γ is traversed counterclockwise.

Since it appears natural to call the orientation of G specified by equations (22) positive, we see that the formal coherence rule for orientations when applied to G and Γ gives the result concordant with the well-known geometrical rule for coherence of the orientations of G and Γ.

However, for a rigorous justification of this conclusion it is also required to apply the above argument to arcs λ_1 and λ_2 of Γ containing, respectively, the points A and B (see Fig. 17.1).

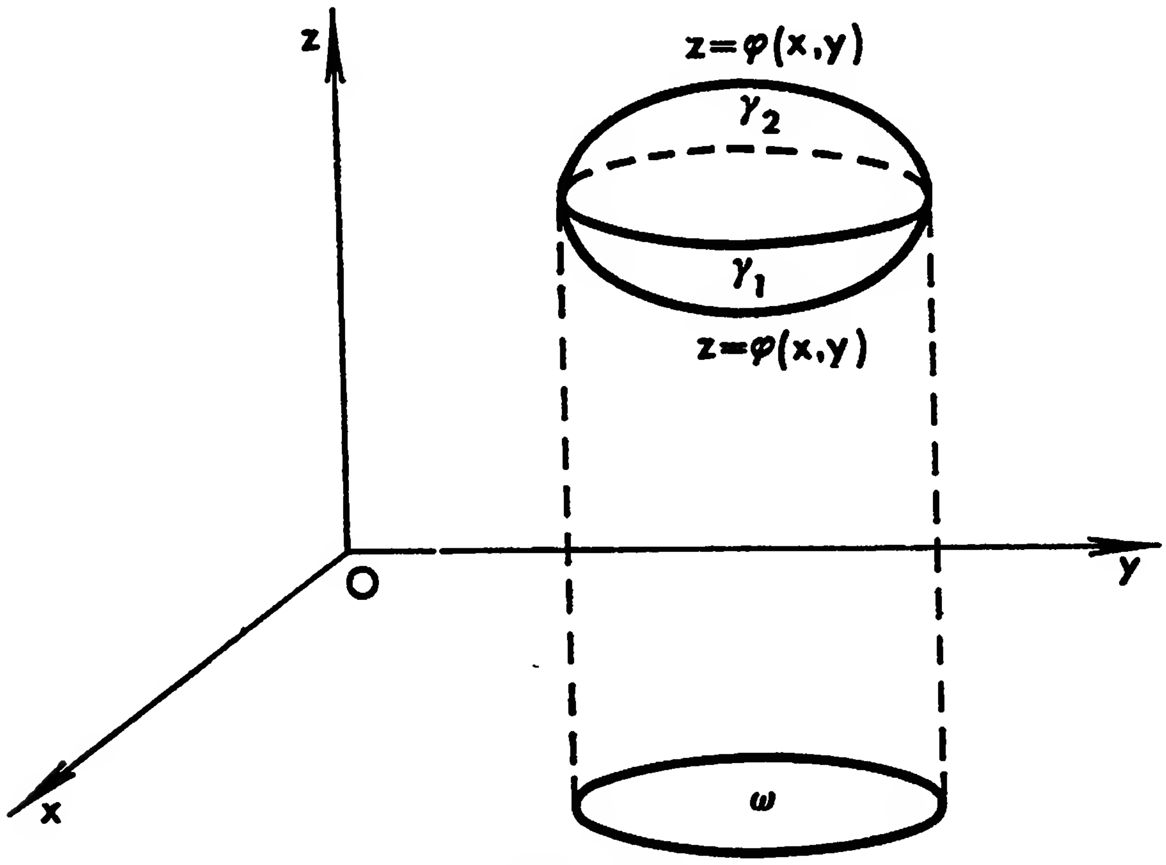

Fig. 17.2

Example 2. Let us consider the convex domain G in the xyz-space with smooth boundary surface Γ shown in Fig. 17.2. Let ω be the projection of G on the xy-plane and let us embed G in the cylindrical solid $\Pi = \{(x, y, z)\colon (x, y) \in \omega\}$ which (together with G) will be regarded as an oriented manifold:

$$x = x,\ y = y,\ z = z,\ (x, y, z) \in \Pi$$

The manifold Π can also be represented in terms of the parameters (v, x', y) $\big((v, x', y) \rightleftarrows (x, y, z)\big)$ by means of the equations

$$x = -x',\ y = y,\ z = -v + \varphi(-x', y),\ (x', y) \in \omega',\ -\infty < v < \infty$$

where ω' is the corresponding domain with the Jacobian

$$\frac{D(x, y, z)}{D(v, x', y)} = \begin{vmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -\varphi'_x(-x', y) & \varphi'_y(-x', y) \end{vmatrix} = 1$$

or by means of the parameters $(v, x, y) \rightleftarrows (x, y, z)$ and the equations

$$x = x,\ y = y,\ z = v+\psi(x, y),\ (x, y) \in \omega,\ -\infty < v < \infty$$

with the Jacobian

$$\frac{D(x, y, z)}{D(v, x, y)} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & \psi'_x & \psi'_y \end{vmatrix} = 1$$

Besides, in the vicinity of γ_1 and γ_2 the corresponding points (v, x', y) and (v, x, y) with $v > 0$ are mapped into the exterior of G, and therefore the equations

$$x = -x',\ y = y,\ z = \varphi(-x', y),\ (x', y) \in \omega'$$

and

$$x = x,\ y = y,\ z = \psi(x, y),\ (x, y) \in \omega$$

determine, respectively, the parts γ_1 and γ_2 of the boundary Γ together with their orientations coherent with the orientation of G chosen in accordance with the stated rule.

Let us agree to define the normal $\boldsymbol{n}$ (not necessarily of unit length) to a surface $\boldsymbol{r} = \boldsymbol{r}(u, v)$ by means of the formula $\boldsymbol{n} = \boldsymbol{r}_u \times \boldsymbol{r}_v$. Then we have $n_z = -1$ for the part γ_1 and $n_z = 1$ for the part γ_2, that is the normal is directed to the exterior of G in both cases. We see that the application of the formal coherence rule for orientations to the concrete case of G and Γ in question leads to the situation in which to the positively oriented three-dimensional domain G there corresponds its boundary surface Γ oriented with the aid of the outer normal.

A rigorous justification of this result also requires a thorough consideration of the parts of Γ covering the boundary of γ_1 (in this case the boundary of γ_1 is simultaneously the boundary of γ_2; see Fig. 17.2).

Example 3. Let us consider a smooth surface S' represented by an equation

$$z = f(x, y),\ (x, y) \in \Omega \tag{25}$$

where Ω is a domain in the xy-plane; let S be a surface belonging to S' and specified by the equation

$$z = f(x, y),\ (x, y) \in \omega \subset \bar{\omega} \subset \Omega$$

where ω is a domain (possibly multiply-connected) with a smooth boundary γ. The boundary of S, that is $\Gamma = \partial S = \bar{S} - S$, is obviously a smooth curve. Let us assume that the surface S' (and therefore the surface S as well) is oriented with the aid of equation (25). Then the projection of its normal $\boldsymbol{n} = \boldsymbol{r}_x \times \boldsymbol{r}_y$ ($\boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+b\boldsymbol{k}$) on the z-axis is equal to 1, that is S is oriented with the aid of the normal forming an acute angle with the positive z-axis. If we apply the screw rule, then in a right-handed coordinate system (x, y, z)

the orientation of S induces the orientation of Γ and of its projection Γ_z on the plane $z = 0$ such that in the motion of the variable point traversing the contour Γ_z in the direction indicated by its orientation the domain ω remains on the left. The same result is obtained if we make the orientations of S and Γ coherent by use of the formal rule.

Indeed, in a neighbourhood of a point of the curve Γ we can describe Γ by an equation $y = \varphi(x)$ or $x = \psi(y)$ considered simultaneously with the equation $z = f(x, y)$. For definiteness, let us take the case of the equations $y = \varphi(x)$, $z = f(x, y)$. The surface in question can also be represented with the preservation of its orientation by means of the parameters $(v, x) \rightleftarrows (x, y)$ connected with (x, y) by the equations

$$x = x, \quad y = -v+\varphi(x), \quad \frac{D(x, y)}{D(v, x)} = \begin{vmatrix} 0 & 1 \\ -1 & \varphi'(x) \end{vmatrix} = 1$$

Let to the points (v, x) with small $v > 0$ there correspond the points of $S'-\bar{S}$. Then the equations $x = x$ and $y = \varphi(x)$ determine the orientation of Γ_z, and, as the parameter x increases, the variable point moves along the contour Γ_z in the direction of its orientation. It is clear that the points $(x, y) \in \omega$ which correspond to $v < 0$ will then remain on the left.

Remark. Let a k-dimensional manifold S lie in a domain Ω belonging to the space R_n of the points $x = (x_1, \ldots, x_n)$ and let Ω go into a domain Ω' belonging to the space R'_n of the points $\xi = (\xi_1, \ldots, \xi_n)$ under the mapping specified by a one-to-one transformation

$$\xi_i = \psi_i(x_1, \ldots, x_n) = \psi_i(x), \quad x \in \Omega, \qquad i = 1, \ldots, n \tag{26}$$

continuously differentiable in both directions.

Then, if equations

$$x_i = \varphi_i(u_1, \ldots, u_k) = \varphi_i(u), \quad u \in \omega, \qquad i = 1, \ldots, n \tag{27}$$

provide a representation of a part $\sigma \subset S$ in the space R_n, the equations

$$\xi_i = \psi_i(\varphi_1(u), \ldots, \varphi_n(u)) = F_i(u), \quad u \in \omega, \qquad i = 1, \ldots, n \tag{28}$$

describe σ in R'_n. Indeed, equations (28) set up a one-to-one correspondence $\xi \rightleftarrows u$, the functions $F_i(u)$ are continuously differentiable together with φ_s and ψ_i and the rank of the matrix $\left\| \frac{\partial F_i}{\partial u_j} \right\|$ is equal to k because its k rows $\left\{\frac{\partial F_1}{\partial u_j}, \ldots, \frac{\partial F_n}{\partial u_j}\right\}$; $j = 1, \ldots, k$, can be regarded as k (n-dimensional) vectors obtained from the k linearly independent vectors $\left\{\frac{\partial \varphi_1}{\partial u_j}, \ldots, \frac{\partial \varphi_n}{\partial u_j}\right\}$ by means of a nondegenerate linear transformation (with the determinant $\left|\frac{\partial \psi_i}{\partial x_s}\right| \neq 0$).

It can easily be seen that if S is oriented in R_n and if the local descriptions of S of form (27) are transformed to the descriptions of type (28), the latter determine the (corresponding) orientation of S in R'_n. Further, if L_k is the boundary of an oriented manifold L_{k+1}, the above rule for the induction of the orientation of L_k by the orientation of L_{k+1} which is applicable both in R_n and in R'_n is such that the above correspondence is not violated.

§ 17.3. Differential Forms

A *differential form of degree* k or, simply, a k-*form* ($k = 1, 2, \ldots,$) defined on an open set $\Omega \subset R_n$ is a finite sum of the type

$$\mathfrak{A} = \sum a(x)\, dx_{i_1} \ldots dx_{i_k}, \qquad x \in \Omega \tag{1}$$

where $a(x)$ are some functions (the *coefficients of the form*) and $dx_{i_1}, \ldots, dx_{i_k}$ are symbols (the differentials) corresponding to the indices $i_1, \ldots, i_k$ which satisfy the inequalities $1 \leqslant i_s \leqslant n$, provided that sum (1) is subjected to some special conditions considered below.

By a *form of degree* 0 (a 0-form) on Ω is meant an arbitrary function $a(x)$ defined on Ω.

In what follows we assume that the coefficients $a(x)$ are continuous and are continuously differentiable as many times as required. Generally speaking, the coefficients $a(x)$ in different terms $a(x)\, dx_{i_1}, \ldots, dx_{i_k}$ may be different, and, in the general case, so are the systems of indices $i_1, \ldots, i_k$.

1. By definition, the following *operations* on a form $\mathfrak{A}$ do not alter it and are called *admissible*:

(a) Rearranging the summands in sum (1).

(b) Replacing a summand

$$a(x)\, dx_{i_1} \ldots dx_{i_k} \tag{2}$$

by a sum

$$a_1(x)\, dx_{i_1} \ldots dx_{i_k} + a_2(x)\, dx_{i_1} \ldots dx_{i_k} \quad \text{where} \quad a_1(x) + a_2(x) = a(x) \tag{3}$$

or, conversely, replacing (3) by (2).

(c) Deleting from (1) a summand involving two equal differentials dx_{i_r} and dx_{i_s} ($i_r = i_s$, $r \neq s$) or a coefficient $a(x) \equiv 0$ (see below).

(d) Interchanging two differentials involved in a term of sum (1) with simultaneous change of the sign of the coefficient $a(x)$ in that term.

(e) If in the ultimate result of the application of operations *(a)*-*(d)* to $\mathfrak{A}$ all the terms of $\mathfrak{A}$ vanish, then $\mathfrak{A}$ is called the *zero form* (not to be confused with a form of degree 0, i.e. with a 0-form which is an arbitrary function $a(x)$!) and is denoted as θ. In particular, this is the case when each of the summands of $\mathfrak{A}$ contains two equal differentials (which is always the case when $k > n$) or when all the coefficients $a(x)$ of the form are identically equal to zero on Ω.

It is obvious that on performing appropriate admissible operations on a k-form $\mathfrak{A}$ we can always reduce it to its *standard form*

$$\mathfrak{A} = \sum_{i_1 < \ldots < i_k} a_{i_1, \ldots, i_k}(x)\, dx_{i_1} \ldots dx_{i_k} \tag{1'}$$

where the sum is extended over all the *ordered* permutations $i_1 < \ldots < i_k$ of the numbers $1, \ldots, n$ taken k at a time.

The standard expression of a given form $\mathfrak{A}$ is obviously determined uniquely to within the order of the summands.

2. If $\mathfrak{A}$ and $\mathfrak{B}$ are two k-forms, their *sum* $\mathfrak{A}+\mathfrak{B}$ is a form obtained by collecting together all the terms of $\mathfrak{A}$ and $\mathfrak{B}$ and regarding them as the terms of $\mathfrak{A}+\mathfrak{B}$; the *difference* $\mathfrak{A}-\mathfrak{B}$ is defined as the sum $\mathfrak{A}+(-\mathfrak{B})$ where $-\mathfrak{B}$ is the form obtained from $\mathfrak{B}$ by changing the signs of all the coefficients of the latter. In particular, if $\mathfrak{A}$ has standard form (1′) and $\mathfrak{B}$ has the standard form

$$\mathfrak{B} = \sum_{i_1<\ldots<i_k} b_{i_1,\ldots,i_k}(x)\, dx_{i_1} \ldots dx_{i_k} \tag{4}$$

then the standard form of $\mathfrak{A}\pm\mathfrak{B}$ is

$$\mathfrak{A}\pm\mathfrak{B} = \sum_{i_1<\ldots<i_k} (a_{i_1,\ldots,i_k} \pm b_{i_1,\ldots,i_k})\, dx_{i_1} \ldots dx_{i_k}$$

We shall not dwell on the formal proof of the fact that these definitions of the sum and difference of k-forms do not depend on the (admissible) representations of $\mathfrak{A}$ and $\mathfrak{B}$.

3. The *product** $\mathfrak{A}\mathfrak{B}$ of a k-form

$$\mathfrak{A} = \sum a\, dx_{i_1} \ldots dx_{i_k} \tag{5}$$

and an s-form

$$\mathfrak{B} = \sum b\, dx_{j_1} \ldots dx_{j_s} \tag{6}$$

(generally, k and s may be different) is defined as the result of the term-by-term multiplication of the expressions on the right-hand sides of (5) and (6) with the retention of the order in which the differentials are arranged. In this operation the coefficients a and b are considered as commuting with the differentials. In particular, in the case of the standard representations (see (1′))

$$\mathfrak{A} = \sum_{i_1<\ldots<i_k} a_{i_1,\ldots,i_k}(x)\, dx_{i_1} \ldots dx_{i_k}$$

and

$$\mathfrak{B} = \sum_{j_1<\ldots<j_s} b_{j_1,\ldots,j_s}\, dx_{j_1} \ldots dx_{j_s}$$

of the forms $\mathfrak{A}$ and $\mathfrak{B}$ we have

$$\mathfrak{A}\mathfrak{B} = \sum_{i_1<\ldots<i_k} \sum_{j_1<\ldots<j_s} a_{i_1,\ldots,i_k} b_{j_1,\ldots,j_s}\, dx_{i_1} \ldots dx_{i_k}\, dx_{j_1} \ldots dx_{j_s} \tag{6′}$$

In the case when $k+s > n$ we obviously obtain $\mathfrak{A}\mathfrak{B} = \theta$.

We shall not dwell on the formal proof of the fact that standard representation (6′) to which the product of two forms $\mathfrak{A}$ and $\mathfrak{B}$ can be reduced by means of a finite number of admissible operations is independent of the original representation of the factors $\mathfrak{A}$ and $\mathfrak{B}$ (for each factor here are meant its representations reducible to one another by admissible operations).

---

* The product of forms is also called their *exterior product*.

4. By the *differential* of k-form* (1) is meant the $(k+1)$-form

$$d\mathfrak{A} = \sum \sum_{j=1}^{n} \frac{\partial a(x)}{\partial x_j}\, dx_j\, dx_{i_1} \dots dx_{i_k} \tag{7}$$

It can readily be checked that this definition is in fact independent of the way in which $\mathfrak{A}$ is represented.

It is evident that

$$dd\mathfrak{A} = d^2\mathfrak{A} = \sum \sum_{s,j} \frac{\partial^2 a}{\partial x_s\, dx_j}\, dx_s\, dx_j\, dx_{i_1} \dots dx_{i_k} = \theta$$

because the summands entering into this sum can be divided into pairs of terms of the form

$$\frac{\partial^2 a}{\partial x_s\, \partial x_j}\, dx_s\, dx_j\, dx_{i_1} \dots dx_{i_k} + \frac{\partial^2 a}{\partial x_j\, \partial x_s}\, dx_j\, dx_s\, dx_{i_1} \dots dx_{i_k}$$

which mutually cancel out.

It is convenient to introduce the symbolic 1-form (operator)

$$d = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\, dx_i$$

Then, obviously, $d\mathfrak{A}$ can be regarded as the (symbolic) product of the forms d and $\mathfrak{A}$. It is natural to put

$$d^2 = dd = \sum_{i,j} \frac{\partial^2}{\partial x_i\, \partial x_j}\, dx_i\, dx_j = \theta$$

Then the relation $d^2\mathfrak{A} = dd\mathfrak{A} = \theta$ can be regarded as the result of the multiplication of the form d^2 by $\mathfrak{A}$.

5. The transformation of a k-form $\mathfrak{A}$ of type (1) to new variables $u = (u_1, \dots, u_n)$ with the aid of continuously differentiable functions

$$x_i = \psi_i(u_1, \dots, u_n) \qquad (i = 1, \dots, n)$$

s carried out according to the following rule (see the explanations below):

$$\mathfrak{A} = \sum a(x)\, dx_{i_1} \dots dx_{i_k} = \sum a(u) \sum_{j_1=1}^{n} \frac{\partial x_{i_1}}{\partial u_{j_1}}\, du_{j_1} \dots \sum_{j_k=1}^{n} \frac{\partial x_{i_k}}{\partial u_{j_k}}\, du_{j_k} =$$

$$= \sum a(u) \sum_{j_1, \dots, j_k} \frac{\partial x_{i_1}}{\partial u_{j_1}} \dots \frac{\partial x_{i_k}}{\partial u_{j_k}}\, du_{j_1} \dots du_{j_k} =$$

$$= \sum a(u) \sum_{s_1 < \dots < s_k} \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u_{s_1}, \dots, u_{s_k})}\, du_{s_1} \dots du_{s_k} \tag{8}$$

Here in the coefficients a the substitution $x_i = \psi_i(u)$ $(i = 1, \dots, n)$ has been made, the result being denoted as $a(u)$.

---

* The differential of a form is also called its *exterior differential.*

The transformation rule from the (vector) variable x to u is expressed by the second equality in (8) which defines the expression obtained as a k-form in the variables $u = (u_1, \ldots, u_n)$. The subsequent equalities automatically follow from that definition. In the fourth member of (8) the functions $\frac{\partial x_{i_r}}{\partial u_{j_r}}$ (the factors by which the original coefficients of the form are multiplied) are written in front of the differentials under the sign of the sum containing the products of the differentials. The terms entering into the sum $\sum\limits_{j_1, \ldots, j_k}$ in the fourth member which correspond to the systems $j_1, \ldots, j_k$ of indices containing equal indices $j_s = j_r$ $(s \neq r)$ can be deleted because the fourth member is a k-form obeying the operation rules stated above. Finally, the passage to the rightmost member of (8) is carried out as follows. On choosing an arbitrary system of k indices $1 \leqslant s_1 < s_2 < \ldots < s_k \leqslant n$ we select those summands entering into the sum $\sum\limits_{j_1, \ldots, j_k}$ which correspond to all the possible permutations of the indices $s_1, \ldots, s_k$ and arrange the differentials in these summands in the increasing order $du_{s_1} \ldots du_{s_k}$. By the properties of the forms, this leads to the appearance of the corresponding signs ± in front of these summands. It can readily be seen that these signs ± are such that after the factors $du_{s_1}, \ldots, du_{s_k}$ are taken outside the brackets the expression in the brackets turns out to be the determinant (Jacobian)

$$\frac{D(x_{i_1}, \ldots, x_{i_k})}{D(u_{s_1}, \ldots, u_{s_k})} \tag{9}$$

On performing such transformations for all the systems of k indices $1 < s_1 < s_2 < \ldots < s_k$ we arrive at the last equality (8).

It should be noted that if

$$a(x)\, dx_{i_1} \ldots dx_{i_k} = a_1(x)\, dx_{i_1} \ldots dx_{i_k} + a_2(x)\, dx_{i_1} \ldots dx_{i_k} \quad (a = a_1 + a_2) \tag{10}$$

then the results of the transformations from x to u in the left-hand side of (10) and in the summands entering into the right-hand side of (10) performed according to rule (8) are in the same relation:

$$\begin{aligned} a(u) \sum_{j_1 < \ldots < j_k} \frac{D(x_{i_1}, \ldots, x_{i_k})}{D(u_{j_1}, \ldots, u_{j_k})}\, du_{j_1} \ldots du_{j_k} = \\ = a_1(u) \sum_{j_1 < \ldots < j_k} \frac{D(x_{i_1}, \ldots, x_{i_k})}{D(u_{j_1}, \ldots, u_{j_k})}\, du_{j_1} \ldots du_{j_k} + \\ + a_2(u) \sum_{j_1 < \ldots < j_k} \frac{D(x_{i_1}, \ldots, x_{i_k})}{D(u_{j_1}, \ldots, u_{j_k})}\, du_{j_1} \ldots du_{j_k} \end{aligned} \tag{11}$$

Further, if the left-hand side of (10) contains two equal differentials then all the determinants entering into the left-hand side of (11) are equal to zero. If two differentials in the left-hand side of (10) are interchanged there appears the minus sign in this expression and the same occurs in the transformed expression. Indeed, such a permutation leads to the permutation of two different rows in each of the determinants. The last property indicates that transformation rule (8) for replacing x by u leads to one and the same differential form (in u) irrespective of the (admissible) representations of the original form $\mathfrak{A}$ (in x).

It should also be noted that the consecutive transformations from x to u and then from u to u' $(u' = (u'_1, \dots, u'_n))$ performed according to rule (8) and the direct transformation from x to u' lead to one and the same result. Indeed, using the transformation law expressed by the fourth member of (8) we can write

$$\mathfrak{A} = \sum a(u) \sum_{j_1, \dots, j_k} \frac{\partial x_{i_1}}{\partial u_{j_1}} \dots \frac{\partial x_{i_k}}{\partial u_{j_k}} du_{j_1} \dots du_{j_k} =$$

$$= \sum a(u') \sum_{j_1, \dots, j_k} \sum_{s_1, \dots, s_k} \frac{\partial x_{i_1}}{\partial u_{j_1}} \dots \frac{\partial x_{i_k}}{\partial u_{j_k}} \frac{\partial u_{j_1}}{\partial u'_{s_1}} \dots \frac{\partial u_{j_k}}{\partial u'_{s_k}} du'_{s_1} \dots du'_{s_k} =$$

$$= \sum a(u') \sum_{s_1, \dots, s_k} \frac{\partial x_{i_1}}{\partial u'_{s_1}} \dots \frac{\partial x_{i_k}}{\partial u_{s_k}} du'_{s_1} \dots du'_{s_k}$$

because

$$\frac{\partial x_{i_m}}{\partial u_{s_m}} = \sum_{j_m=1}^{n} \frac{\partial x_{i_m}}{\partial u_{j_m}} \frac{\partial u_{j_m}}{\partial u'_{s_m}}$$

Rule (8) is invariant with respect to the operations $\mathfrak{A} \pm \mathfrak{B}$, $\mathfrak{A}\mathfrak{B}$ and $d\mathfrak{A}$. For instance, in the case of $\mathfrak{A}\mathfrak{B}$, this assertion means that the product of the results of the transformations of $\mathfrak{A}$ and $\mathfrak{B}$ performed in accordance with rule (8) is equal to the result of the transformation of the product $\mathfrak{A}\mathfrak{B}$ by that rule. To justify this assertion we again use the transformation law expressed by the fourth member of (8), and it is sufficient to confine ourselves to the standard representations of the original forms. We thus obtain (see the explanations below):

$$\sum_{i_1 < \dots < i_k} a_{i_1, \dots, i_k}(u) \sum_{\mu_1, \dots, \mu_k} \frac{\partial x_{i_1}}{\partial u_{\mu_1}} \dots \frac{\partial x_{i_k}}{\partial u_{\mu_k}} du_{\mu_1} \dots du_{\mu_k} \times$$

$$\times \sum_{j_1 < \dots < j_s} b_{j_1, \dots, j_s}(u) \sum_{\nu_1, \dots, \nu_s} \frac{\partial x_{j_1}}{\partial u_{\nu_1}} \dots \frac{\partial x_{j_s}}{\partial u_{\nu_s}} du_{\nu_1} \dots du_{\nu_s} =$$

$$= \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_s} a_{i_1, \dots, i_k}(u) \, b_{j_1, \dots, j_s}(u) \sum_{\mu_1, \dots, \mu_k} \sum_{\nu_1, \dots, \nu_s} \frac{\partial x_{i_1}}{\partial u_{\mu_1}} \dots$$

$$\dots \frac{\partial x_{i_k}}{\partial u_{\mu_k}} \frac{\partial x_{j_1}}{\partial u_{\nu_1}} \dots \frac{\partial x_{j_s}}{\partial u_{\nu_s}} du_{\mu_1} \dots du_{\mu_k} du_{\nu_1} \dots du_{\nu_s} =$$

$$= \sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_s} a_{i_1, \dots, i_k}(x) \, b_{j_1, \dots, j_s}(x) \, dx_{i_1} \dots dx_{i_k} \, dx_{j_1} \dots dx_{j_s} = \mathfrak{A}\mathfrak{B}$$

In the first member of the last equalities we have written the product of the forms $\mathfrak{A}$ and $\mathfrak{B}$ expressed in terms of u. The second member is the result of the term-by-term multiplication of $\mathfrak{A}$ and $\mathfrak{B}$ performed in accordance with the multiplication rule for forms (in u). According to (8) (see the second member in (8)), this result is a $(k+s)$-form in the variables $u = (u_1, \dots, u_n)$ which can be obtained by means of the transformation from x to u (according to rule (8)) of the form written as the third member of the last equalities, that is of $\mathfrak{A}\mathfrak{B}$.

Let

$$\mathfrak{A} = \sum a \, dx_{i_1} \dots dx_{i_k}$$

be a k-form and

$$\mathfrak{B} = \sum b \, dx_{j_1} \dots dx_{j_l}$$

an arbitrary form. Then

$$d(\mathfrak{A}\mathfrak{B}) = d\mathfrak{A} \cdot \mathfrak{B} + (-1)^k \mathfrak{A} \, d\mathfrak{B} \tag{12}$$

Indeed,

$$\begin{aligned} d(\mathfrak{A}\mathfrak{B}) &= \sum \sum \sum_j \frac{\partial(ab)}{\partial x_j} dx_j \, dx_{i_1} \dots dx_{i_k} \, dx_{j_1} \dots dx_{j_l} = \\ &= \sum \sum \sum_j \frac{\partial a}{\partial x_j} dx_j \, dx_{i_1} \dots dx_{i_k} b \, dx_{j_1} \dots dx_{j_l} + \\ &\quad + (-1)^k \sum \sum \sum_j a \, dx_{i_1} \dots dx_{i_k} \frac{\partial b}{\partial x_j} dx_j \, dx_{j_1} \dots dx_{j_l} = \\ &= d\mathfrak{A} \cdot \mathfrak{B} + (-1)^k \mathfrak{A} \cdot d\mathfrak{B} \end{aligned}$$

Now we can proceed to the proof of an important fact, namely of the invariance of the above definition (see (7)) of the differential of a form $\mathfrak{A}$ with respect to any variables u $(x_i = \psi_i(u_1, \dots, u_n);\ i = 1, \dots, n)$. In other words, the transformation of the form $d\mathfrak{A}$ expressed in terms of x to the new variable u results in the expression

$$d\mathfrak{A} = \sum_{j_1} \sum \dots \sum_{j_k} \sum_{l} \frac{\partial}{\partial u_l} \left(a \frac{\partial x_{i_1}}{\partial u_{j_1}} \dots \frac{\partial x_{i_k}}{\partial u_{j_k}} \right) du_l \, du_{j_1} \dots du_{j_k}$$

For $k = 0$ the invariance is trivial: if

$$\mathfrak{A} = f(x) = f(x_1, \dots, x_n)$$

then

$$\begin{aligned} d\mathfrak{A} &= \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \sum_{s=1}^{n} \frac{\partial x_j}{\partial u_s} du_s = \\ &= \sum_{s=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \frac{\partial x_j}{\partial u_s} \right) du_s = \sum_{s=1}^{m} \frac{\partial f}{\partial u_s} du_s \end{aligned}$$

Now let us suppose that the invariance of $d\mathfrak{A}$ has been proved for some $k \geqslant 0$ and show that then it holds for $k+1$. To this end it is sufficient to consider the simplest $(k+1)$-form (see the explanations below)

$$\mathfrak{A} = a(x)\, dx_{i_1} \dots dx_{i_{k+1}} = a(x)\, dx_{i_1} \dots dx_{i_k} \cdot dx_{i_{k+1}}$$

By formula (12), taking into account that $d^2 = 0$, we obtain

$$d\mathfrak{A} = d\big(a(x)\, dx_{i_1} \dots dx_{i_k}\big)\, dx_{i_{k+1}} + \\ +(-1)^k a(x)\, dx_{i_1} \dots dx_{i_k}\, d(dx_{i_{k+1}}) = d\big(a(x)\, dx_{i_1} \dots dx_{i_k}\big)\, dx_{i_{k+1}}$$

We have thus represented $d\mathfrak{A}$ as the product of the differential of a k-form by the differential of the 0-form $x_{i_{k+1}}$. Since both factors are invariant and, as we know, the product of any forms is invariant, this completes the proof.

Now let us consider an oriented k-dimensional differentiable manifold L_k described by some functions

$$x_i = f_i(u_1, \dots, u_k) \ (i = 1, \dots, n), \ (u_1, \dots, u_k) \in G \rightleftarrows L_k \qquad (13)$$

possessing uniformly continuous partial derivatives on a bounded open set G. We shall suppose that $L_k \subset \Omega \subset R_n$ where Ω is a domain on which a (differential) k-form $\mathfrak{A}$ is defined (see (1); the coefficients $a(x)$ are supposed to be continuous on $\bar{\Omega}$).

The integral of the form $\mathfrak{A}$ *over* L_k is defined as the ordinary multiple integral

$$\int_{L_k} \mathfrak{A} = \sum \int_G a(u) \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u_1, \dots, u_k)}\, du_1 \dots du_k \qquad (14)$$

$$\big(a(u) = a(f_1(u), \dots, f_n(u))\big)$$

which is understood in Lebesgue's sense in the general case and in Riemann's sense when G is Jordan measurable.

This definiton is independent of the choice of the admissible parameters $u' = (u'_1, \dots, u'_k)$ *preserving the orientation of* L_k, that is connected with u with the aid of a transformation $G' \ni u' = (u'_1, \dots, u'_k) \rightleftarrows u = (\tilde{u}_1, \dots, \tilde{u}_k) \in$ $\in G$ continuously differentiable in both directions and having a positive Jacobian $\left(\text{the derivatives } \frac{\partial u'_j}{\partial u_s} \text{ are supposed to be } \textit{uniformly continuous} \text{ on } G\right)$. Indeed, we have

$$\int_{G'} a(u') \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u'_1, \dots, u'_k)}\, du'_1 \dots du'_k =$$

$$= \int_G a(u) \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u'_1, \dots, u'_k)} \frac{D(u'_1, \dots, u'_k)}{D(u_1, \dots, u_k)}\, du_1 \dots du_k =$$

$$= \int_G a(u) \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u_1, \dots, u_k)}\, du_1 \dots du_k \qquad (15)$$

(see Theorem 1 in § 12.16 and, for the case of Lebesgue's integrals, property 22 in § 19.3).

If the Jacobian is negative, then in the second member of (15) and also in the third member there appears the minus sign, and therefore when the orientation of L_k is reversed the integral of $\mathfrak{A}$ over L_k changes sign.

It is also important that if $\mathfrak{A}$ is transformed from x to x' by means of a transformation $x \rightleftarrows x'$ continuously differentiable in both directions, the integral $\int\limits_{L_k} \mathfrak{A}$ expressed in terms of x' remains invariant, that is it remains equal to the right-hand side of (14). Indeed, the form $\mathfrak{A}$ is written in terms of the variables x' as

$$\mathfrak{A} = \sum a(x') \sum_{j_1<\dots<j_k} \frac{D(x_{i_k}, \dots, x_{i_k})}{D(x'_{j_1}, \dots, x'_{j_k})} dx'_{j_1} \dots dx'_{j_k}$$

and the integral of $\mathfrak{A}$ over L_k (with respect to the variables $x' = (x'_1, \dots, x'_n)$) is equal to

$$\sum \int\limits_G a(u) \sum_{j_1<\dots<j_k} \frac{D(x_{i_1}, \dots, x_{i_k})}{D(x'_{j_1}, \dots, x'_{j_k})} \frac{D(x'_{j_1}, \dots, x'_{j_k})}{D(u_1, \dots, u_k)} du_1 \dots du_k =$$

$$= \sum \int\limits_G a(u) \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u_1, \dots, u_k)} du_1 \dots du_k$$

that is it is equal to the integral of $\mathfrak{A}$ with respect to the variables $x = (x_1, \dots, x_n)$.

We can also consider a part σ of L_k described by functions (13) when the parameter u ranges over a measurable subset $e \subset G$ and define the integral of the k-form $\mathfrak{A}$ over σ by means of the formula

$$\int\limits_\sigma \mathfrak{A} = \sum \int\limits_e a(u) \frac{D(x_{i_1}, \dots, x_{i_k})}{D(u_1, \dots, u_k)} du_1 \dots du_k \tag{16}$$

analogous to (14).

This definition is obviously also invariant with respect to any change of variables $x \rightleftarrows x'$ by means of a transformation continuously differentiable in both directions and is independent of the choice of the admissible parameters (see property 22 in § 19.3). In this case it is natural to call σ *a measurable part of L_k oriented coherently with L_k*.

Now let us suppose that an oriented manifold L_k can be covered by a finite number of submanifolds σ_s oriented coherently with L_k:

$$L_k = \sum_{s=1}^{N} \sigma_s$$

Let each of the manifolds σ_s be described by some functions

$$x_i = f_i^s(u^s), \quad u^s \in G^s; \qquad i = 1, \dots, n, \quad s = 1, \dots, N$$

possessing uniformly continuous partial derivatives on a bounded domain G^s; $s = 1, \dots, N$. Then the integral of a k-form $\mathfrak{A}$ over L_k can be defined as follows. We choose pairwise disjoint measurable parts $\lambda_s \subset \sigma_s$ (oriented coherently with σ_s; $s = 1, \ldots N$, or, which is the same, with L_k) whose union is $L_k \left(L_k = \sum_{s=1}^{N} \lambda_s \right)$ and put

$$\int_{L_k} \mathfrak{A} = \sum_{s=1}^{N} \int_{\lambda_s} \mathfrak{A}$$

The parts λ_s can be defined by the relations

$$\lambda_1 = \sigma_1, \quad \lambda_2 = \sigma_2 - \sigma_2\sigma_1, \quad \ldots, \quad \lambda_N = \sigma_N - \sigma_N\sigma_1 - \ldots - \sigma_N\sigma_{N-1}$$

For instance, it is clear that $\lambda_2 \in \sigma_2$ and that $\sigma_1\sigma_2$ is a manifold described by the parameter $u^2 \in \omega$ where $\omega \subset G^2$ is an open set (see Lemma 4 in § 17.1). Then λ_2 is described by the parameter $u^2 \in G^2 - \omega$ where $G^2 - \omega$ is a measurable set belonging to G^2 and λ_2 is a measurable part of L_k oriented coherently with L_k.

For any other representation

$$L_k = \sum_{j=1}^{M} \lambda_j', \quad \lambda_j'\lambda_i' = 0, \qquad j \neq i$$

of the indicated type we have

$$\sum_{s=1}^{N} \int_{\lambda_s} \mathfrak{A} = \sum_{s=1}^{N} \sum_{j=1}^{M} \int_{\lambda_s\lambda_j'} \mathfrak{A} = \sum_{j=1}^{M} \int_{\lambda_j'} \mathfrak{A}$$

Here one should take into account that if some parts λ_s and λ_j' intersect then they belong, respectively, to some intersecting manifolds σ_s and σ_j' described by the corresponding parameters $u^s \in G^s$ and $u^{j'} \in G^{j'}$, and the parts λ_s and λ_j' themselves are described by $u^s \in e^s \subset G^s$ and $u^{j'} \in e^{j'} \subset G^{j'}$, respectively, where e^s and $e^{j'}$ are measurable. By the lemma we have mentioned, the manifold $\sigma_s\sigma_j'$ can be described by the parameter $u^s \in \omega^s$ or by the parameter $u^{j'} \in \omega^{j'}$. Both parameters are in a one-to-one correspondence (uniformly) continuously differentiable in both directions, the Jacobian of the transformation $u^s \rightleftarrows u^{j'}$ being positive (σ^s and $\sigma^{j'}$ are oriented coherently!).

The intersection $e^s\omega^s$ is a measurable set of points u^s which goes, under this transformation, into the set $(e^s\omega^s)'$ of points $u^{j'}$ which is also measurable (see property 22 in § 19.3). The intersection of $(e^s\omega^s)'$ with $e^{j'}$ is in its turn a measurable set of points $u^{j'}$ to which there obviously corresponds the set $\lambda_s\lambda^{j'} \subset L_k$. Consequently, $\lambda_s\lambda^{j'}$ is a part of L_k.

In conclusion we write the useful equality

$$\int_{L_k} (\alpha\mathfrak{A} + \beta\mathfrak{B}) = \alpha \int_{L_k} \mathfrak{A} + \beta \int_{L_k} \mathfrak{B}$$

where α and β are numbers and $\mathfrak{A}$ and $\mathfrak{B}$ are k-forms.

Remark. It should be noted that the integral $\int\limits_{L_k} \mathfrak{A}$ of a k-form $\mathfrak{A}$ over an oriented k-dimensional manifold L_k defined above is invariant with respect to any transformation of coordinates $x \rightleftarrows x'$ continuously differentiable in both directions. This follows from the invariance of the terms $\int\limits_{\lambda_s} \mathfrak{A}$ of which the integral in question consists.

The indicated invariance property of the integral $\int\limits_{L_k} \mathfrak{A}$ is one of the fundamental properties of differential forms.

Example 1. The integral of a 1-form

$$\mathfrak{A} = P(x, y, z)\, dx + Q(x, y, z)\, dy + R(x, y, z)\, dz$$

in the (three-dimensional) xyz-space over an oriented one-dimensional differentiable manifold L_1 described by equations

$$x = \varphi(t),\ y = \psi(t),\ z = \chi(t) \qquad (0 < t < l)$$

where φ, ψ and χ possess the derivatives φ', ψ' and χ' continuous on $[0, l]$ is equal to

$$\int\limits_{L_1} \mathfrak{A} = \int\limits_0^l \left(P(\varphi, \psi, \chi)\,\varphi' + Q(\varphi, \psi, \chi)\,\psi' + R(\varphi, \psi, \chi)\,\chi'\right) dt =$$

$$= \int\limits_{L_1} (P\, dx + Q\, dy + R\, dz)$$

that is it is the line integral of the vector $\boldsymbol{a} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ over the oriented curve L_1.

Example 2. The integral of a 2-form

$$\mathfrak{B} = P(x, y, z)\, dy\, dz + Q(x, y, z)\, dz\, dx + R(x, y, z)\, dx\, dy$$

in the (three-dimensional) xyz-space over an oriented two-dimensional manifold L_2 described by equations

$$x = \varphi(u, v),\ y = \psi(u, v),\ z = \chi(u, v) \qquad (u, v) \in \omega$$

where φ, ψ and χ possess continuous partial derivatives on $\bar{\omega}$ is equal to

$$\int\limits_{L_2} \mathfrak{B} = \iint\limits_{\omega} \left[P(\varphi, \psi, \chi)\frac{D(y, z)}{D(u, v)} + Q(\varphi, \psi, \chi)\frac{D(z, x)}{D(u, v)} + R(\varphi, \psi, \chi)\frac{D(x, y)}{D(u, v)}\right] du\, dv =$$

$$= \iint\limits_{L_2} (P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy)$$

that is it is the surface integral over the surface $\boldsymbol{r} = \varphi\boldsymbol{i} + \psi\boldsymbol{j} + \chi\boldsymbol{k}$ oriented with the aid of the normal $\boldsymbol{n} = \boldsymbol{r}_u + \boldsymbol{r}_v$.

Example 3. The triple integral

$$\int_G f(x, y, z)\, dx\, dy\, dz = \int_{L_3} \mathfrak{L}$$

over a domain G in the xyz-space can be interpreted as the integral of the 3-form

$$\mathfrak{L} = f(x, y, z)\, dx\, dy\, dz$$

over the three-dimensional manifold $L_3 = G$ equipped with positive orientation ($x = x, y = y, z = z$).

§ 17.4. Stokes' Theorem

Stokes' Theorem. *Let L'_{k+1} be a connected oriented $(k+1)$-dimensional manifold and $L_{k+1} \subset \bar{L}_{k+1} \subset L'_{k+1}$ be its part which is also a $(k+1)$-dimensional manifold. Let L_{k+1} have a boundary $\partial L_{k+1} = L_k$ which is a connected k-dimensional manifold oriented coherently with L_{k+1} (see the coherence rule for the orientation of a boundary in § 17.2 (18) and (19)[*]).*

Then for an arbitrary differential k-form $\mathfrak{A}$ defined in a domain $\Omega \subset R_n$ ($\bar{L}_{k+1} \subset \Omega$) there holds Stokes' formula

$$\int_{L_{k+1}} d\mathfrak{A} = \int_{\partial L_{k+1}} \mathfrak{A} \tag{1}$$

Proof.

1. We shall begin with the simplest case when the oriented part L_{k+1} of L'_{k+1} is the $(k+1)$-dimensional cube specified by the following functions of the variables $x_1, \dots, x_{k+1}$ written in the indicated order:

$$\left.\begin{array}{lll} x_j = x_j, & 0 \leqslant x_j \leqslant \delta, & j = 1, \dots, k+1 \\ x_j = 0, & j = k+2, \dots, n & \end{array}\right\} \tag{2}$$

In the case of a rectangle the proof is quite analogous.

The boundary ∂L_{k+1} consists of the $2(k+1)$ oriented parts:

$$L_k = \sum_{s=1}^{k+1} (L_0^s + L_\delta^s)$$

These parts coincide, to within the orientation, with the parts determined by the equations

$$\left.\begin{array}{l} L_0^{*s} = \{x_j = x_j, 0 \leqslant x_j \leqslant \delta, j = 1, \dots, s-1, s+1, \dots \\ \qquad \dots, k+1; x_j = 0, j = s, k+2, \dots, n\} \\ L_0^{*s} = \{x_j = x_j, 0 \leqslant x_j \leqslant \delta, j = 1, \dots, s-1, s+1, k+1, \\ \qquad x_s = \delta, x_j = 0,\ j = k+2, \dots, n\} \end{array}\right\} \tag{3}$$

---

[*] Here L_{k+1} must not necessarily be 1-connected; see § 17.2 (20) and (21).

To find the relationship between the orientations of these parts let us renumber the variables $x_s, x_1, \ldots, x_{s-1}, x_{s+1}, \ldots x_n$ by denoting them respectively as $x'_1, \ldots, x'_n$. The Jacobian of this transformation is equal to $(-1)^{s-1}$. On the other hand, equations (2) in which x_j with $j = 1, \ldots, k+1$ are arbitrary determine the points of L_{k+1} for $x_s < \delta$ and the points of the exterior of L_{k+1} for $x_s > \delta$, and therefore the parts L^s_δ and L^{*s}_δ have the same or reverse orientations depending on whether the number $s-1$ is even or odd. Arguing in the same manner we conclude that the parts L^s_0 and L^{*s}_0 have the same orientations for odd $s-1$ and reverse orientations for even $s-1$.

It suffices to carry out the proof for the simplest form

$$\mathfrak{A} = a(x)\,dx_{i_1}, \ldots, dx_{i_k} \qquad (1 \leqslant i_s \leqslant n)$$

By what has been said, equality (1) which we have to prove reduces to the following equality connecting ordinary multiple integrals*:

$$\int\limits_{\Delta} \sum_{j=1}^{n} \left(\frac{\partial a}{\partial x_j}\right)_0 \frac{D(x_j, x_{i_1}, \ldots, x_{i_k})}{D(x_1, \ldots, x_{k+1})}\, dx_1, \ldots, dx_{k+1} =$$

$$= \sum_{s=1}^{k+1} (-1)^{s+1} \left(\int\limits_{L^{*s}_\delta} a_0 \frac{D(x_{i_1}, \ldots, x_{i_k})}{D(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_{k+1})} \times \right.$$

$$\times dx_1 \ldots dx_{s-1}\, dx_{s+1} \ldots dx_{k+1} -$$

$$- \int\limits_{L^{*s}_0} a_0(x) \frac{D(x_{i_1}, \ldots, x_{i_k})}{D(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_{k+1})} \times$$

$$\left. \times dx_1 \ldots dx_{s-1}\, dx_{s+1} \ldots dx_{k+1} \right) \tag{4}$$

where $a_0 = a(x_1, \ldots, x_{k+1}, 0, \ldots, 0)$, $\left(\frac{\partial a}{\partial x_j}\right)_0 = \frac{\partial a}{\partial x_j}(x_1, \ldots, x_{k+1}, 0, \ldots, 0)$ and $\Delta = \{0 \leqslant x_j \leqslant \delta;\ j = 1, \ldots, k+1\}$ is a cube in the space $(x_1, \ldots, x_{k+1})$. It is quite obvious that formula (4) is sure to hold in the following cases:

(i) when for some pair (m, l), $m \neq l$ we have $i_m = i_l$ because in that case all the Jacobians in (4) are equal to zero;

(ii) when $i_m > k+1$ for at least one m because in this case $x_{i_m} \equiv 0$ (see (2) and (3)).

Therefore we have to prove (4) for different indices i_m satisfying the inequalities $1 \leqslant i_m \leqslant k+1$. Equality (4) is not violated when these indices are rearranged in it in an increasing order because under such an operation all the three Jacobians in (4) either simultaneously retain their values or simultaneously change sign. Therefore it suffices to prove (4), for instance, under the assumption that the indices i_s are arranged as $1, 2, \ldots, l-1, l+1, \ldots, k+1$. In the latter case all the summands entering into the sum on the left-hand side of (4) corresponding to the indices $j \neq l$ are equal to zero (because the corresponding Jacobians on the left-hand side of (4) are equal to zero), and therefore the left-hand side of (4) is still more simplified:

---

* The integrals $\int\limits_{L^{*s}_\delta}$ and $\int\limits_{L^{*s}_0}$ are reducible (see § 17.3 (14)) to multiple integrals over the projection of L^{*s}_0 on the plane $x_1, \ldots, x_{s-1}, x_s, \ldots, x_{k+1}$ by putting, respectively, $x_s = \delta$ and $x_s = 0$ in a_0.

$$\int\limits_{L_{k+1}} d\mathfrak{A} = (-1)^{l-1} \int\limits_{\Delta} \left(\frac{\partial a}{\partial x_l}\right)_0 dx_1 \ldots dx_{k+1} =$$

$$= (-1)^{l-1} \int\limits_0^\delta \ldots \int\limits_0^\delta \left(\frac{\partial a}{\partial x_l}\right)_0 dx_1\, dx_{k+1} =$$

$$= (-1)^{l-1} \int\limits_0^\delta \ldots \int\limits_0^\delta [a(x_1, \ldots, x_{l-1}, \delta, x_{l+1}, \ldots, x_{k+1}, 0, \ldots, 0) -$$

$$- a(x_1, \ldots, x_{l-1}, 0, x_{l+1}, \ldots, x_{k+1}, 0, \ldots, 0)]\, dx_1 \ldots$$

$$\ldots dx_{l-1} dx_{l+1} \ldots dx_{k+1} = \int\limits_{L_\delta^l} \mathfrak{A} + \int\limits_{L_0^l} \mathfrak{A} = \int\limits_{L_k} \mathfrak{A} \tag{5}$$

where the last equality holds since for $j \neq l$ we have

$$\int\limits_{L_0^j} \mathfrak{A} = \int\limits_{L_0^j} a\, dx_1 \ldots dx_{l-1}\, dx_{l+1} \ldots dx_{k+1} =$$

$$= \int\limits_0^\delta \ldots \int\limits_0^\delta a \frac{D(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{k+1})}{D(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k+1})} \times$$

$$\times dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_{k+1} = 0$$

and, similarly, $\int\limits_{L_\delta^j} \mathfrak{A} = 0$ for the Jacobian on the right-hand side contains a column whose elements are the derivatives $\frac{\partial x_j}{\partial x_1}, \ldots, \frac{\partial x_j}{\partial x_{j-1}}, \frac{\partial x_j}{\partial x_{j+1}}, \ldots, \frac{\partial x_j}{\partial x_{k+1}}$ which are equal to zero.

2. Let us agree that continuously differentiable functions and continuously differentiable in both directions mappings will be simply called functions and mappings, respectively.

(a) Let a point x^0 belong to L_{k+1}. Then after the coordinates are appropriately renumbered we can find a rectangle

$$\Delta = \{|x_j - x_j^0| < \delta,\ j = 1, \ldots, k+1;\ |x_j - x_j^0| < \sigma, \quad j = k+2, \ldots, n\}$$

and functions $x_j = f_j(x^{k+1})$, $j = k+2, \ldots, n$, $x^{k+1} = (x_1, \ldots x_{k+1}) \in \Delta^{(k+1)} = \{|x_j - x_j^0| < \delta, j = 1, \ldots, k+1\}$ describing the part $L_{k+1}\Delta$. Hence, $|x_j - f_j(x^{k+1})| < \sigma$, and if δ is decreased, then moreover

$$|x_j - f_j(x^{k+1})| < \sigma,\ x^{k+1} \in \overline{\Delta^{(k+1)}}, \qquad j = k+2, \ldots, n$$

By the closedness and boundedness of $\overline{\Delta^{(k+1)}}$, the left members of the last inequalities do not in fact exceed a positive number $\sigma_1 < \sigma$, and therefore if $\lambda = \sigma - \sigma_1$, then the points x of the form

$$x_j = x_j,\ j = 1, \dots, k+1;\quad x_j = z_j + f_j(x^{k+1}) \tag{6}$$

$$x^{k+1} \in \Delta^{(k+1)},\ |z_j| < \lambda,\ j = k+2, \dots, n \tag{6'}$$

constitute a set $\Lambda \subset \Delta$. Under (6), open rectangle (6') which we denote Λ', is mapped onto Λ and $\bar{\Lambda}'$ onto $\bar{\Lambda}$. Under this mapping the $(k+1)$-dimensional rectangle $\Delta^{(k+1)}$ $(z_j = 0)$ belonging to Λ' goes into $L_{k+1}\Delta = L_{k+1}\Lambda$ and $\overline{\Delta^{(k+1)}}$ into $\overline{L_{k+1}\Lambda}$. The change of variables $x_j - x_j^0 = \delta u_j; j = 1, \dots, k+1$; $z_j = \lambda u_j$; $j = k+1, \dots, n$, produces a mapping of Λ' onto unit cube of points u. It should also be noted that the renumbering of the coordinates reduces to the mapping $x_j' = x_{s_j}$ $(j = 1, \dots, n;\ 1 \leq s_j \leq n)$.

(b) Now let $x^0 \in L_k$. Then there holds Theorem 1, § 17.2, in which we should put $\Gamma = L_k$ and $S = L_{k+1}$. Arguing as above, we see that if the number δ_2 in § 17.2 (1) is diminished and δ_1 is chosen accordingly (less than before), we arrive at a rectangle, which we again denote Δ, possessing the following property. There are numbers λ, $\mu > 0$ such that the set Λ consisting of the points x of the form

$$\left.\begin{aligned} x_j &= x_j,\ j = 1, \dots, k;\ x^k \in \Delta^{(k)} \\ x_{k+1} &= v + \varphi(x^k),\ |v| < \lambda \\ x_j &= z_j + f_j\big(x_1, \dots, x_k, v + \varphi(x^k)\big),\ |z_j| < \mu,\ j = k+2, \dots, n \end{aligned}\right\} \tag{7}$$

belongs to Δ.

Equations (7) specify a mapping under which the rectangle Λ' $(x^k \in \Delta^{(k)}$, $|v| < \lambda$, $|z_j| < \mu)$ lying in the space $(x_1, \dots, x_k, v, z_{k+1}, \dots, z_n)$ goes into the domain Λ and $\bar{\Lambda}'$ into $\bar{\Lambda}$. Under the mapping one of the rectangles $(x^k \in \Delta^{(k)}, 0 < v < \lambda)$ and $(x^k \in \Delta^{(k)}, -\lambda < v < 0)$ goes into $L_{k+1}\Delta = L_{k+1}\Lambda$ (see § 17.2 (11), (12)). It can be mapped onto a cube by means of a simple transformation.

It follows from (a) and (b) that an arbitrary point $x^0 \in \overline{L_{k+1}}$ can be covered by a domain $G_{x^0} = \Lambda$ which is mapped onto a domain Λ' belonging to another n-dimensional space R_n' and its closure $\bar{\Lambda}$ onto $\bar{\Lambda}'$ so that $\overline{L_{k+1}\Lambda}$ is mapped onto a $(k+1)$-dimensional cube $\bar{\omega} \in R_n'$. Therefore (see the remark at the end of § 17.3) for the part $L_{k+1}\Lambda$ there holds Stokes' formula

$$\int_{L_{k+1}\Lambda} d\mathfrak{A} = \int_{\partial(L_{k+1}\Lambda)} \mathfrak{A}$$

because it holds for a cube.

Note that in the cases under consideration the boundary L_k of the manifold L_{k+1} is not a manifold but the closure of the sum of a finite number of pairwise nonintersecting and coherently oriented manifolds, and the integral

$\int\limits_{L_k} \mathfrak{A}$ is defined in a natural way as the sum of the integrals of $\mathfrak{A}$ over these manifolds. This fact should be taken into account in what follows.

Since L_{k+1} is a bounded closed set, there exists its finite covering $G_1, \ldots, G_N$ by the sets of the type of G_{x_0}. By virtue of the lemma on the partition of unity (see § 18.4), there is a system of infinitely differentiable finite functions $\varphi_1(x), \ldots, \varphi_N(x)$ possessing the following properties:

(i) $0 \leqslant \varphi_j(x) \leqslant 1$,

(ii) the support of φ_j $(j = 1, \ldots, N)$ is a (closed) set belonging to the (open) set G_j,

(iii) $\sum\limits_1^N \varphi_j(x) = 1$ on L_{k+1}.

Consequently (see the explanations below)

$$\int\limits_{L_{k+1}} d\mathfrak{A} = \int\limits_{L_{k+1}} d\left(\sum_1^N \varphi_j\mathfrak{A}\right) = \sum_1^N \int\limits_{L_{k+1}} d(\varphi_j\mathfrak{A}) = \sum_1^N \int\limits_{L_{k+1}G_j} d(\varphi_j\mathfrak{A}) =$$

$$= \sum_1^N \int\limits_{\partial(L_{k+1}G_j)} \varphi_j\mathfrak{A} = \sum_1^N \int\limits_{L_kG_j} \varphi_j\mathfrak{A} = \sum_1^N \int\limits_{L_k} \varphi_j\mathfrak{A} = \int\limits_{L_k} \sum_1^N \varphi_j\mathfrak{A} = \int\limits_{L_k} \mathfrak{A}$$

In the first of these equalities we have used property (iii); the second equality is apparent; the third one holds because the form $\varphi_j\mathfrak{A}$ $(j = 1, \ldots$ $\ldots, N)$ and, together with it, the form $d(\varphi_j\mathfrak{A})$ are equal to zero outside G_j. The fourth equality holds by virtue of Stokes' theorem whose proof for elementary parts $L_{k+1}G_j$ has been already carried out and the fifth equality holds because at the points not belonging to the intersection of the set $\partial(L_{k+1}G_j)$ with L_k the form $\varphi_j\mathfrak{A}$ is equal to zero. Finally, the sixth equality holds since the form $\varphi_j\mathfrak{A}$ is equal to zero outside G_j, the seventh is obvious and the last one holds by virtue of (iii).

Examples. Stokes' general theorem (formula (1)) implies as special cases the formulas (see the explanations below)

$$\iint\limits_{L_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int\limits_{\partial L_2} (P\, dx + Q\, dy) \tag{8}$$

$$\iiint\limits_{L_3} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\, dy\, dz = \iint\limits_{\partial L_3} (P\, dy\, dz + Q\, dz\, dx + R\, dx\, dy) \tag{9}$$

and

$$\iint\limits_{L_2} \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) dy\, dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) dz\, dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy\right] =$$

$$= \int\limits_{\partial L_2} (P\, dx + Q\, dy + R\, dz) \tag{10}$$

In formula (8) L_2 is a bounded two-dimensional manifold in the xy-plane and in (10) it is a bounded two-dimensional manifold in the three-dimensional space (x, y, z). L_3 is a bounded three-dimensional manifold in the three-dimensional space (x, y, z). Further, it is assumed that L_2 and ∂L_2 and also L_3 and ∂L_3 satisfy the conditions of Theorem 2 in § 17.2 or the more general conditions (see (20) and (21) in § 17.2) allowing L_2 and L_3 to be multiply-connected. The validity of equalities (8), (9) and (10) follows from Stokes' general theorem and from the fact that the form under the integral sign on the left-hand side of each of these equalities is the differential of the form under the integral sign on the corresponding right-hand side. For instance, we have

$$d(P\,dx+Q\,dy) = \left(\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy\right)dx+\left(\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy\right)dy =$$
$$= \frac{\partial P}{\partial y}dy\,dx+\frac{\partial Q}{\partial x}dx\,dy = \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy \quad .$$

It should also be noted that the integral on the left-hand side of (8) is an ordinary integral over the oriented domain L_2 while the integral on the right-hand side of (8) is an ordinary line integral taken over the contour ∂L_2 oriented coherently with L_2 (see examples in § 17.3), and hence (8) is nothing but Green's formula. Similar considerations show that (9) is the Gauss-Ostrogradsky formula for the domain L_3 with the smooth coherently oriented boundary ∂L_3 while (10) coincides with Stokes' ordinary formula for the smooth oriented surface L_2 in the three-dimensional space pulled over the contour ∂L_2.

In the elementary derivation of the Green, Gauss-Ostrogradsky and Stokes formulas (§§ 13.5, 13.10 and 13.11) it was assumed that the corresponding domains and surfaces can be cut into parts of some special form. In the derivation of these formulas given here such requirements are not imposed on the geometrical figures in question, and in this sense the results obtained are more general.

Stokes' theorem can be extended to a manifold $L_{k+1} \subset \bar{L}_{k+1} \subset L'_{k+1}$ whose closure can be cut with the aid of smooth surfaces into a finite number of parts each of which can be regarded as the result of a continuous deformation of a cube by means of a transformation continuously differentiable in both directions. Their orientations should be coherent in the sense of the explanation given in § 13.7 for the special case of a piecewise smooth surface.

The functions P, Q and R in formulas (8), (9) and (10) are supposed to be continuously differentiable in the corresponding domains containing L_2 or L_3.

CHAPTER 18

Supplementary Topics

§ 18.1. Generalized Minkowski's Inequality

For any finite number N of elements $x^1, x^2, \ldots, x^N$ of a normed linear space E we have the inequality

$$\left\| \sum_{k=1}^{N} x^k \right\| \leq \sum_{k=1}^{N} \|x^k\| \tag{1}$$

obtained by induction from the basic inequality $\|x+y\| \leq \|x\|+\|y\|$ $(x, y \in E)$. Further, if x is the sum of a series $\sum_1^\infty x^k$ $(x, x^k \in E)$, i.e.

$$x = x^1+x^2+ \ldots = \sum_1^\infty x^k$$

convergent in norm in the space E, which means that $\left\| x - \sum_1^N x^k \right\| \to 0$ $(N \to \infty)$ then (see § 6.3)

$$\|x\| = \lim_{N\to\infty} \left\| \sum_1^N x^k \right\| \leq \lim_{N\to\infty} \sum_1^N \|x^k\| = \sum_1^\infty \|x^k\|$$

This can be written in the form

$$\|x\| \leq \|x^1\|+\|x^2\|+ \ldots \tag{2}$$

where the series on the right-hand side may be divergent to infinity.

Now, applying (1) and (2) to elements of the spaces l_p and L_p' (or L_p) we obtain (Minkowski's) inequalities

$$\left(\sum_{k=1}^\infty \left| \sum_{l=1}^m a_{kl} \right|^p \right)^{1/p} \leq \sum_{l=1}^m \left(\sum_{k=1}^\infty |a_{kl}|^p \right)^{1/p} \tag{3}$$

and

$$\left(\int_\Omega \left| \sum_{l=1}^m f_l(x) \right|^p dx \right)^{1/p} \leq \sum_{l=1}^m \left(\int_\Omega |f_l(x)|^p \, dx \right)^{1/p} \tag{4}$$

($1 \leq p \leq \infty$) where it is allowable to put $m = \infty$ on condition that in this case the sums of the series $\sum\limits_{l=1}^{\infty} a_{kl} = a_k$ are understood as numbers a_k such that $\sum\limits_{1}^{\infty} \left| a_k - \sum\limits_{l=1}^{m} a_{kl} \right|^p \to 0$ ($m \to \infty$) and the sum $\sum\limits_{l=1}^{m} f_l(x)$ is understood as a function $F(x) \in L'_p$ (L_p) to which the finite sum $\sum\limits_{l=1}^{m} f_l(x)$ tends, as $m \to \infty$, with respect to the metric of L_p, i.e. $\int\limits_{\Omega} \left| F(x) - \sum\limits_{l=1}^{m} f_l(x) \right|^p dx \to 0$ ($m \to \infty$).

Inequality (4) involves the summation with respect to the discrete index $l = 1, 2, 3, \ldots$ on which $f_l(x)$ depends. As an analogue of (4) we can write generalized Minkowski's inequality

$$\left(\int\limits_{G} \left(\int\limits_{\Omega} |F(x, y)|\, dy \right)^p dx \right)^{1/p} \leq \int\limits_{\Omega} \left(\int\limits_{G} |F(x, y)|^p\, dx \right)^{1/p} dy \tag{5}$$

$$(\Omega \subset R_n,\ G \subset R_m)$$

If $F(x, y) = 0$ outside $\Omega \times G$ inequality (5) can be rewritten as

$$\left(\int\limits_{R_m} \left(\int\limits_{R_n} |F(x, y)|\, dy \right)^p dx \right)^{1/p} \leq \int\limits_{R_n} \left(\int\limits_{R_m} |F(x, y)|^p\, dx \right)^{1/p} dy \tag{6}$$

In generalized Minkowski's inequalities (5) and (6) the role of the summation index l entering into (4) is played by the parameter y with respect to which the integration (instead of the summation) is carried out.

The most general form of inequality (6) involves integrals understood in Lebesgue's sense; in this case the condition that the right-hand member of (6) makes sense guarantees the existence of the left-hand member and the validity of the inequality.

For the sake of simplicity, we shall confine ourselves to the case of a function $F(x, y)$ of two (scalar) variables x and y. Without violating generality we can assume that $F(x, y) \geq 0$. Let us begin with the special case when $F(x, y)$ is a bounded function defined on a square $\Delta_N = \{|x|, |y| \leq N\}$ and integrable on it in Riemann's or Lebesgue's sense.

We can write (see the explanations below)

$$\begin{aligned}
\int \left(\int F(x, y)\, dy \right)^p dx &= \int \left(\int F(x, y)\, dy \right)^{p-1} \int F(x, y)\, dy\, dx = \\
&= \int\int \left(\int F(x, y)\, dy \right)^{p-1} F(x, y)\, dy\, dx = \\
&= \int \left[\int \left(\int F(x, y)\, dy \right)^{p-1} F(x, y)\, dx \right] dy \leq \\
&\leq \int \left(\int \left(\int F(x, y)\, dy \right)^p dx \right)^{(p-1)/p} \left(\int F(x, y)^p\, dx \right)^{1/p} dy = \\
&= \left(\int \left(\int F(x, y)\, dy \right)^p dx \right)^{(p-1)/p} \int \left(\int F(x, y)^p\, dx \right)^{1/p} dy
\end{aligned} \tag{7}$$

$\left(\text{where } \int = \int_{-N}^{N}\right)$, whence follows the desired inequality

$$\left(\int \left(\int F(x,y)\,dy\right)^p dx\right)^{1/p} \leqslant \int \left(\int F(x,y)^p\,dx\right)^{1/p} dy \tag{8}$$

In the second equality (7) we have written the integral $\left(\int F\,dy\right)^{p-1}$ which is independent of y as a factor under the integral sign in $\int F(x, y)\,dy$. In the third equality we have changed the order of integration, which is legitimate for Lebesgue integrable nonnegative (measurable) functions (see § 19.3, property 19, Fubini's theorem). When F is Riemann integrable on Δ_N this is also legitimate. For, in the latter case the integral $\int F(x, y)\,dy$* is an integrable function of x on $[-N, N]$ and therefore the $(p-1)$th power of its modulus is also integrable on $[-N, N]$ and, consequently, on Δ_N as well. The $(p-1)$th power of the modulus of that integral is multiplied by the function $F(x, y)$ integrable on Δ_N, and hence under the sign $\int\int$ in the third member of (7) there is a Riemann integrable function and therefore it is legitimate to change the order of integration. In the fourth relation (inequality) in (7) we have used Hölder's inequality with respect to x.

In our further discussion we shall deal with the integration in Lebesgue's sense. Suppose that we are given an arbitrary nonnegative measurable function $F(x, y)$ (generally speaking, unbounded) for which the integral on the right-hand side of (8) is finite. Let us put

$$F_N(x, y) = \begin{cases} F(x, y) & \text{for } (x, y) \in \Delta_N \text{ and } F \leqslant N \\ 0 & \text{for the other } (x, y) \end{cases}$$

The function F_N is nonnegative, bounded, measurable and equal to zero outside Δ_N. As was already proved, a function of this kind satisfies the inequality

$$\left(\int \left(\int F_n(x,y)\,dy\right)^p dx\right)^{1/p} \leqslant \int \left(\int [F_N(x,y)]^p\,dx\right)^{1/p} dy$$

for any N, whence follows the validity of (8) because, as is proved in the theory of Lebesgue's integral (see § 19.3, property 14), it is legitimate to pass to the limit as $N \to \infty$ in the last inequality since F_N is monotone in N and

$$F_N(x, y) \to F(x, y) \qquad (N \to \infty,\ F_N \leqslant F)$$

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* Or the corresponding lower integral (see Theorem 1, § 12.12).

§ 18.2. Sobolev's* Regularization of Function

Let us denote by σ_ε the ball of radius ε with centre at the origin in the space $R_n = R$:

$$\sigma_\varepsilon = \{|x| \leqslant \varepsilon\}, \qquad \sigma_1 = \sigma$$

Let $\psi(t)$ be an even nonnegative infinitely differentiable function of one variable $t\,(-\infty < t < \infty)$ equal to zero for $|t| \geqslant 1$ and such that the n-fold multiple integral $\int \psi(|x|)\,dx$ is equal to unity:

$$\int \psi(|x|)\,dx = \int\limits_{|x|\leqslant 1} \psi(|x|)\,dx = 1 \tag{1}$$

where

$$x = (x_1, \ldots, x_n), \quad |x|^2 = \sum_1^n x_j^2, \quad \text{and} \quad \int = \int\limits_R$$

For instance, as ψ we can take the function

$$\psi(t) = \begin{cases} \dfrac{1}{\lambda_n} e^{1/(t^2-1)} & \text{for } 0 \leqslant |t| < 1 \\ 0 & \text{for } |t| \geqslant 1 \end{cases}$$

(see § 16.6, Example 2) where the constant λ_n is chosen so that condition (1) holds.

The function

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi\left(\frac{x}{\varepsilon}\right) \quad \text{where} \quad \varphi(x) = \psi(|x|), \; \varepsilon > 0 \tag{2}$$

is infinitely differentiable on R and its support belongs to σ_ε. For this function there holds the condition

$$\int \varphi_\varepsilon(x)\,dx = \frac{1}{\varepsilon^n}\int \varphi\left(\frac{x}{\varepsilon}\right)dx = \int \varphi(u)\,du = \int \psi(|u|)\,du = 1 \tag{3}$$

Let $\Omega \subset R$ be an open set and let f be a function belonging to the class $L_p(\Omega)$, $1 \leqslant p \leqslant \infty$**. On extending f to the whole space R by putting $f = 0$ on $R-\Omega$ we construct the function

$$f_\varepsilon(x) = f_{\Omega,\varepsilon} = \int \varphi_\varepsilon(x-u) f(u)\,du = \int \varphi_\varepsilon(u) f(x-u)\,du \tag{4}$$

which is called *(Sobolev's) regularization of f.*

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* Academician S. L. Sobolev (born 1908), a distinguished Soviet mathematician.

** The class $L_\infty(\Omega)$ is the set of the functions measurable on Ω and equipped with the norm $\|f\|_\infty = \sup\limits_{x\in\Omega} |f(x)|$ or $\|f\|_\infty = \operatorname{ess\,sup}\limits_{x\in\Omega} |f(x)|$ (for the meaning of "ess sup" see the footnote on page 333, § 18.3).

Since φ is a finite function the computation of the integrals in (3) and (4) reduces to the integration over some n-dimensional balls. For instance, it suffices to extend the integration in the first integral in (4) over the ball $|x-u| \leq \varepsilon$ and in the second integral over the ball $|u| \leq \varepsilon$.

Let us dwell on some properties of f_ε. We shall use the notation

$$\|\cdot\|_p = \|\cdot\|_{L_p(R)}, \qquad 1 \leq p \leq \infty$$

(i) *If* $f \in L_p(R)$, $1 \leq p < \infty$, *then*

$$\|f_\varepsilon - f\|_p \to 0, \qquad \varepsilon \to 0 \tag{5}$$

For, by (3), we have

$$f_\varepsilon(x) - f(x) = \frac{1}{\varepsilon^n} \int \varphi\left(\frac{x-u}{\varepsilon}\right) [f(u) - f(x)]\, du = \int \varphi(v) [f(x-\varepsilon v) - f(x)]\, dv$$

whence, applying generalized Minkowski's inequality and taking into account that $\varphi \geq 0$ and that (3) holds, we derive the relation

$$\begin{aligned} \|f_\varepsilon - f\|_p &\leq \int \varphi(v) \|f(x-\varepsilon v) - f(x)\|_p\, dv \leq \\ &\leq \sup_{|v| \leq \varepsilon} \|f(x-v) - f(x)\|_p \to 0, \qquad \varepsilon \to 0 \end{aligned} \tag{6}$$

In the case $p = \infty$ property (i) does not hold. However, if we suppose that $\Omega = R$ and that $f(x)$ is uniformly continuous on R then (6) can be written in the form

$$\|f_\varepsilon - f\|_\infty \leq \sup_{|v| < \varepsilon} |f(x-v) - f(x)| \to 0, \quad \varepsilon \to 0$$

(ii) *If* $f \in L_p(R)$, $1 \leq p \leq \infty$, *then*

$$\|f_\varepsilon\|_p \leq \int \varphi_\varepsilon(x-u) \|f\|_p\, du = \|f\|_p \tag{7}$$

(iii) *If f is a locally integrable function on R, that is if $f \in L(V)$ for any ball $V \subset R$, then f_ε is an infinitely differentiable function on R, and for any non-negative integral vector* $s = (s_1, \ldots, s_n)$ (i.e., $s_j \geq 0$ are integers) *we have*

$$f_\varepsilon^{(s)}(x) = \int \varphi_\varepsilon^{(s)}(x-u)\, f(u)\, du$$

Proof. If f is a continuous finite function (in R), the last assertion is a direct consequence of the classical theorems on the continuity of an integral with respect to the parameter and on the differentiability under the integral sign. It should be taken into account that $\varphi_\varepsilon(x-u)$ as function of u is finite and infinitely differentiable.

Now let us assume that f is locally integrable. In this case, denoting by e_1 the unit vector of the x_1-axis, we can write (see the explanations below)

$$\left|\frac{f_\varepsilon(x+he_1)-f_\varepsilon(x)}{h}-\int \frac{\partial}{\partial x_1}\varphi_\varepsilon(x-u)f(u)\,du\right| =$$

$$= \left|\int \left\{\frac{1}{h}[\varphi_\varepsilon(x+he_1-u)-\varphi_\varepsilon(x-u)]-\frac{\partial}{\partial x_1}\varphi_\varepsilon(x-u)\right\}f(u)\,du\right| =$$

$$= \left|\int_A \left[\frac{\partial}{\partial x_1}\varphi_\varepsilon(x+\theta he_1-u)-\varphi_\varepsilon(x-u)\right]f(u)\,du\right| \leqslant$$

$$\leqslant \omega(|h|)\int_A |f(u)|\,du \to 0 \qquad (h \to 0) \tag{8}$$

where $0 < \theta < 1$, which proves that

$$\frac{\partial}{\partial x_1}f_\varepsilon(x) = \int \frac{\partial}{\partial x_1}\varphi_\varepsilon(x-u)f(u)\,du$$

In the second equality (8) we have used the mean value theorem (for the variable x_1). Here it is allowable to assume that $|h|$ does not exceed a given number δ and that A is a closed ball in the space R of the points u with a sufficiently large radius so that the function $\varphi_\varepsilon(x+he_1-u)$ vanishes outside A for all h with $|h| \leqslant \delta$. The existence of such a ball follows from the fact that the function $\varphi_\varepsilon(u)$ has a compact support. This accounts for the fact that the integral in the third member of (8) is extended over the set A.

Now let V be a closed ball in R with centre at x (the points of the ball will be denoted x'). Then the function $\varphi_\varepsilon(x'-u)$ is continuously differentiable on the bounded and closed set $V\times A$ of points (x', u). The modulus of continuity $\omega(t)$ of the partial derivative of the function $\varphi_\varepsilon(x'-u)$ with respect to x_1 on the set $V\times A$ satisfies the condition $\omega(t) \to 0$ $(t \to 0)$, whence follows the last relation (8).

(iv) *If $f \in L_p(\Omega)$ $(1 \leqslant p < \infty)$ then there is a sequence of finite functions ψ_k infinitely differentiable in Ω for which*

$$\|f-\psi_k\|_p \to 0 \quad (k \to \infty) \quad \text{and} \quad |\psi_k(x)| \leqslant \sup_{x\in\Omega}|f(x)|$$

Proof. Let us choose an arbitrary $\eta > 0$ and choose a bounded open set $g \subset \bar{g} \subset \Omega$ such that

$$\|f\|_{L_p(\Omega-g)} < \frac{\eta}{2}$$

Let d be the distance from g to the boundary of Ω ($d > 0$; if $\Omega = R_n$ then $d = \infty$). We shall need the function

$$f_g(x) = \begin{cases} f(x) & \text{for } x \in g \\ 0 & \text{for } x \notin g \end{cases}$$

and its regularization $f_{g,\varepsilon} = \psi$. For $\varepsilon < d$ the regularization ψ is an infinitely differentiable function finite in Ω and satisfying the condition

$$\|f-f_{g,\varepsilon}\|_{L_p(\Omega)} \leqslant \|f-f_g\|_{L_p(\Omega)}+\|f_g-f_{g,\varepsilon}\|_{L_p(\Omega)} =$$
$$= \|f\|_{L_p(\Omega-g)}+\|f_g-f_{g,\varepsilon}\|_{L_p(\Omega)} < \frac{\eta}{2}+\frac{\eta}{2} = \eta$$

provided that ε is sufficiently small.

Further (see (7) in which we put $p = \infty$ and $f(x) = 0$ for $x \notin \Omega$), we have

$$|f_{g,\varepsilon}(x)| \leqslant \sup_{x\in R} |f_g(x)| \leqslant \sup_{x\in R} |f(x)|$$

Now, taking $\eta = \eta_k \to 0$ and putting $\varepsilon = \varepsilon_k$ and $g = g_k$ we see that the functions $\psi_k = f_{g_k,\varepsilon_k}$ satisfy the required conditions.

The assertions of property (iv) we have proved here strengthen Theorems 3 and 4 proved in § 14.4. The first of these assertions means that *the set of infinitely differentiable finite functions on Ω is dense in $L_p(\Omega)$* $(1 \leqslant p < \infty)$.

(v) *A function $f(x)$ defined and nondecreasing on a closed interval $[a, b]$ can be approximated with an arbitrary accuracy relative to the metric of $L_p(a, b)$ by a nondecreasing infinitely differentiable function (generally speaking, not finite).*

Indeed, let us extend $f(x)$ to the entire real axis by putting

$$f(x) = \begin{cases} 0 & \text{for } x < a-\varepsilon,\ b+\varepsilon < x \\ f(a) & \text{for } a-\varepsilon \leqslant x < a \\ f(b) & \text{for } b < x \leqslant b+\varepsilon \end{cases}$$

Then the regularization $f_\varepsilon(x)$ of the extended function $f(x)$ is a nondecreasing function on the interval $[a, b]$. For, taking into account that $\varphi_\varepsilon(u)$ is even and nonnegative and considering the values of x satisfying the inequalities $a \leqslant x < x_1 \leqslant b$ we can write

$$f_\varepsilon(x) = \int\limits_{x-\varepsilon}^{x+\varepsilon} \varphi_\varepsilon(u-x) f(u)\, du = \int\limits_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) f(x+t)\, dt \leqslant$$
$$\leqslant \int\limits_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) f(x_1+t)\, dt = f_\varepsilon(x_1)$$

because for $t \in (-\varepsilon, \varepsilon)$ there holds the inequality $f(x+t) \leqslant f(x_1+t)$. The infinite differentiability of $f_\varepsilon(x)$ was proved in (iii). Further, taking into account (1), we obtain for $n = 1$ the relations

$$\|f(x)-f_\varepsilon(x)\|_{L_p(a,b} \leqslant \int\limits_{-\varepsilon}^{\varepsilon} \varphi_\varepsilon(t) \|f(x)-f(x+t)\|_{L_p(a,b)}\, dt \leqslant$$
$$\leqslant \sup_{|t|\leqslant\varepsilon} \|f(x)-f(x+t)\|_{L_p(-\infty,\infty)} \to 0, \qquad \varepsilon \to 0$$

which shows that f_ε is an approximating function for f possessing property (v).

§ 18.3. Convolution

In this section the integration will be understood in the Lebesgue sense and we shall be interested in functions belonging to the space $L_p = L_p(R_n)$ $(1 \leqslant p \leqslant \infty)$. For a finite p, a function f belongs to L_p if it is Lebesgue measurable and possesses the finite norm

$$||f||_{L_p} = \left(\int\limits_{R_n} |f(x)|^p\, dx\right)^{1/p}$$

$\left(\text{where } \int = \int\limits_{R_n}\right)$.

The space L_∞ is defined as the class of the functions f bounded and measurable on R_n equipped with the norm[*]

$$||f||L_\infty = \sup_{x \in R} |f(x)|$$

For any function $K(x) = K(x_1, \ldots, x_n) \in L = L_1$ makes sense the *convolution* (or *faltung*) defined with the aid of the Lebesgue integral (see § 16.8 (16)):

$$K{*}f = \frac{1}{(2\pi)^{n/2}} \int K(u) f(x-u)\, du = \frac{1}{(2\pi)^{n/2}} \int K(x-u) f(u)\, du \tag{1}$$

Integral (1) exists for almost all $x \in R_n$ and is a function of x belonging to L_p; this function satisfies generalized Minkowski's inequality (see §18.1 (6)):

$$||K{*}f||_{L_p} \leqslant \frac{1}{(2\pi)^{n/2}} \int |K(u)|\, ||f(x-u)||_{L_p}\, du = \frac{1}{(2\pi)^{n/2}} ||K||_L\, ||f||_{L_p} \tag{2}$$

where the norm $||f(x-u)||_{L_p} = ||f(x)||_{L_p}$ is taken with respect to x. For $p = \infty$ inequality (2) is quite obvious.

The function $K{*}f \in L_p$, in its turn, can be subjected to the operation of forming its convolution with a function $K_1 \in L_p$, and there holds the commutativity property

$$K_1{*}K{*}f = K{*}K_1{*}f \qquad (K, K_1 \in L,\ f \in L_p) \tag{3}$$

(which follows from Fubini's theorem proved in the theory of Lebesgue's integral).

We have defined the convolution of two ordinary measurable functions $K \in L$ and $f \in L_p$ as an *ordinary function* $K{*}f$ belonging to L_p and computed by formula (1) involving Lebesgue's integral.

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[*] We note that L_∞ more often denotes the totality of the so-called *essentially bounded* functions f measurable on R_n and possessing the (finite) norm

$$||f||_\infty = \operatorname*{ess\,sup}_{x \in R} |f(x)|$$

where $\operatorname*{ess\,sup}_{x \in R} |f(x)|$ (also denoted as $\operatorname*{vrai\,sup}_{x \in R} |f(x)|$ or $\operatorname*{vrai\,max}_{x \in R} |f(x)|$) is the so-called *essential supremum of* f defined as the smallest number M for which the set $\{x: M < f(x)\}$ is of Lebesgue measure zero.

However, K, f and $K*f$ can also be regarded as generalized functions (belonging to the space S'). Therefore the generalized functions $\widetilde{K*f}$ and $\widehat{K*f}$ belonging to S' (they must not necessarily be ordinary functions) make sense. This allows us to put, by definition,

$$\tilde{K}\tilde{f} = \widetilde{K*f} \quad \text{and} \quad \hat{K}\hat{f} = \widehat{K*f} \tag{4}$$

It should be stressed that $\tilde{K}$ and $\hat{K}$ are continuous functions (since $K \in L$).

The first equality (4) defines the product of $\tilde{K}$ by the generalized function $\tilde{f}$ on condition that $f \in L_p$.

Definitions (4) automatically imply the equalities

$$K*f = \widehat{\tilde{K}\tilde{f}} = \widetilde{\hat{K}\hat{f}} \qquad (K \in L, f \in L_p)$$

(which generalize relations (16) in § 16.8 where it was assumed that $f = \varphi \in S \in L_p$, $1 \leqslant p \leqslant \infty$).

It should be noted that if $\tilde{\mu} = K \in L$ and at the same time μ is an infinitely differentiable function of polynomial growth, then the product $\mu\tilde{f}$ $(f \in L_p)$ can be defined in two different ways, namely, on the one hand, it is the functional

$$(\mu\tilde{f}, \varphi) = (\tilde{f}, \mu\varphi) \qquad (\varphi \in S)$$

and, on the other hand, as was defined in (4), it is the functional $\mu\tilde{f} = \widetilde{\hat{\mu}*f}$. Let us show that these functionals are in fact equal. If $f \in S$, the relation

$$(\tilde{f}, \mu\varphi) = \left(\widetilde{\hat{\mu}*f}, \varphi\right) \qquad (\varphi \in S) \tag{5}$$

expresses the equality of ordinary integrals of ordinary functions which can be proved by analogy with § 16.8 (16). In case $f \in L_p$ is an arbitrary function there is a sequence of finite functions $f_l \in S$ convergent to f in the sense of L_p and moreover in the sense (S') (see § 18.2, property (iv). For each $l = 1, 2, \ldots$ we have $(\tilde{f}_l, \mu\varphi) = \left(\widetilde{\hat{\mu}*f_l}, \varphi\right)$ $(\varphi \in S)$, whence, on passing to the limit as $l \to \infty$, we obtain (5). For, $f_l \to f$ (L_p) implies $\hat{\mu}*f_l \to \hat{\mu}*f$ (L_p) and moreover $\hat{\mu}*f_l \to \hat{\mu}*f(S')$, and therefore $\widetilde{\hat{\mu}*f_l} \to \widetilde{\hat{\mu}*f}$ (S').

Let us consider the convolution of the generalized function $\mathrm{P}.\frac{1}{x} \in S'$ and a function $f \in L_p$ $(1 < p < \infty$; see Example 2 in § 16.7). It is defined as the limit

$$F(x) = \mathrm{P}.\frac{1}{t}*f = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \frac{1}{\sqrt{2\pi}} \int\limits_{\varepsilon < |t| < 1/\varepsilon} \frac{f(x-t)}{t}\,dt = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} \frac{f(x-t)}{t}\,dt$$

in the sense of the space $L_p = L_p(-\infty, \infty)$. Thus,

$$\lim_{\varepsilon \to 0} \left\| F(x) - \left(\mathrm{P}_\varepsilon.\frac{1}{t}*f\right) \right\|_{L_p} = \lim \left\| (\mathrm{P}.-\mathrm{P}_\varepsilon.)\frac{1}{t}*f \right\|_{L_p} = 0$$

where

$$P_{\varepsilon}.\frac{1}{t} = \begin{cases} \frac{1}{t} & \text{for } \varepsilon < |t| < \frac{1}{\varepsilon} \\ 0 & \text{for the other values of } t \end{cases}$$

It can be proved that this limit exists for any function $f \in L_p$, whence it follows that $F \in L_p$. Moreover, there exists a constant C_p dependent on p $(1 < p < \infty)$ and independent of f such that there holds the inequality

$$\|F\|_{L_p} \leq C_p \|f\|_{L_p}$$

It is important to note the commutativity property

$$K*P.\frac{1}{t}*f = P.\frac{1}{t}*K*f \qquad (K \in L,\ f \in L_p,\ 1 < p < \infty) \tag{6}$$

Relation (6) follows from the equality

$$K*P_{\varepsilon}.\frac{1}{t}*f = P_{\varepsilon}.\frac{1}{t}*K*f \qquad (\varepsilon > 0)$$

$\left(P_{\varepsilon}.\frac{1}{t} \in L\right)$ if we pass in it to the limit in the sense of L_p. Indeed, we have

$$\left\|\left(K*P.\frac{1}{t}*f\right) - \left(K*P_{\varepsilon}.\frac{1}{t}*f\right)\right\|_{L_p} = \left\|K*(P.-P_{\varepsilon}.)\frac{1}{t}*f\right\|_{L_p} \leq$$
$$= C_p\left\|(P.-P_{\varepsilon}.)\frac{1}{t}*f\right\|_{L_p} \to 0$$

and

$$\left\|\left(P.\frac{1}{t}*K*f\right) - \left(P_{\varepsilon}.\frac{1}{t}*K*f\right)\right\|_{L_p} = \left\|(P.-P_{\varepsilon}.)\frac{1}{t}*(K*f)\right\|_{L_p} \to 0 \qquad (\varepsilon \to 0)$$

§ 18.4. Partition of Unity

Lemma 1. *For any two bounded open sets g and g' such that $g \subset \bar{g} \subset g'$ there is a function $\varphi(x)$ infinitely differentiable on g', equal to* 1 *on g and satisfying the inequalities $0 \leq \varphi(x) \leq 1$.*

Proof. Let us denote by g^{δ} the set of all points x whose distances from $\bar{g}$ do not exceed δ $\left(r(x,\bar{g}) \leq \delta\right)$. This is obviously a closed set, and $g \subset g^{\delta} \subset g^{2\delta} \subset$ $\subset g'$ provided that δ is sufficiently small. Let us put

$$\psi(x) = \begin{cases} 1 & \text{for } x \in g^{\delta} \\ 0 & \text{for } x \notin g^{\delta} \end{cases} \tag{1}$$

and

$$\varphi(x) = \psi_{\delta}(x) = \int \varphi_{\delta}(y-x)\,\psi(y)\,dy \tag{2}$$

The function $\varphi = \psi_{\delta}$ is the regularization of the function ψ.

As we know, the function $\varphi(x)$ is infinitely differentiable. Besides, by virtue of (1) and since integral (2) is in fact taken over the ball of radius δ

with centre at x, we have

$$\varphi(x) = \begin{cases} 1 & \text{on } g \\ 0 & \text{outside } g^{2\delta} \end{cases}$$

It should be noted that integral (2) always exists in Lebesgue's sense while in Riemann's sense it may not exist since the set g^{δ} and its intersection with the ball are not necessarily Jordan measurable.

Taking into account that $0 \leqslant \psi(x) \leqslant 1$ and that φ_{δ} is nonnegative we finally arrive at the inequalities

$$0 \leqslant \varphi(x) \leqslant \int \varphi_{\delta}(y-x)\, dy = 1$$

To prove the lemma in terms of Riemann's integral we make use of a network cutting R_n into equal cubes of diameter δ. Let Λ be the set of those cubes each of which contains at least one point $x \in g$. For a sufficiently small δ we have $g \subset \Lambda \subset g'$. The further course of the proof is analogous to the above (g should be everywhere replaced by Λ). The set Λ^{δ} is obviously Jordan measurable.

Lemma 2 (on the partition of unity). *Let a bounded closed set $F \subset R_n$ be covered by open sets $g_1, \ldots, g_N$.*

Then there is a system of infinitely differentiable functions $\varphi_1(x), \ldots, \varphi_N(x)$ possessing the following properties:

(i) $0 \leqslant \varphi_j(x) \leqslant 1$,

(ii) φ_j *is finite in* g_j,

(iii) $\sum\limits_{1}^{N} \varphi_j(x) = 1$ *on* F.

Proof. Let us construct open sets $g_1', \ldots, g_N'$ such that

$$g_j' \subset \bar{g}_j' \subset g_j, \qquad \sum_{1}^{N} g_j = G' \supset F$$

and, using the foregoing lemma, determine nonnegative infinitely differentiable functions $\lambda_j(x)$ finite in g_j and equal to 1 on g_j'. Next we put

$$\psi_j(x) = \frac{\lambda_j(x)}{\lambda_1(x) + \ldots + \lambda_N(x)}$$

The functions ψ_j are obviously infinitely differentiable on G' and satisfy on G' the conditions $0 \leqslant \psi_j(x) \leqslant 1$ and $\sum\limits_{1}^{N} \psi_j(x) = 1$. However, they are not defined at the points x where all λ_j's turn into zero. Since F is a bounded closed set and $G' \supset F$ is an open set, we can construct (see the remark below) a second open set G'' such that $G' \supset \bar{G}'' \supset G'' \supset F$ and an infinitely differentiable function $\varkappa(x)$ finite in G', equal to 1 on G'' and satisfying the inequalities $0 \leqslant \varkappa(x) \leqslant 1$. Now it can readily be shown that the functions

$$\varphi_j(x) = \varkappa(x)\, \psi_j(x) \qquad (j = 1, \ldots, N)$$

satisfy all the assertions of the lemma if we agree that $\varphi_j(x) = 0$ at the points where $\varkappa(x) = 0$ even when $\psi_j(x)$ is not defined. Properties (i), (ii) and (iii) can easily be checked. If $x \in \bar{G}'$ then $x \in \bar{g}'_{j_0}$ for some j_0 and $\lambda_{j_0}(x) = 1$, and therefore in some neighbourhood of x the function ψ_j and, together with it, the function φ_j are infinitely differentiable. In case $x \notin \bar{G}'$, there is a neighbourhood of x where $\varkappa$ is identically equal to zero, and consequently φ_j is infinitely differentiable in that neighbourhood.

Remark. The existence of the set G'' mentioned above can be established as follows.

Let us put

$$G'_n = \bigcup_{j=1}^{m} g'_{j,n} \qquad (n = 1, 2, \ldots)$$

where $g'_{j,n}$ denotes the totality of all those points of the set g'_j whose distances from the boundary of that set are greater than $1/n$. It is evident that $g'_{j,n}$ and G'_n are open sets and that

$$G'_1 \subset \bar{G}'_1 \subset G'_2 \subset \bar{G}'_2 \subset \ldots$$

and

$$\bigcup_{1}^{\infty} G'_n = G' \supset F$$

Consequently, by Lemma 2, § 12.22, we have $G'_{n_0} \supset F$ for some $n = n_0$. Therefore, on putting $G'_{n_0} = G''$, we obtain $F \subset G'' \subset \bar{G}'' \subset G'$.

CHAPTER 19

Lebesgue Integral

§ 19.1. Lebesgue Measure

In this section we shall introduce the notion of the Lebesgue (n-dimensional) measure for bounded sets forming the class of Lebesgue measurable sets. It generalizes the notion of the Jordan measure since it turns out that every Jordan measurable set is also Lebesgue measurable, the corresponding measures being equal.

In §§ 12.2 and 12.5 we defined the (Jordan) exterior (outer) measure (content) m_eE, interior (inner) measure m_iE and the Jordan measure $mE = |E|$ of a set E. In particular, it was proved that the Jordan inner and outer measures of an arbitrary bounded set exist and are invariant (with respect to any choice of the system of rectangular coordinates).* For the Lebesgue exterior (outer) measure, interior (inner) measure and the Lebesgue measure of a set E we shall use the symbols μ_eE, μ_iE, and $\mu E = |E|$.

The symbol $|E|$ for the notation of the measure will be used in those cases when it is already known that the set E in question is such that its Lebesgue and Jordan measures, provided both of them exist, are equal.

In this section we shall only deal with bounded sets belonging to the n-dimensional space R_n, and therefore when speaking of a set we shall mean a set of the space R_n which is bounded. In some cases when the set in question is unbounded or can a priori be unbounded we shall always make the necessary stipulations.

As in the case of the Jordan measure, to any (bounded) set E (belonging to R_n) are attributed two numbers μ_iE and μ_eE which are *the Lebesgue inner* and *outer measures of* E respectively. In the case when these numbers coincide their common value $\mu E = \mu_iE = \mu_eE$ is called *the Lebesgue measure of* E, and the set E is said to be *Lebesgue measurable*. We shall begin with the definition of the Lebesgue measure for the special classes of open and closed sets without resorting to the definition of their outer and inner measures.

By the symbols $G, G_1, G', G_2, \dots$ we shall only denote open sets while the symbols $F, F_1, F', F_2, \dots$ will be used for closed sets exclusively. This convention allows us to avoid sometimes the stipulations that the sets under consideration denoted by these symbols are open or closed. We shall also agree that the symbols Δ and σ will be used, respectively, for cubes and for figures formed of cubes (or, generally, rectangles) with edges parallel to the coordinate axes (see § 12.2).

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* In the present section we only need these very properties of Jordan measure.

The Lebesgue measure of a (bounded) open set G is defined by the relation

$$\mu G = m_i G = \sup_{\sigma \subset G} |\sigma|$$

where the supremum is extended over the volumes of the figures σ belonging to G.

An open set G can be represented as a countable union of closed cubes with no interior points in common (i.e. any two of them can only intersect along some parts of their boundaries; see § 12.5. Theorem 2):

$$G = \sum_1^{\infty} \Delta_k = \sum \Delta_k \tag{1}$$

It follows that

$$\mu G = m_i G = \sum |\Delta_k| \tag{2}$$

where $|\Delta_k|$ is the volume of Δ_k. Lemma 2 proved below implies that the measure of G in (2) is independent of the way in which G is represented as sum (1) (where Δ_q and Δ_p with $q \neq p$ have no interior points in common; $p, q = 1, 2, \ldots$).

The Lebesgue measure of a (bounded) closed set F is defined by the relation

$$\mu F = m_e F = \inf_{\sigma \supset F} |\sigma|$$

where the infimum is extended over the volumes of the figures σ containing F

Both numbers μG and μF are invariant with respect to any choice of the rectangular coordinates because the numbers $m_i G$ and $m_e F$ possess this property (see what was said in § 12.2 after formula (6)), whence, as will be shown, follows the invariance of $\mu_i E$ and $\mu_e E$ for an arbitrary set E, which implies the invariance of μE provided that E is Lebesgue measurable.

Let us prove a number of lemmas establishing some properties of measures of closed and open sets.

Lemma 1. *Let $\sigma_k (k = 1, 2, \ldots)$ be closed figures with no interior points in common (any two of them can only intersect along some parts of their boundaries) and let $\sigma'_k (k = 1, 2, \ldots)$ be closed figures such that*

$$\sum \sigma_k \subset \sum \sigma'_k \tag{3}$$

Then

$$\sum |\sigma_k| \leq \sum |\sigma'_k| \tag{4}$$

where the inequality becomes an equality if and only if the figures σ'_k have no interior points in common.

Proof. Let us assume that the right-hand member of inequality (4) is finite because, if otherwise, the inequality becomes trivial. Let us choose an arbitrary $\varepsilon > 0$ and consider open figures $\sigma_k'' \supset \sigma_k'$ such that

$$\sum |\sigma_k''| < \sum |\sigma_k'| + \varepsilon$$

The bounded closed set $\sum\limits_{1}^{N} \sigma_k$ is covered, for any N, by open figures σ_k'', and therefore it is possible to select a finite number of these figures, say $\sigma_{k_1}'', \ldots, \sigma_{k_s}''$, which also cover this set; it follows that

$$\sum_{k=1}^{N} |\sigma_k| = \left|\sum_{1}^{N} \sigma_k\right| \leq \sum_{j=1}^{s} |\sigma_{k_j}''| < \sum |\sigma_k'| + \varepsilon \tag{5}$$

where in the first relation (equality) we have taken into account that any two figures σ_k have no interior points in common. The right-hand side of (5) being independent of N, we obtain

$$\sum_{1}^{\infty} |\sigma_k| \leq \sum_{1}^{\infty} |\sigma_k'| + \varepsilon$$

whence, by the arbitrariness of $\varepsilon > 0$, follows (4).

If any two of the figures σ_k' have no interior points in common and $\sum \sigma_k = \sum \sigma_k'$ we can interchange the roles of σ_k and σ_k' in this argument to conclude that in this case (4) is in fact an exact equality. In case the intersection of some figures σ_k' and σ_l' is a nondegenerate rectangle we can replace σ_l' $(k < l)$ by the figure $\sigma_l' - \sigma_k'\sigma_l'$ in this argument to conclude that in this case relation (4) is in fact a strict inequality.

When saying that there is a (bounded) open set

$$G = \sum_k \sigma_k \quad \text{with} \quad \mu G = \sum_k |\sigma_k|$$

where σ_k are closed figures (most often, $\sigma_k = \Delta_k$ are cubes), we shall automatically mean (see Lemma 1) that the figures σ_k can only intersect along some parts of their boundaries. The possibility of such a representation of G was proved in § 12.5 (Theorem 2). All the other representations $G = \sum \sigma_k'$ where the figures σ_k' also have no interior points in common lead, in accordance with Lemma 1, to the equality $\mu G = \sum\limits_k |\sigma_k'|$.

Lemma 2. *Let G^j; $j = 1, 2, \ldots$, be a finite or countable system of open sets belonging to a cube Δ. Then the sum $G = \sum\limits_j G^j$ is an open set whose measure satisfies the inequality*

$$\mu G \leq \sum_j \mu G^j \tag{6}$$

which becomes an equality if and only if the sets G^j are pairwise disjoint.

Proof. The fact that the set G is open is quite obvious. Let us consider two representations of G of the form

$$G=\sum_k \Delta_k,\quad \mu G=\sum_k |\Delta_k| \quad \text{and} \quad G^j=\sum_k \Delta_k^j,$$
$$\mu G^j=\sum_k |\Delta_k^j|$$

where Δ_k and Δ_k^j are closed cubes. We have $\sum \Delta_k=\sum_j\sum_k \Delta_k^j$ and, by Lemma 1,

$$\mu G=\sum |\Delta_k| \leqslant \sum_j\sum_k |\Delta_k^j| = \sum_j \mu G^j \tag{7}$$

If G^j are pairwise disjoint, any two of the sets Δ_k^j can only intersect along some parts of their boundaries, and, by virtue of Lemma 1, there holds the equality

$$\sum |\Delta_k| = \sum_j\sum_k |\Delta_k^j|$$

whence it follows that in this case (6) is in fact an equality. If the intersection G^sG^j is nonempty for some s, j $(s \neq j)$ then there are cubes Δ_μ^s and Δ_ν^j with common interior points, and, by Lemma 1, (7) is a strict inequality (with the sign "$<$" instead of "$\leqslant$") and therefore so is (6).

Lemma 3. *If F_1 and F_2 are two nonintersecting closed sets then*

$$\mu(F_1+F_2) = \mu F_1+\mu F_2 \tag{8}$$

Proof. Since the sets F_1 and F_2 are bounded, closed and disjoint the distance between them (see § 17.10, Exercise 5) is positive:

$$\varrho = \varrho(F_1, F_2) > 0$$

Let us cover each point $x \in F_1$ by a (closed or open) cube Δ_x with centre at x and diameter less than $\varrho/2$ and then select a finite system of these cubes also covering F_1. The union of the cubes of this system is a figure σ_1' covering F_1. In a similar way we construct a figure σ_2' covering F_2. The figures σ_1' and σ_2' are obviously disjoint. Next we choose an arbitrary $\varepsilon > 0$ and construct two figures σ_1'' and σ_2'' such that

$$F_1 \subset \sigma_1'',\quad F_2 \subset \sigma_2'',\quad |\sigma_1''| \leqslant |\mu F_1|+\varepsilon \quad \text{and} \quad |\sigma_2''| \leqslant |\mu F_2|+\varepsilon$$

For the figures $\sigma_1 = \sigma_1'\sigma_1''$ and $\sigma_2 = \sigma_2'\sigma_2''$ there obviously hold the same relations

$$F_1 \subset \sigma_1,\quad F_2 \subset \sigma_2,\quad |\sigma_1| \leqslant \mu F_1+\varepsilon \quad \text{and} \quad |\sigma_2| \leqslant \mu F_2+\varepsilon$$

and, besides, they are nonintersecting.

Since $\sigma_1+\sigma_2$ is a figure containing the closed set F_1+F_2, we have

$$\mu(F_1+F_2) \leqslant |\sigma_1+\sigma_2| = |\sigma_1|+|\sigma_2| \leqslant \mu F_1+\mu F_2+2\varepsilon \tag{9}$$

Now let us choose a figure $\sigma \supset F_1+F_2$ such that $\mu(F_1+F_2)+\varepsilon > |\sigma|$. It follows that

$$\mu(F_1+F_2)+\varepsilon > |\sigma| \geqslant |\sigma(\sigma_1+\sigma_2)| = |\sigma\sigma_1|+|\sigma\sigma_2| \geqslant \mu F_1+\mu F_2 \tag{10}$$

Here we have used the fact that the figures $\sigma\sigma_1$ and $\sigma\sigma_2$ are disjoint and contain the sets F_1 and F_2 respectively. Finally, taking into account that $\varepsilon > 0$ is arbitrary, we conclude that (9) and (10) imply (8).

Lemma 4. *If F is a bounded and closed nonempty set, G is an open set (not necessarily bounded) and $F \subset G$, then there is a figure σ (closed or open) such that*

$$F \subset \sigma \subset G, \quad \mu F < |\sigma| < \mu G \tag{11}$$

Proof. Let us cover every point $x \in F$ by cubes $\Delta_x^{(1)} \subset \Delta_x^{(2)} \subset \Delta_x^{(3)} \subset G$ with parallel faces and centres at x, the lengths $\delta_x^{(1)}$, $\delta_x^{(2)}$ and $\delta_x^{(3)}$ of whose edges satisfy the relation $\delta_x^{(1)} < \delta_x^{(2)} < \delta_x^{(3)}$. Let the cubes $\Delta_x^{(1)}$ be open and $\Delta_x^{(2)}$ be closed. By Heine-Borel lemma, there is a finite number of cubes $\Delta_x^{(1)}$, say $\Delta_{x,1}^{(1)}, \ldots, \Delta_{x,N}^{(1)}$, covering F. Then the (closed) figure

$$\sigma = \sum_{k=1}^{N} \Delta_{x,k}^{(2)} \tag{12}$$

obviously satisfies conditions (11) and, together with it, so is the open figure obtained from σ by deleting its boundary. It should be taken into account that the figure $\sum_1^N \Delta_{x,k}^{(1)} \supset F$ is strictly inside σ and the latter is in its turn strictly inside the figure $\sum_1^N \Delta_{x,k}^{(3)} \subset G$.

Lemma 5. *For a closed set F belonging to a bounded open set G there holds the relation*

$$\mu(G-F) = \mu G-\mu F \tag{13}$$

Proof. Let us represent the open sets G and $G' = G-F$ as sums of the form

$$G = \sum_k \Delta_k \quad \text{and} \quad G' = \sum \Delta_k' \qquad (\mu G = \sum |\Delta_k|, \ \mu G' = \sum |\Delta_k'|)$$

consisting of closed cubes. Let $\sigma \subset G$ be an arbitrary figure covering F. Then $G = \sum \Delta_k \subset \sigma+\sum \Delta_k'$, and, by virtue of Lemma 1,

$$\mu G = \sum |\Delta_k| \leqslant |\sigma|+\sum |\Delta_k'| = |\sigma|+\mu G'$$

Whence, taking the infimum of the right-hand member over all $\sigma \supset F$ we obtain

$$\mu G \leqslant \mu F+\mu G'$$

On the other hand, the sets F and $\sum_1^N \Delta_k'$ are closed and disjoint, and therefore. by Lemmas 1 and 4,

$$\mu F+\sum_{1}^{N}|\Delta_k'| = \mu F+\mu\sum_{1}^{N}\Delta_k' = \mu\left(F+\sum_{1}^{N}\Delta_k'\right) \leqslant \mu G$$

whence

$$\mu F+\mu G' = \mu F+\sum_{1}^{\infty}|\Delta_k'| \leqslant \mu G$$

which proves (13).

Now let us consider an arbitrary (bounded) set E.

The Lebesgue inner (interior) measure of E is defined by the equality $\mu_i E = \sup\limits_{F\in E} \mu F$ where the supremum of the Lebesgue measures μF is taken over all the closed sets F belonging to E. The existence of the supremum follows from the possibility of embedding E in a closed cube Δ, that is $F \subset E \subset \Delta$, whence

$$\mu F = m_e F \leqslant |\Delta|$$

The Lebesgue outer (exterior) measure of E is defined by the equality $\mu_e E = \inf\limits_{G\subset E} \mu G$ where the infimum of the Lebesgue measures μG is taken over all the open sets G containing E. The existence of $\mu_e E \geqslant 0$ is obvious since $\mu G \geqslant 0$.

Finally, a set E is said to be *Lebesgue measurable* if its inner and outer (Lebesgue) measures coincide; in this case the common value $\mu E = \mu_i E = \mu_e E$ of the Lebesgue outer and inner measures of E is called *the Lebesgue measure of E.*

Let us prove the important inequalities

$$m_i E \leqslant \mu_i E \leqslant \mu_e E \leqslant m_e E \tag{14}$$

To this end we make use of the fact that the Jordan exterior measure $m_e E$ can be regarded as the infimum of the volumes of the open figures $\sigma \supset E$. The result is the same irrespective of whether the Jordan exterior measure $m_e E$ is evaluated with the aid of open or closed figures $\sigma \supset E$. Since the open figures $\sigma \supset E$ are a special case of the open sets $G \supset E$ we see that $\mu_e E \leqslant m_e E$. As to the Jordan interior measure $m_i E$, we can regard it as the supremum of the volumes of the closed figures $\sigma \subset E$; since such figures σ are a special case of the closed sets $F \subset E$, we see that $m_i E \leqslant \mu_i E$.

It follows from (14) that *if E is Jordan measurable it is sure to be Lebesgue measurable, and $mE = \mu E$.*

Now it can easily be seen that the sets F and G are Lebesgue measurable.* This follows from (14) and from the equalities**

$$m_i G = \mu G = \mu_e G, \quad \mu_i F = \mu F = m_e F$$

---

* Note that a bounded closed set F and a bounded open set G are not necessarily Jordan measurable.

** Because we have, for instance $\mu G_0 = m_i G_0$ and $\mu_e G_0 = \inf\limits_{G\supset G_0} \mu G$.

In § 12.3 we considered some important examples of sets of Jordan measure zero which, consequently, are also of Lebesgue measure zero.

Theorem 1. *A set E is (Lebesgue) measurable if and only if, given any $\varepsilon > 0$, there are sets F and G such that*

$$F \subset E \subset G \quad \text{and} \quad \mu G - \mu F < \varepsilon \tag{15}$$

Proof. The assumption that such sets F and G exist implies that

$$\mu F \leqslant \mu_i E \leqslant \mu_e E \leqslant \mu G$$

whence $\varepsilon > \mu_e E - m_i E$, and, since $\varepsilon > 0$ is arbitrary, $\mu_i E = \mu_e E$. Conversely, if E is measurable then for any $\varepsilon > 0$ there are sets F and G such that

$$F \subset E \subset G \quad \text{and} \quad \mu E - \frac{\varepsilon}{2} < \mu F \leqslant \mu E \leqslant \mu G < \mu E + \frac{\varepsilon}{2}$$

whence follows (15).

Theorem 2. *The sum, difference and intersection of measurable sets E_1 and E_2 are also Lebesgue measurable.*

Proof. Let us take an arbitrary $\varepsilon > 0$ and choose some sets F_1, F_2, G_1 and G_2 so that the relations (see Theorem 1 and Lemma 5)

$$F_1 \subset E_1 \subset G_1, \quad \mu G_1 - \mu F_1 = \mu(G_1 - F_1) < \varepsilon$$

and

$$F_2 \subset E_2 \subset G_2, \quad \mu G_2 - \mu F_2 = \mu(G_2 - F_2) < \varepsilon$$

hold.

It follows that

$$\left.\begin{array}{c} F_1 + F_2 \subset E_1 + E_2 \subset G_1 + G_2 \\ F_1 - G_2 \subset E_1 - E_2 \subset G_1 - F_2, \quad F_1 F_2 \subset E_1 E_2 \subset G_1 G_2 \end{array}\right\} \tag{16}$$

The assertions of the theorem now follow from the calculations (see the explanations below)

$$\mu(G_1 + G_2) - \mu(F_1 + F_2) = \mu\big((G_1 + G_2) - (F_1 + F_2)\big) \leqslant$$
$$\leqslant \mu\big((G_1 - F_1) + (G_2 - F_2)\big) \leqslant \mu(G_1 - F_1) + \mu(G_2 - F_2) < 2\varepsilon$$
$$\mu(G_1 - F_2) - \mu(F_1 - G_2) = \mu\big((G_1 - F_2) - (F_1 - G_2)\big) \leqslant$$
$$\leqslant \mu\big((G_1 - F_1) + (G_2 - F_2)\big) \leqslant \mu(G_1 - F_1) + \mu(G_2 - F_2) < 2\varepsilon$$

and

$$\mu G_1 G_2 - \mu F_1 F_2 = \mu(G_1 G_2 - F_1 F_2) \leqslant \mu\big(G_1(G_2 - F_2)\big) + \mu\big(F_2(G_1 - F_1)\big) \leqslant$$
$$\leqslant \mu(G_2 - F_2) + \mu(G_1 - E_1) < 2\varepsilon$$

The first relations (equalities) in these calculations hold by virtue of Lemma 5 because the right-hand members of (16) are open sets while the left-hand members are closed sets. The second relations (inequalities) are implied by Lemma 2 and by the following set-theoretic inclusion relations:

$$\left.\begin{array}{ll}\text{(a)} & G_1+G_2-(F_1+F_2)\\ \text{(b)} & (G_1-F_2)-(F_1-G_2)\\ \text{(c)} & G_1G_2-F_1F_2\end{array}\right\}\subset (G_1-F_1)+(G_2-F_2)$$

In order to prove these inclusion relations let us denote by A and B their left-hand and right-hand sides respectively.

(a) Let $x \in A$, then $x \in G_1+G_2$ and, simultaneously, $x \notin F_1$ and $x \notin F_2$. Therefore if $x \in G_1$ then $x \in G_1-F_1 \subset B$ and if $x \in G_2$ then $x \in G_2-F_2 \subset B$.

(b) Let $x \in A$, then $x \in G_1$, $x \notin F_2$ and $x \notin F_1-G_2$. Consequently, for $x \in G_2$ we have $x \in G_2-F_2 \subset B$ and for $x \in G_2$, by virtue of the condition $x \notin F_1-G_2$, we conclude that $x \notin F_1$, whence it follows that $x \in G_1-F_1 \subset B$.

(c) Let $x \in A$, then $x \in G_1G_2$ and $x \notin F_1F_2$, that is at least one of the relations $x \notin F_1$ and $x \notin F_2$ holds. If the former holds then $x \in G_1-F_1 \subset B$ and if the latter holds then $x \in G_2-F_2 \subset B$.

It should also be noted that the measurability of E_1E_2 follows from the measurability of E_1+E_2 and E_1-E_2. For, if Δ is a cube containing E_1+E_2, then

$$E_1E_2 = \Delta-[(\Delta-E_1)+(\Delta-E_2)]$$

Theorem 3. *If E_1 and E_2 are Lebesgue measurable sets and their intersection is empty** ($E_1E_2 = 0$) *then*

$$\mu(E_1+E_2) = \mu E_1+\mu E_2 \tag{17}$$

Proof. Let us take an arbitrary $\varepsilon > 0$ and choose some sets F_1, F_2, G_1 and G_2 such that

$$\begin{array}{ll} F_1 \subset E_1 \subset G_1, & F_2 \subset E_2 \subset G_2\\ \mu E_1-\varepsilon < \mu F_1, & \mu G_1 < \mu E_1+\varepsilon\\ \mu E_2-\varepsilon < \mu F_2, & \mu G_2 < \mu E_2+\varepsilon \end{array}$$

Since $F_1+F_2 \subset E_1+E_2 \subset G_1+G_2$, we have (see Lemma 2)

$$\mu(E_1+E_2) \leqslant \mu(G_1+G_2) \leqslant \mu G_1+\mu G_2 \leqslant \mu E_1+\mu E_2+2\varepsilon \tag{18}$$

Further,

$$\mu E_1+\mu E_2 \leqslant \mu F_1+\mu F_2+2\varepsilon = \mu(F_1+F_2)+2\varepsilon \leqslant \mu(E_1+E_2)+2\varepsilon \tag{19}$$

(since the sets F_1 and F_2 are closed and $F_1F_2 = 0$; see Lemma 3).

By the arbitrariness of $\varepsilon > 0$, relations (18) and (19) imply (17).

With the aid of Theorems 2 and 3 we can readily prove by induction that if $e_1, \ldots, e_N$ are Lebesgue measurable pairwise disjoint sets, then their sum is also Lebesgue measurable and

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* Equality (17) also holds in the case when E_1E_2 is nonempty but $\mu(E_1E_2) = 0$. For in that case $\mu(E_1+E_2) = \mu(E_1+(E_2-E_1E_2)) = \mu E_1+\mu(E_2-E_1E_2) = \mu E_1+\mu E_2-\mu(E_1E_2) = \mu E_1+\mu E_2$ (see Theorem 3 and Theorem 4 below).

$$\mu(e_1+\ldots+e_N)=\mu e_1+\ldots+\mu e_N$$

Theorem 4. *If E_1 and E_2 are Lebesgue measurable sets and $E_1\supset E_2$ then*

$$\mu(E_1-E_2)=\mu E_1-\mu E_2 \tag{20}$$

Proof. The measurability of E_1-E_2 was already established in Theorem 2. Equality (20) itself follows from Theorem 3.

Theorem 5. *A bounded set*

$$E=\bigcup_{k=1}^{\infty} e_k = e_1+e_2+e_3+\ldots, \tag{21}$$

where e_k are Lebesgue measurable pairwise disjoint sets is Lebesgue measurable and

$$\mu E=\sum_{k=1}^{\infty}\mu e_k \tag{22}$$

Proof. The set E being bounded, its inner and outer measures $\mu_i E$ and $\mu_e E$ make sense.

Therefore, for any natural number N, we have

$$\sum_1^N \mu e_k=\mu\left(\sum_1^N e_k\right)=\mu_i\left(\sum_1^N e_k\right)\leqslant\mu_i E$$

$\left(\text{since }\sum_1^{\infty} e_k\subset E\right)$. It follows that series (22) is convergent and inequality

$$\sum_1^{\infty}\mu e_k\leqslant\mu_i E \tag{23}$$

holds.

On the other hand, since the set E is bounded, it can be embedded in an open cube Δ, and for any $\varepsilon>0$ and any natural number k there is a set $G_k\subset\Delta$ such that

$$e_k\subset G_k,\ \mu G_k<\mu e_k+\varepsilon\cdot 2^{-k}\qquad (k=1,2,\ldots)$$

From the last relation and from the fact that $\sum_1^{\infty} G_k\subset\Delta$ is an open set we conclude (see Lemma 2) that

$$\mu_e E\leqslant\mu\left(\sum_1^{\infty} G_k\right)\leqslant\sum_1^{\infty}\mu G_k\leqslant\sum_1^{\infty}\mu e_k+\varepsilon\sum_1^{\infty}2^{-k}=\sum_1^{\infty}\mu e_k+\varepsilon$$

Now, since $\varepsilon>0$ is arbitrary, we obtain (see (23)) the inequalities

$$\mu_e E\leqslant\sum_1^{\infty}\mu e_k\leqslant\mu_i E$$

Finally, taking into account that $\mu_i E \leq \mu_e E$, we see that the set E is measurable and equality (22) holds.

Theorem 5 can be restated in an equivalent form as follows.

Theorem 6. *Let $E_1, E_2, E_3, \ldots$ be measurable sets forming a nondecreasing sequence*

$$E_1 \subset E_2 \subset E_3 \subset \ldots$$

whose union $E = \bigcup_{k=1}^{\infty} E_k$ is bounded. Then E is a measurable set and

$$\lim_{k\to\infty} \mu E_k = \mu E$$

Proof. Let us put $e_1 = E_1$, and $e_N = E_N - E_{N-1}$ $(N = 2, 3, \ldots)$, then the sets e_k are measurable and pairwise disjoint and

$$\bigcup_{k=1}^{\infty} e_k = E$$

Therefore, by virtue of Theorem 5, the set E is measurable and

$$\mu E_N = \mu(e_1 + \ldots + e_N) = \sum_{k=1}^{N} \mu e_k \to \mu E, \qquad N \to \infty$$

We shall also state one more theorem reducible to Theorems 4 (or (6)).

Theorem 7. *Let $E_1, E_2, E_3, \ldots$ be measurable sets forming a nonincreasing sequence*

$$E_1 \supset E_2 \supset \ldots$$

Then their intersection $E = \bigcap_{k=1}^{\infty} E_k$ is a measurable set and $\mu E = \lim_{n\to\infty} \mu E_n$.

Proof. We can write $E = \bigcap_{k=1}^{\infty} E_k = E_1 - \bigcup_{k=1}^{\infty} (E_1 - E_k)$ and therefore (see the explanations below)

$$\mu E = \mu E_1 - \mu \bigcup_{k=1}^{\infty} (E_1 - E_k) = \mu E_1 - \lim_{k\to\infty} \mu(E_1 - E_k) = \lim_{k\to\infty} \mu E_k$$

Indeed, the sets $E_1 - E_k \subset E_1$ $(k = 1, 2, \ldots)$ are measurable and form a nondecreasing sequence, and hence, by Theorem 6, their union is measurable and its Lebesgue measure is

$$\lim_{k\to\infty} \mu(E_1 - E_k) = \mu E_1 - \lim_{k\to\infty} \mu E_k$$

Here the existence of the limit on the right-hand side follows from that of the limit on the left-hand side.

Theorem 8. *If the union of a (finite or countable) system of measurable sets $E_1, E_2, \ldots$ is bounded, then it is measurable and*

$$\mu\Big(\bigcup_k E_k\Big) \leqslant \sum_k \mu E_k \tag{24}$$

Proof. The measurability of the union $\bigcup_k E_k$ follows from the equality

$$\bigcup_k E_k = E_1 + (E_2 - E_1) + (E_3 - E_1 - E_2) + \ldots \tag{25}$$

where the summands on the right-hand side are measurable pairwise disjoint sets. Further, as we know, the measure of the set on the left-hand side of (25) exactly coincides with the sum of the measures of the sets forming the series on the right-hand side; the measure of the kth set in this series obviously does not exceed μE_k, whence follows (24).

Theorem 9. *The intersection of a finite or countable number of measurable sets $E_1, E_2, \ldots$ is measurable.*

Proof. The assertion of the theorem follows from Theorems 2 and 8 if we take into account that

$$\bigcap_k E_k = E_1 - \bigcup_k (E_1 - E_k)$$

It should be noted that a single-point set (in the space R_n) is both Jordan and Lebesgue measurable and has measure zero. By Theorem 5, a bounded countable set of points (belonging to R_n) is a Lebesgue measurable set of measure zero although it must not necessarily be Jordan measurable. For example, the set Δ' of rational points (i.e. with rational coordinates) belonging to a cube Δ is of Lebesgue measure zero but is nonmeasurable in Jordan's sense. The set of points $x = (x_1, \ldots, x_n) \in \Delta$ whose not all coordinates are rational has obviously the Lebesgue measure equal to $|\Delta|$.

We also note that if a set E is Jordan measurable, the addition of its boundary to it does not alter its measure (i.e. $mE = m\bar{E}$) while in the case of the Lebesgue measure this is no longer so; for instance, for the sets Δ' and Δ considered above we have $\mu\Delta' = 0$, $\Delta = \bar{\Delta}'$ and $|\Delta| > 0$.

§ 19.2. Measurable Functions

Lebesgue measurable sets E $(E \subset R_n)$ will be simply called measurable. They are always bounded.

A function $f = f(x) = f(x_1, \ldots, x_n)$ is said to be *measurable on a given set* E $(E \subset R_n)$ if the set E is measurable, the function f is finite* on E and the set

$$\{x\colon x \in E,\ f(x) < A\} = \{f < A\} \tag{1}$$

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* Here and henceforth we often use the expression "a finite function on E" in the sense that f assumes on E finite values (not equal to $\pm\infty$ or to ∞) and not in the sense of a function with compact support (see footnote on page 156), which we hope will not lead to a

(consisting of the points $x \in E$ at which $f(x) < A$) is measurable for any (real) number A.

The rightmost expression in (1) does not indicate explicitly that the points x in question belong to $E\,(x \in E)$ but this is tacitly implied. Similar expressions encountered below should be understood in the same sense.

Let $A < B$ be arbitrary numbers. There hold the following set-theoretic equalities:

$$\{f \leqslant A\} = \bigcap_{k=1}^{\infty} \left\{f < A + \frac{1}{k}\right\} \tag{2}$$

$$\{f = A\} = \{f \leqslant A\} - \{f < A\} \tag{3}$$

$$\{A \leqslant f\} = E - \{f < A\} \tag{4}$$

$$\{A < f\} = \{A \leqslant f\} - \{A = f\} \tag{5}$$

$$\{A < f < B\} = \{f < B\} - \{f \leqslant A\} \tag{6}$$

$$\{A \leqslant f < B\} = \{f < B\} - \{f < A\} \tag{7}$$

$$\{A \leqslant f \leqslant B\} = \{A \leqslant f < B\} + \{f = B\} \tag{8}$$

and

$$\{A < f \leqslant B\} = \{A < f < B\} + \{f = B\} \tag{9}$$

The measurability of f on E implies the measurability of every set on the left-hand sides of these relations. Indeed, the measurability of each of the sets

$$\left\{f < A + \frac{1}{k}\right\} = \left\{x\colon x \in E, f(x) < A + \frac{1}{k}\right\} \qquad (k = 1, 2, \ldots)$$

implies, by virtue of (2), the measurability of their intersection which exactly coincides with the set $\{f \leqslant A\}$. This implies the measurability of the minuend and subtrahend in (3), whence it follows that their difference is measurable as well. In this way we consecutively prove all the formulas (4)-(9).

It is also important that, for arbitrary A and B, the measurability of any set entering into the left-hand sides of relations (2) and (4)-(9) implies the measurability of f on E provided that the set E is measurable (this property does not concern (3)). For instance, let it be known that E is measurable and that the set $\{f \leqslant A\}$ is measurable for any number A. Then obviously the set

$$\{f < A\} = \bigcup_{k=1}^{\infty} \left\{f \leqslant A - \frac{1}{k}\right\}$$

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confusion. Hence, in this definition it is assumed that $f(x)$ is a (finite) number for any $x \in E$. The case when $f(x)$ is allowed to take on the values $\pm\infty$ (or ∞) is of interest when f is the limit superior or the limit inferior of a sequence of functions finite on E. This case will be dealt with in Theorem 2 below. Generally, if in a problem in question it is convenient to assign to $f(x)$, $x \in E$, not only finite but also infinite values $\pm\infty$, $-\infty$ and ∞ it is natural to consider f as being measurable on E if each of the sets $\{f = +\infty\}$ and $\{f = -\infty\}$ (or $\{f = \infty\}$) is measurable and if on the remaining part of E the function f assumes finite values and is measurable in the sense indicated above.

is also measurable. Also, let, for instance, the set E and all the sets $\{A < f < B\}$ (for any A and B such that $A < B$) be measurable. Then the set

$$\{f < A\} = \bigcup_{k=1}^{\infty} \{A-k < f < A\}$$

is measurable as a sum of a countable number of measurable sets.

A function f continuous on a bounded closed set F or on a bounded open set G is measurable on it.

Indeed, both F and G are measurable. Besides, for any A, the set $\{x: x \in F, f(x) \leq A\}$ is bounded and closed, and, consequently, measurable, and the set $\{x: x \in G, f(x) < A\}$ is bounded and open and, consequently, also measurable.

There hold the following propositions:

If $e \subset E$ is a measurable set and f is a measurable function on E then f is also measurable on e.

Indeed, the set $\{x: x \in e, f(x) < A\}$ is the intersection of the two measurable sets e and $\{x: x \in E, f(x) < A\}$.

If a function f is measurable on each of the sets e_k ($k = 1, 2, \ldots$) and if the union $E = \bigcup_k e_k$ is a bounded set then f is also measurable on E.

Indeed, E as a bounded union of measurable sets e_k is measurable. Further, the set $\{x: x \in E, f(x) < A\}$ is bounded and is the sum of the measurable sets $\{x: x \in e_k, f(x) < A\}$ ($k = 1, 2, \ldots$), which completes the proof of the proposition.

Theorem 1. *If f and φ are measurable functions on E then the functions*

$$\text{(i) } f+\varphi, \quad \text{(ii) } -\varphi, \quad \text{(iii) } f\varphi \quad \text{and} \quad \text{(iv) } 1/\varphi$$

are also measurable on condition that in Case (iv) *the function $\varphi(x)$ is different from zero $(\varphi(x) \neq 0)$ for all $x \in E$.*

Proof. (i) Let us choose an arbitrary number A. There holds the equality

$$\{f+\varphi < A\} = \bigcup_{r+\varrho<A} \{f < r\}\{\varphi < \varrho\} \tag{10}$$

where the union is extended over all the pairs (r, ϱ) of rational numbers whose sum is less than A. Indeed, if $x \in E$, $f(x) < r$, $\varphi(x) < \varrho$ and $r+\varrho < A$ then $f(x)+\varphi(x) < A$, and the right-hand member of (10) belongs to the left-hand one. If $x \in E$, $f(x)+\varphi(x) < A$ and $\delta = A-f(x)-\varphi(x)$ then there are rational numbers r and ϱ exceeding, respectively, $f(x)$ and $\varphi(x)$ by less than $\delta/2$, and therefore $f(x)+\varphi(x) < r+\varrho < A$, that is the left-hand member of (10) is contained in the right-hand member. Since these pairs (r, ϱ) of rational numbers (for which $r+\varrho < A$) form a countable set, the measurability of f and φ on E implies that the right-hand member of (10) is a measurable set, and hence so is the left-hand member.

(ii) In this case the assertion of the theorem follows from the equality $\{-\varphi < A\} = \{\varphi > -A\}$.

(iii) Let $f(x)$, $\varphi(x) \geqslant 0$ for all $x \in E$. If $A \leqslant 0$, the set $\{f\varphi < A\}$ is empty and, consequently, measurable. Let us suppose that $A > 0$. If $f(x) < r$, $\varphi(x) < \varrho$ and $r\varrho < A$ then $f(x)\varphi(x) < A$. Conversely, if $f(x)\varphi(x) < A$ then there are rational numbers $r, \varphi > 0$ such that $r\varrho < A, f(x) < r$ and $\varphi(x) < \varrho$. Therefore

$$\{f\varphi < A\} = \bigcup_{\substack{r,\varrho<A \\ r,\varrho>0}} \{f < r\}\{\varphi < \varrho\} \tag{11}$$

where the right-hand member and, together with it, the left-hand member are measurable.

Now it readily follows that the set $\{f\varphi < A\}$ is measurable when each of the functions f and φ retains sign on E.

The general case can be reduced to the above cases with the aid of the representation

$$\{f\varphi < A\} = \{f\varphi < A; f, \varphi \geqslant 0\} + \{f\varphi < A; f \geqslant 0, \varphi \leqslant 0\} + \\ + \{f\varphi < A; f \leqslant 0, \varphi \geqslant 0\} + \{f\varphi < A; f, \varphi \leqslant 0\}$$

(iv) If $\varphi(x) > 0$ for all $x \in E$ then the set $\{1/\varphi < A\}$ is empty for $A \leqslant 0$ and thus measurable while for $A > 0$ its measurability follows from the equality $\{1/\varphi < A\} = \{1/A < \varphi\}$. The case when $\varphi(x) < 0$ on E is treated similarly. The general case can be reduced to the above two cases with the aid of the representation

$$\left\{\frac{1}{\varphi} < A\right\} = \left\{\frac{1}{\varphi} < A; \varphi > 0\right\} + \left\{\frac{1}{\varphi} < A; \varphi < 0\right\}$$

Theorem 2. *The limit superior of a sequence of functions $f_n(x)$ measurable and finite (i.e. assuming finite values) on E, that is the function*

$$\psi(x) = \overline{\lim_{n\to\infty}} f_n(x), \qquad x \in E \tag{12}$$

is measurable on E in the sense of the definition stated in the footnote in the foregoing section.

Proof. The set E can be split into the three pairwise disjoint sets E_0, E_+ and E_- defined by conditions (14) below:

$$E = E_0 + E_+ + E_- \tag{13}$$

where

$$\left.\begin{array}{l} \psi(x) \text{ is finite on } E_0, \text{ is equal to } +\infty \text{ on } E_+ \text{ and} \\ \text{is equal to } -\infty \text{ on } E_- \end{array}\right\} \tag{14}$$

Let $\psi(x)$ be finite on E (i.e. $E_+ = E_- = 0$). Then for any real number A and any natural number k we have the inclusion relations

$$\left\{\psi < A - \frac{1}{k}\right\} \subset \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{f_n < A - \frac{1}{k}\right\} \subset \left\{\psi \leqslant A - \frac{1}{k}\right\} \tag{15}$$

For, if $x \in \left\{\psi < A-\frac{1}{k}\right\}$ then, by virtue of (12), there is N such that the inequality

$$f_n(x) < A-\frac{1}{k}, \qquad n \geqslant N \tag{16}$$

holds, and consequently

$$x \in \bigcap_{n=N}^{\infty} \left\{f_n < A-\frac{1}{k}\right\} \tag{17}$$

Then, moreover, x belongs to the set entering as the middle term in (15). Further, if x belongs to that set then there is N such that (17) holds for this x and thus (16) holds as well. Now, since (12) is fulfilled, we have $\psi(x) \leqslant A-\frac{1}{k}$, which means that x belongs to the set on the right-hand side of (15).

It is easily seen that (see (15))

$$\begin{aligned}\{\psi < A\} &= \bigcup_{k} \left\{\psi < A-\frac{1}{k}\right\} = \bigcup_{k} \left\{\psi \leqslant A-\frac{1}{k}\right\} = \\ &= \bigcup_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{f_n < A-\frac{1}{k}\right\}\end{aligned} \tag{18}$$

whence it follows that, since the functions f_n are measurable on E, the right-hand member of (18) is a measurable set and hence so is the left-hand member.

Let us pass to the general case when E_+ and E_- are not necessarily empty. For $x \in E_-$ we have

$$\varlimsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n(x) = -\infty \tag{19}$$

whence follows the equality

$$E_- = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n < -k\} \tag{20}$$

showing that E_- is measurable. Indeed, if $x \in E_-$ then for any natural number k there is a natural number N such that

$$f_n(x) < -k, \qquad n \geqslant N \tag{21}$$

that is

$$x \in \bigcap_{n=N}^{\infty} \{f_n < -k\} \tag{22}$$

and, moreover,

$$x \in \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \{f_n < -k\} \tag{23}$$

Since (23) holds for any k, the point x belongs to the right-hand member of (20). Conversely, if x belongs to the right-hand member of (20), it follows that (23) holds for any k and, for some N, the relation (22) also holds; this implies the validity of (21), whence follows (19), which means that the right-hand side of (20) belongs to the left-hand side.

Finally, we have the equality

$$E_+ = \bigcap_{k=1}^{\infty} \bigcap_{s=1}^{\infty} \bigcup_{n=s}^{\infty} \{f_n > k\} \tag{24}$$

showing that E_+ is measurable. Indeed, if $x \in E_+$ then for any k there is N such that

$$f_n(x) > k \tag{25}$$

for an infinite number of values of $n > N$, and then

$$x \in \bigcap_{s=1}^{\infty} \bigcup_{n=s}^{\infty} \{f_n > k\} \tag{26}$$

for any k, that is x belongs to the right-hand member of (24). Conversely, if the last property holds then (26) is valid for any k, and therefore inequality (25) is fulfilled for any k and for an infinite number of values of n, that is $x \in E_+$.

Thus, the sets E_+ and E_- are measurable, and consequently the set $E_0 = E - E_+ - E_-$ is measurable as well; since the function ψ is finite on E_0, it is measurable in the sense of the definition stated above, which completes the proof of the theorem.

Remark. In Theorem 2 it is allowable to replace the limit superior by the limit inferior because

$$\varphi(x) = \varliminf_{n\to\infty} f_n(x) = -\varlimsup_{n\to\infty} (-f_n(x))$$

and the functions $h(x)$ and $-h(x)$ are measurable simultaneously.

Let the set E_0 be the same as above and let E_0' be the set on which φ is finite. The function $\psi(x)-\varphi(x)$ is measurable on the set E_0E_0' containing the important measurable subset

$$E_* = \{\psi(x)-\varphi(x) = 0\}$$

of the set E consisting of those points $x \in E$ for which the finite limit

$$\lim_{n\to\infty} f_n(x) = \varphi(x) = \psi(x)$$

exists.

Definition. A sequence of finite functions f_k measurable on E is said to be *convergent in measure* to a function f measurable on E if, given any $\delta > 0$, the measure of the set

$$e_k = \{x \in E: |f(x)-f_k(x)| > \delta\} \tag{27}$$

tends to zero as $k \to \infty$ ($\mu e_k \to 0$, $k \to \infty$).

Theorem 3. *If a sequence of functions f_k finite and measurable on E converges on E to a finite function f it also converges to f in measure.*

Proof. Indeed, assume the contrary. Then for some $\delta > 0$ there must be a number $\lambda > 0$ and a sequence of indices $k_1, k_2, \dots$ such that $\mu e_{k_j} \geqslant \lambda$. Let us put

$$e = \bigcap_{s=1}^{\infty} (e_{k_s} + e_{k_{s+1}} + \dots)$$

then

$$\mu e = \lim_{s\to\infty} \mu(e_{k_s} + e_{k_{s+1}} + \dots) \geqslant \varliminf_{s\to\infty} \mu e_{k_s} \geqslant \lambda$$

and hence e is a nonempty set. But any point $x \in e$ obviously belongs to an infinite sequence of sets $e_{\nu_1}, e_{\nu_2}, e_{\nu_3}, \dots$ $(\nu_1 < \nu_2 < \dots)$ and therefore

$$|f_{\nu_s}(x) - f(x)| > \delta$$

which contradicts the relation

$$\lim_{k\to\infty} f_k(x) = f(x)$$

Theorem 4. *A Riemann integrable function f on a set E is Lebesgue measurable on E.*

Proof. The condition of the theorem automatically implies the Jordan measurability and hence the Lebesgue measurability of E. Let E'' be the set of the points of continuity of f. By Lebesgue's theorem (see §§ 12.8 and 12.10), $\mu E'' = \mu E$. Let us take an arbitrary $\varepsilon > 0$ and determine, for any natural k, closed sets $F_k \subset E''$ such that $\mu F_k > \mu E - \varepsilon \cdot 2^{-k}$ and $F_k \subset F_{k+1}$. The function f is continuous on the set F_k and, consequently, is measurable on it. Let us consider the functions

$$f_k = \begin{cases} f & \text{for } x \in F_k \\ 0 & \text{for } x \in E - F_k \end{cases}$$

which are obviously measurable on E. We have

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{on} \quad E' = \bigcup_k F_k \quad \text{and} \quad \mu E' = \mu E''$$

and consequently f is measurable on E'. Since $E' \subset E'' \subset E$ and $\mu(E - E') = 0$, the function f is also measurable on E.

Theorem 5. *If a function f is measurable and positive on a set E of positive measure, then there are a positive number λ and a subset $e \subset E$ of positive measure such that $f(x) \geqslant \lambda$ on e.*

Proof. Indeed, let us consider the sets

$$e_0 = \{x\colon x \in E, 1 \leqslant f\} \quad \text{and} \quad e_k = \left\{x\colon x \in E, \frac{1}{k+1} \leqslant f < \frac{1}{k}\right\},\ k = 1, 2, \dots$$

which are obviously pairwise disjoint and such that

$$E = \sum_0^\infty e_k \quad \text{and} \quad \mu E = \sum_0^\infty \mu e_k$$

Since $\mu E > 0$, there is k for which $\mu e_k > 0$. For that value of k we can put $e = e_k$ and $\lambda = 1/(k+1)$ to satisfy the theorem.

§ 19.3. Lebesgue Integral

A two-way sequence of real numbers

$$\ldots < p_{-2} < p_{-1} < p_0 < p_1 < p_2 < \ldots, p_k \to \infty, p_{-k} \to -\infty, k \to +\infty \tag{1}$$

satisfying the condition

$$\sup_k (p_{k+1} - p_k) = \delta_R < \infty \tag{2}$$

will be said to specify a *partition* R of the (entire) real axis.

Let a finite measurable function f be defined on a (measurable) set E (in the space R_n). Let us consider the (measurable) sets

$$e_k = \{x : x \in E, p_k \leqslant f(x) < p_{k+1}\} = \{p_k \leqslant f(x) < p_{k+1}\}$$
$$(k = 0, \pm 1, \pm 2, \ldots)$$

and the two-way series

$$\left.\begin{aligned} \underline{S}_R(f) &= \sum_{-\infty}^{\infty} p_k \mu e_k = \sum_k p_k \mu e_k \\ \overline{S}_R(f) &= \sum_{-\infty}^{\infty} p_{k+1} \mu e_k = \sum_k p_{k+1} \mu e_k \end{aligned}\right\} \tag{3}$$

which we shall call, respectively, the *(Lebesgue) lower* and *upper integral sums of f* (corresponding to the given partition R).

We shall agree that the symbols $\underline{S}_R(f)$ and $\overline{S}_R(f)$ serve for the notation of these series and, in case the series are convergent*, let them also denote the sums of the series themselves (which are definite numbers).

If f is a bounded function on E then, for sufficiently large N, all the sets e_k with $|k| > N$ are empty and series (3) become finite sums. But if f is unbounded on E series (3) may be convergent or divergent.

H. Lebesgue proved that *if at least one of the series* (3) *is convergent for some partition R then the other is also convergent; moreover, these series*

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* A two-way series $\sum_{-\infty}^{\infty} u_k$ is said to be convergent (absolutely convergent) if each of the series $\sum_{-\infty}^{-1} u_k$ and $\sum_0^{\infty} u_k$ is convergent (absolutely convergent). The sum of the sums of these two series is called the sum of the original series $\sum_{-\infty}^{\infty} u_k$. By (1) and by the fact that $\mu e_k \geqslant 0$, series (3), provided it is convergent, is necessarily absolutely convergent.

then automatically converge for any other partition R, and the finite limits

$$\lim_{\delta_R\to 0} \underline{S}_R(f) = \lim_{\delta_R\to 0} \bar{S}_R(f) = \int_E f(x)\,dx \tag{4}$$

exist and are equal to one and the same number which we shall call the Lebesgue integral of the function f on the set E.

Hence, in particular, *for any bounded measurable function on E there exists its Lebesgue integral.*

The first and the second terms in (4) are definitions* of the Lebesgue integral of f on E, the third term being a symbol for its notation. This notation does not differ from that of the Riemann integral. Generally speaking, here there is no danger of confusion because if a function f is Riemann integrable on E or even absolutely integrable in the (Riemann) improper sense it is also Lebesgue integrable, the values of the two integrals coinciding. By the way, if the Riemann improper integral of f on E is convergent but not absolutely, then f is not Lebesgue integrable on E, and this is the only case when the corresponding stipulation may be needed to avoid confusion.

The assertions stated will be justified later, and we shall also give two other definitions of the Lebesgue integral equivalent to the former. One of these new definitions is formulated below.

A function φ will be called a *simple function* (or a *step function* with a finite or countable number of "steps") on a measurable set E if it is defined by equalities

$$\varphi(x) = c_j,\ x \in \alpha_j, \qquad j = 1, 2, \ldots \tag{5}$$

where c_j are (real) constant numbers and α_j are measurable pairwise disjoint sets whose sum coincides with E. (Cf. § 14.4 where we considered finitely-valued functions and the footnote on page 156.)

A simple function φ is said to be *Lebesgue integrable* if the series

$$\sum_j c_j \mu\alpha_j = \int_E \varphi(x)\,dx \tag{6}$$

is absolutely convergent. The sum of that series is called *the Lebesgue integral of φ* and is denoted as written on the right-hand side of (6).

The second definition reads: a function f is said to be *Lebesgue integrable on E* if there exists a sequence of simple functions $f_k(x)$ integrable (in the sense of (6)) on E which is uniformly convergent to $f(x)$ on E. It is readily proved that for such a function there automatically exists the limit

$$\lim_{k\to\infty} \int_E f_k(x)\,dx = \int_E f(x)\,dx \tag{7}$$

independent of the choice of the sequence $\{f_k\}$ of the indicated type. This limit is called *the Lebesgue integral of f on E*. The integral $\int_E f_k\,dx$ on the left-hand side of (7) is understood in the sense of (6).

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* Later on some other (equivalent) definitions will be stated. When comparing the definitions of the Lebesgue integral, we shall refer to the definition expressed by (4) as the first one.

The equivalence of the first and second definitions of the Lebesgue integral will be proved below. The equivalence of the definitions is understood in the sense that a function f satisfying the conditions of one of the definitions satisfies the conditions of the other definition as well, the corresponding limits in (4) and (7) being equal to each other.

Lebesgue integrable functions are also termed *summable.*[*]

H. Lebesgue himself used the following terminology.

By an integrable function on E he meant only a bounded measurable function defined on E. For such a function f there exist finite sums $\underline{S}_R(f)$ and $\overline{S}_R(f)$ (which are finite numbers) for any partition R, and finite limit (4) which is the (Lebesgue) integral of f on E. Therefore the evaluation of the Lebesgue integral of a bounded measurable function reduces to finding the (one-fold) limit as $\delta_R \to 0$.

As to the unbounded functions for which limits (4) exist, H. Lebesgue called them *summable* in order to stress that the computation of the integral of such a function involves a *two-fold passage to the limit*, namely, in the first place, the sums of the infinite series $\underline{S}_R(f)$ and $\overline{S}_R(f)$ should be evaluated and, in the second place, limits (4) should be found.

In this connection we can say that the Lebesgue integral of an unbounded function is an *improper integral*; moreover, it is natural to consider it an *absolutely convergent* improper integral.

Before proceeding to the proofs of the assertions stated we shall dwell on some properties of simple functions.

An arbitrary simple function φ on E specified by relations

$$\varphi(x) = c_i, \quad x \in \alpha_i, \quad \sum \alpha_i = E, \quad \alpha_i\alpha_j = 0 \qquad (i \neq j) \tag{8}$$

is measurable on E because for any real number A the set

$$\{\varphi < A\} = \sum_{c_i < A} \alpha_i$$

is a finite or countable sum of measurable sets and therefore is itself measurable.

The definition of an integrable (in the sense of relation (6)) simple function φ and the value of its integral are in fact independent of the way in which the function is represented. For instance, if the function φ in question specified by (8) is also representable by equalities $\varphi(x) = c_j'$, $x \in \alpha_j'$, $\sum_j \alpha_j' = E$, $\alpha_i'\alpha_j' = 0$ $(i \neq j)$ then the conditions

$$c_i\mu(\alpha_i\alpha_j') = c_j'\mu(\alpha_i\alpha_j') \tag{9}$$

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[*] Later on the notion of the integrability (summability) will be extended to functions finite (i.e. assuming finite values) *almost everywhere* on E. At present we deal with functions finite everywhere on E.

hold automatically because, for a given pair (i, j), either $\mu(\alpha_i\alpha'_j) = 0$ or, if otherwise, there is a point $x \in \alpha_i\alpha'_j$, for which we have $\varphi(x) = c_i = c'_j$. Therefore if the series $\sum\limits_i c_i\mu\alpha_i$ is absolutely convergent, so are the following series having one and the same sum (see the explanations below):

$$\sum_i c_i\mu\alpha_i = \sum_i \sum_j c_i\mu(\alpha_i\alpha'_j) = \sum_j \sum_i c'_j\mu(\alpha_i\alpha'_j) = \sum_j c'_j\mu\alpha'_j$$

(see § 11.9). The absolute convergence of the double series standing as the second member in these equalities and the first equality follow from the absolute convergence of the first series and from the fact that $|c_i\mu\alpha_i| = \sum\limits_j |c_i\mu(\alpha_i\alpha'_j)|$. The second equality holds because in the absolutely convergent double series $\sum\limits_i \sum\limits_j c_j\mu(\alpha_i\alpha'_j)$ it is allowable to interchange the summations with respect to the indices i and j, and there holds (9). The third equality is explained like the first one.

When dealing with more than one simple function simultaneously, for instance with the function φ determined by equalities (8) and with another simple function $\psi(x)$ determined by equalities

$$\psi(x) = d_j, \quad x \in \beta_j, \quad \sum_j \beta_j = E, \quad \beta_j\beta_s = 0 \qquad (j \neq s)$$

it is convenient to unify their representations by putting

$$\varphi(x) = c_i, \quad x \in \alpha_i\beta_j \quad \text{and} \quad \psi(x) = d_j, \quad x \in \alpha_i\beta_j$$

The measurable sets $\alpha_i\beta_j$ entering into this representation are pairwise disjoint and their sum is equal to E (an empty set is of course regarded as nonintersecting with any other set).

It is evident that if the functions φ and ψ are integrable then, for any real numbers A and B, the simple function

$$A\varphi(x)+\beta\psi(x) = Ac_i+Bd_j, \quad x \in (\alpha_i, \beta_j)$$

is also integrable and there hold the equalities

$$\begin{aligned}\int_E (A\varphi+\beta\psi)\,dx &= \sum_i\sum_j (Ac_i+Bd_j)\,\mu(\alpha_i\beta_j) = \\ &= A\sum_i c_i \sum_j \mu(\alpha_i\beta_j)+B\sum_j d_j \sum_i \mu(\alpha_i\beta_j) = \\ &= A\sum_i c_i\mu\alpha_i+\beta\sum_j d_j\mu\beta_j = A\int_E \varphi\,dx+B\int_E \psi\,dx \qquad (10)\end{aligned}$$

Further, if the simple functions φ and ψ satisfy the inequality $\varphi(x) \leqslant \psi(x)$ then $c_i\mu(\alpha_i\beta_j) \leqslant d_j\mu(\alpha_i\beta_j)$, and therefore, if φ and ψ are integrable, then

$$\int_E \varphi\,dx \leqslant \int_E \psi\,dx \qquad (\varphi(x) \leqslant \psi(x)) \qquad (11)$$

Besides, in this case, if $\varphi(x) \geqslant 0$ and ψ is integrable, the function φ is automatically integrable.

We also note that the definition of the integrability of a simple function φ implies that φ and $|\varphi|$ are simultaneously integrable or nonintegrable, and, in the case of integrability, there holds the inequality (see (6))

$$\left|\int\limits_E \varphi\,dx\right| = \left|\sum_k c_k \mu e_k\right| \leqslant \sum_k |c_k|\,\mu e_k = \int\limits_E |\varphi|\,dx$$

It is also important that if a simple function φ specified by relations (8) is bounded (that is $|c_i| \leqslant M$; $i = 1, 2, \ldots$), it is integrable and its integral satisfies the inequality

$$\left|\int\limits_E \varphi\,dx\right| \leqslant \int\limits_E |\varphi|\,dx \leqslant \sum_k |c_k|\,\mu e_k \leqslant M\mu E \tag{12}$$

Let us also discuss another property important for our further aims. Let a measurable function f be defined on a measurable set E, and let us consider a partition R of type (1) of the real axis and the corresponding sets

$$e_k = \{p_k \leqslant f(x) < p_{k+1}\}, \qquad k = 0, \pm 1, \pm 2, \ldots$$

Using these sets we can construct the two simple functions $\underline{f}_R(x)$ and $\bar{f}_R(x)$ determined by the relations

$$\underline{f}_R(x) = p_k,\ x \in e_k \quad \text{and} \quad \bar{f}_R(x) = p_{k+1},\ x \in e_k$$

which will be referred to as the *lower* and the *upper simple functions of the function f, respectively, corresponding to the given partition R*. It is obvious that $\underline{f}_R(x) \leqslant f(x) < \bar{f}_R(x)$ and (see (2))

$$|\bar{f}_R(x) - \underline{f}_R(x)| \leqslant \delta_R$$

for all $x \in E$, and therefore we also have

$$|f(x) - \underline{f}_R(x)| < \delta_R \quad \text{and} \quad |f(x) - \bar{f}_R(x)| < \delta_R$$

It follows that both *the lower and the upper simple functions (corresponding to the partition R) uniformly tend to $f(x)$ on E as $\delta_R \to 0$, that is for any function f measurable on E we have*

$$\lim_{\delta_R \to 0} \underline{f}_R(x) = \lim_{\delta_R \to 0} \bar{f}_R(x) = f(x)$$

where the passage to the limit is uniform.

Note that the functions $\underline{f}_R(x)$ and $\bar{f}_R(x)$ are integrable (in the sense of (6)) if the corresponding series, that is the lower and upper integral sums

$$\underline{S}_R(f) = \sum_k p_k \mu e_k = \int\limits_E \underline{f}_R(x)\,dx$$

$$\bar{S}_R(f) = \sum_k p_{k+1} \mu e_k = \int\limits_E \bar{f}_R(x)\,dx$$

are absolutely convergent.

What has been said allows us to restate Lebesgue's proposition stated above (its proof is given in Theorem 1 below) as follows: *if for at least one partition R one of the functions $\underline{f}_R(x)$ and $\bar{f}_R(x)$ is integrable on E (in the sense of* (6)) *then the other is also integrable and, moreover, all such functions constructed for any other partition R are also integrable on E and there exist the limits*

$$\lim_{\delta_R\to 0}\int_E \underline{f}_R(x)\,dx = \lim_{\delta_R\to 0}\int_E \bar{f}_R(x)\,dx = \int_E f(x)\,dx$$

equal to one and the same number which is called the Lebesgue integral of f on E.

Theorem 1. *Let a function f be measurable on a set E and let one of the sums* (3) *(lower or upper) be convergent for some partition R of the real line. Then the other sum, and also the upper and lower sums corresponding to an arbitrary partition R', are convergent as well. Besides, all the limits of type* (4) *are equal to one another.*

Proof. Let us consider the lower and upper simple functions corresponding to the partition R:

$$\underline{f}_R(x) = p_k, \qquad x \in e_k = \{p_k \leqslant f(x) < p_{k+1}\}$$

and

$$\bar{f}_R(x) = p_{k+1} \qquad x \in e_k$$

where p_k are the points of division forming the partition R. Let us also consider the lower simple function $\underline{f}_{R'}(x)$ corresponding to some other partition R'.

For definiteness, let us suppose that the series $\underline{S}_R(f)$ is convergent. This automatically implies that it is absolutely convergent, and therefore the integral of $\underline{f}_R(x)$ on E (understood in the sense of (6)) exists and is expressed by the relation

$$\underline{S}_R(f) \sum_k p_k \mu e_k = \int_E \underline{f}_R(x)\,dx$$

The inequalities

$$|\bar{f}_R(x) - \underline{f}_R(x)| \leqslant \delta_R$$

and

$$|\underline{f}_{R'}(x) - \underline{f}_R(x)| \leqslant |\underline{f}_{R'}(x) - f(x)| + |f(x) - \underline{f}_R(x)| < \delta_{R'} + \delta_R$$

show that the simple functions $\bar{f}_R(x) - \underline{f}_R(x)$ and $\underline{f}_{R'}(x) - \underline{f}_R(x)$ are bounded and therefore integrable, whence it follows that the function

$$\bar{f}_R(x) = \underline{f}_R(x) + [\bar{f}_R(x) - \underline{f}_R(x)]$$

and also the function

$$\underline{f}_{R'}(x) = \underline{f}_R(x) + [\underline{f}_{R'}(x) - \underline{f}_R(x)]$$

are integrable since each of them is a sum of two integrable simple functions. This means that the numbers (the sums of the corresponding series) $\bar{S}_R(f)$

and $\underline{S}_{R'}(f)$ make sense. This completes the proof of the first part of the theorem.

Further, by virtue of (12), we have the relation

$$|\underline{S}_R(f)-\underline{S}_{R'}(f)| \leqslant \int\limits_E |\underline{f}_R(x)-\underline{f}_{R'}(x)|\,dx \leqslant (\delta_R+\delta_{R'})\,\mu_E \to 0 \qquad (\delta_R,\ \delta_{R'} \to 0)$$

which is nothing but the condition of Cauchy's criterion, whence it follows that the first limit (4) exists.

Finally,

$$|\overline{S}_R(f)-\underline{S}_R(f)| = \int\limits_E |\overline{S}_R(x)-\underline{S}_R(x)|\,dx \leqslant \delta_R \mu E \to 0 \qquad (\delta_R \to 0)$$

and therefore the second limit (4) also exists and is equal to the first one, which completes the proof of the theorem.

Earlier we stated the second definition of the Lebesgue integral based on the approximation of the given function with arbitrary integrable simple functions, not necessarily with the upper or the lower simple function. Its validity follows from the theorem below.

Theorem 2. *For a function f to be Lebesgue integrable on E it is necessary and sufficient that there should exist a sequence of integrable simple functions $\lambda_k(x)$ $(k = 1, 2, \ldots)$ uniformly convergent to f on E. When such a sequence exists it turns out automatically that*

$$\lim_{k\to\infty} \int\limits_E \lambda_k(x)\,dx = \int\limits_E f\,dx$$

and hence this limit is independent of the particular choice of the sequence $\{\lambda_k\}$.

Proof. Let a function f be Lebesgue integrable on E and let $\underline{f}_{R_m}(x)$ be its lower simple function corresponding to a partition R_m of the real line with $\delta_{R_m} < 1/m$ $(m = 1, 2, \ldots)$. Then

$$|\underline{f}_{R_m}(x)-f(x)| < \frac{1}{m} \to 0, \qquad m \to \infty$$

which means that the sequence of the functions $\underline{f}_{R_m}(x)$ is uniformly convergent to $f(x)$ on E and, besides, as is known from Theorem 1,

$$\int\limits_E \underline{f}_{R_m}(x)\,dx = \underline{S}_{R_m}(f) \to \int\limits_E f\,dx \qquad (m \to \infty)$$

We have thus proved that for any Lebesgue integrable function f on E there is a sequence possessing the properties indicated in Theorem 2, namely, we can put $\lambda_m(x) = \underline{f}_{R_m}(x)$.

Conversely, let us suppose that there is a sequence of integrable (in the sense of (6)) simple functions $\lambda_m(x)$ on E which is uniformly convergent to a function $f(x)$ on E, that is

$$|\lambda_m(x)-f(x)|<\varepsilon_m\to 0, \qquad m\to\infty$$

(where ε_m are independent of $x\in E$). Then the function f is measurable and finite on E (see § 19.2, Theorem 2), and the sequence of its lower simple functions $f_{R_m}(x)$ (defined as above) is also uniformly convergent to f:

$$|f(x)-f_{R_m}(x)|<\frac{1}{m}\to 0, \qquad m\to\infty$$

It follows that

$$|\lambda_m(x)-f_{R_m}(x)|<|\lambda_m(x)-f(x)|+|f(x)-f_{R_m}(x)|\leqslant$$
$$\leqslant\varepsilon_m+\frac{1}{m}\to 0, \qquad m\to\infty \tag{13}$$

The function $f_{R_m}(x)$ is integrable (in the sense of (6)) for any m since it is representable as the sum

$$f_{R_m}(x)=\lambda_m(x)+\big(f_{R_m}(x)-\lambda_m(x)\big)$$

of two integrable simple functions. Indeed, the function λ_m is integrable by the hypothesis, and the function $f_{R_m}-\lambda_m$ is bounded. The integrability of f_{R_m} (in the sense of (6)) means that there exists the lower integral sum of the function f corresponding to the partition R_m, and therefore Theorem 1 implies that the given function f is Lebesgue integrable on E and

$$\int_E f_{R_m}(x)\,dx=S_{R_m}(f)\to\int_E f\,dx$$

where the left-hand member is the integral of the simple function $f_{R_m}(x)$ in the sense of definition (6) and the right-hand member is its integral in the sense of the first definition (4).

Finally, taking into account that (see (13))

$$\left|\int_E[\lambda_m(x)-f_{R_m}(x)]\,dx\right|<\left(\varepsilon_m+\frac{1}{m}\right)\mu E$$

we obtain (see (10))

$$\int_E\lambda_m\,dx=\int_E f_{R_m}\,dx+\int_E(\lambda_m-f_{R_m})\,dx\to\int_E f\,dx$$

The integral of an integrable simple function $\varphi(x)$ understood in the sense of (6) coincides with its integral in the sense of the first definition because we can assume in Theorem 2 that the function φ is approximated with the functions $\lambda_m=\varphi$ $(m=1,2,\ldots)$.

Now we proceed to the basic properties of the Lebesgue integral.

1. *If a function f is Lebesgue integrable on E then it can be redefined in an arbitrary manner on a set of Lebesgue measure zero without violating its*

integrability and altering the value of its integral, that is

$$\int_E f\,dx = \int_E f_1\,dx$$

where f_1 is the new (redefined) function. A function of the type of f_1 coinciding with f almost everywhere is said to be equivalent to f. Hence, we can also say that the replacement of the integrand function f in the Lebesgue integral by an equivalent function f_1 does not alter the integral.

This property is an obvious consequence of the equality $S_R(f) = S_R(f_1)$ holding for any partition R of the real axis.

When we say that a function f is (Lebesgue) integrable on E we mean that it is finite (assumes finite values) on E, that is the functional relationship $y = f(x)$ assigns to each point $x \in E$ a definite (finite) number y. However, in the theory of the Lebesgue integral it proves useful to introduce the notion of a *Lebesgue integrable function f defined (and finite) almost everywhere on E*, that is at all the points of the set E except possibly on a subset (of E) of Lebesgue measure zero. This means that such a function is not defined at those points x of the set E which belong to that subset; in particular, it may happen that it is natural to put $f(x) = \infty$ at some (or all) of these points.

More precisely, *a function f defined (and finite) almost everywhere on a set E is said to be Lebesgue integrable on E* if it is Lebesgue integrable on the subset $E' \subset E$ consisting of the points at which it assumes finite values. For such a function we put, by definition,

$$\int_E f\,dx = \int_{E'} f\,dx$$

This definition implies that E' is a measurable set, and therefore the set E also turns out to be measurable because $\mu(E-E') = 0$.

The class of all functions f which are finite almost everywhere on a set E and Lebesgue integrable on E is usually denoted as $L(E)$. In particular, the class $L(E)$ contains as its part the totality of all finite Lebesgue integrable functions on E which, in its turn, contains as its part the collection of all bounded measurable functions on E.

For example, if E is the (one-dimensional) closed interval $[0, 1]$, we can say the function $1/\sqrt{x}$ is finite almost everywhere on $[0, 1]$ since it assumes finite values on the half-open interval $(0, 1]$ differing from $[0, 1]$ by a single-point set (consisting of the point $x = 0$) whose measure is equal to zero. As will be seen later (property 15), the Lebesgue integral of this function exists and coincides with its (improper) Riemann integral:

$$\int_{[0,1]} \frac{dx}{\sqrt{x}} = \int_{(0,1]} \frac{dx}{\sqrt{x}} = \int_0^1 \frac{dx}{\sqrt{x}} = 2$$

Thus, the function under consideration belongs both to $L([0, 1])$ and to $L((0, 1])$.

The expedience of the definition of a measurable function finite almost everywhere on a given set can also be demonstrated by the following situation. It may turn out that a given sequence of measurable finite functions $f_k(x)$ on a set E is convergent to a finite function not at all the points of E but only almost everywhere on E. At those points of the set E where the convergence does not take place and which form a set of measure zero there exists either an (improper) limit equal to ∞ or no limit (finite or infinite) at all.

If F and Φ are two functions finite almost everywhere and measurable on E and A and B are real numbers, the function $AF+B\Phi$ is defined as follows. Let E' and E'' be the subsets of E on which F and Φ are finite, respectively; then both functions are finite on the (measurable) intersection $E_* = E'E''$ of E' and E''. If we now define the function $AF+B\Phi$ on E_* in the ordinary way it will be automatically defined almost everywhere on E because $\mu(E-E_*) = 0$.

2. *If $F, \Phi \in L(E)$ and $F(x) \leqslant \Phi(x)$ almost everywhere on E then*

$$\int_E F\,dx \leqslant \int_E \Phi\,dx \tag{14}$$

Proof. Let us begin with the case when both functions F and Φ are finite on E. For any partition R of the real line we have

$$\underline{F}_R(x) \leqslant F(x) \leqslant \Phi(x) \leqslant \bar{\Phi}_R(x), \qquad x \in E$$

Since $F, \Phi \in L(E)$ it follows that the simple functions $\underline{F}_R$ and $\bar{\Phi}_R$ are integrable on E and, since $\underline{F}_R(x) \leqslant \bar{\Phi}_R(x)$, we obtain (see (11))

$$\int_E \underline{F}_R(x)\,dx \leqslant \int_E \bar{\Phi}_R(x)\,dx_R$$

On passing to the limit in this inequality as $\delta_R \to 0$ we arrive at (14).

In the general case we consider the greatest subset $E' \subset E$ on which both F and Φ are finite and the inequality $F \leqslant \Phi$ is fulfilled. For this set E' we can readily prove (14) with E replaced by E', and, since $\mu E' = \mu E$, inequality (14) holds for E as well because, according to our formal convention, in such cases we put

$$\int_E F\,dx = \int_{E'} F\,dx \quad \text{and} \quad \int_E \Phi\,dx = \int_{E'} \Phi\,dx$$

3. *If $F, \Phi \in L(E)$ and A and B are real numbers then $AF+B\Phi \in L(E)$ and*

$$\int_E (AF+B\Phi)\,dx = A\int_E F\,dx + B\int_E \Phi\,dx \tag{15}$$

Proof. Let E' ($\mu E' = \mu E$) be the greatest subset of E on which both functions F and Φ are finite. By Theorem 2, there exist two sequences of simple functions F_k and Φ_k ($k = 1, 2, \ldots$) integrable on E' which are uniformly convergent to F and Φ respectively. By (10), we have

$$\int_{E'} (AF_k+B\Phi_k)\,dx = A\int_{E'} F_k\,dx+B\int_{E'}\Phi_k\,dx \tag{16}$$

for these functions, and since F_k, Φ_k and $AF_k+B\Phi_k$ are integrable simple functions whose sequences are uniformly convergent to finite functions F, Φ and $AF+B\Phi$ respectively, we see that not only F and Φ but also $AF+B\Phi$ belong to $L(E')$, and it is legitimate to pass to the limit under the integral sign in equality (16) as $k\to\infty$. Finally, since $\mu E = \mu E'$, we obtain (15).

In particular, on putting $A = 1$ and $B=\pm 1$ in (15), we obtain

$$\int_E (F\pm\Phi)\,dx = \int_E F\,dx\pm\int_E \Phi\,dx$$

(where the existence of the integrals on the right-hand side implies the existence of the integral on the left-hand side).

It can readily be proved by induction that

$$\int_E \sum_1^l f_k\,dx = \sum_1^l \int_E f_k\,dx \qquad (l = 1, 2, \ldots)$$

4. *If $E_1 \subset E$, the set E_1 is measurable and a function f belongs to $L(E)$, then $f\in L(E_1)$, that is the function f is finite almost everywhere on the set E_1 and is Lebesgue integrable on it.*

Proof. Let E' be the greatest subset of E on which f is finite and let $E_1' = E'E_1$. Then

$$e_k' = \{x\colon\ x\in E_1',\ p_k\leqslant f(x)<p_{k+1}\}\subset\{x\colon\ x\in E',\ p_k\leqslant f(x)<p_{k+1}\} = e_k$$

for any partition R (see (1)). Therefore, since the function f is finite on E' and belongs to $L(E')$, we have

$$\sum_k |p_k|\,\mu e_k' \leqslant \sum_k |p_k|\,\mu e_k < \infty$$

It follows, by Theorem 1, that $f\in L(E_1')$. Consequently $f\in L(E_1)$ because $\mu E_1 = \mu E_1'$.

5. *If $f\in L(E_1)$, $f\in L(E_2)$ and $\mu(E_1E_2) = 0$ then $f\in L(E_1+E_2)$ and*

$$\int_{E_1+E_2} f\,dx = \int_{E_1} f\,dx+\int_{E_2} f\,dx \tag{17}$$

Proof. Let us begin with the case when f is finite on E_1+E_2. Taking an arbitrary partition R of the real line we introduce the sets

$$e_k' = \{x:\ x\in E_1,\ p_x\leqslant f(x)<p_{k+1}\}$$
$$e_k'' = \{x:\ x\in E_2,\ p_k\leqslant f(x)<p_{k+1}\}$$

and

$$e_k = \{x:\ x\in E_1+E_2,\ p\leqslant f(x)<p_{k+1}\}$$

Taking into account that $\mu(E_1E_2) = 0$, we see that $\mu e_k = \mu e'_k + \mu e''_k$. It follows that for the lower integral sums of f corresponding to the sets E_1+ $+E_2$, E_1 and E_2 the equality

$$\sum_k p_k \mu e_k = \sum_k p_k \mu e'_k + \sum_k p_k \mu e''_k$$

is fulfilled. By the conditions of the theorem, the series on the right-hand side are convergent, which implies the convergence of the series on the left-hand side, whence $f \in L(E_1+E_2)$. On passing to the limit in the last equality as $\delta_R \to 0$ we arrive at (17).

In the general case we consider the greatest subsets $E'_1 \subset E_1$ and $E'_2 \subset E_2$ $(\mu E'_1 = \mu E_1$ and $\mu E'_2 = \mu E_2)$ on which f is finite. For these subsets there holds equality (17), and consequently it holds for E_1 and E_2 as well.

It can readily be proved by induction that

$$\int\limits_{\bigcup\limits_1^N E_k} f\,dx = \sum_{k=1}^{N} \int\limits_{E_k} f\,dx \qquad (\mu(E_kE_l) = 0; \quad k \neq l)$$

6. *If a bounded function f is Riemann integrable on a set E it is also Lebesgue integrable on that set and the values of the integrals of f on E (taken in both senses) coincide.*

Indeed, the function f is bounded and Riemann integrable on E; therefore it is measurable (see Theorem 4, § 19.2) and hence Lebesgue integrable on E. Now we can represent the set E in the form of the union $E = \bigcup\limits_k E_k$ of a finite number of Jordan measurable (and, consequently, Lebesgue measurable) sets E_k with no interior points in common (i.e. any two of them can only intersect along some parts of their boundaries), which means that their intersection is of Jordan measure zero (and therefore of Lebesgue measure zero). Therefore, for the Lebesgue integral of f on E we obtain (see property 2) the relations

$$\sum_k m_k |E_k| \leqslant \int\limits_E f\,dx = \sum_k \int\limits_{E_k} f\,dx \leqslant \sum_k M_k |E_k|$$

where m_k and M_k are, respectively, the infimum and the supremum of f on E_k. Since the leftmost and rightmost members of these relations tend to the Riemann integral of f on E for $\max\limits_k d(E_k) \to 0$, this integral is equal to the corresponding Lebesgue integral.

7. *Let f be a nonnegative measurable function on E (not necessarily finite, that is $f(x) \leqslant +\infty$ for $x \in E$) and let*

$$(f)_N = \begin{cases} f(x) & \textit{for} \quad 0 \leqslant f(x) \leqslant N \\ N & \textit{for} \quad N < f(x) \end{cases} \tag{18}$$

Then, if $f \in L(E)$, there holds the limiting relation

$$\lim_{N\to\infty} \int_E (f)_N \, dx = \int_E f \, dx \tag{19}$$

Conversely, if the limit on the left-hand side of (19) *exists, then $f \in L(E)$.*

Proof. Let $f \in L(E)$, then the set $\{x: f(x) = +\infty\}$ is of Lebesgue measure zero and the function $(f)_N$ is measurable and bounded on E because for $A \leq N$ the set $\{(f_N) < A\}$ coincides with $\{f < A\}$ and for $A > N$ it coincides with E. Thus, $(f)_N \in L(E)$, and, besides, $(f_N)(x) \leq (f)_{N+1}(x) \leq f(x)$. Consequently,

$$\sum_{p_{k+1} \leq N} p_k \mu e_k \leq \int_E (f)_N \, dx \leq \int_E f \, dx \tag{20}$$

for any partition R (see (1)), whence, on passing to the limit for $N \to \infty$, we obtain

$$S_R(f) \leq \lim_{N\to\infty} \int_E (f_N) \, dx \leq \int_E f \, dx \tag{21}$$

Since the leftmost member in (21) can be made arbitrarily close to the rightmost member, there holds equality (19).

If we suppose now that the limit on the left-hand side of (19) exists and is finite, it will follow automatically that the set $e = \{x: x \in E, f(x) = +\infty\}$ is of measure zero. For we have the inequality

$$\int_E (f)_N \, dx \geq \int_e (f)_N \, dx = N \mu e$$

where the leftmost member has a finite limit as $N \to \infty$ which is only possible when $\mu e = 0$. Therefore the first inequality in (21) implies (we put $p_0 = 0$) that

$$S_R(f) = \sum_{k=1}^{\infty} p_k \mu e_k \leq \lim_{N\to\infty} \int (f)_N \, dx$$

Consequently the function f is integrable on the set E' at whose points it assumes finite values, that is $f \in L(E')$, or, which is the same, $f \in L(E)$.

It should be noted that the integrability of f on E' automatically implies that E' is measurable and therefore so is E.

It follows from what has been said that *the Lebesgue integral of a nonnegative function f measurable on E (but not necessarily finite) can be defined as limit* (19).

In some courses the theory of the Lebesgue integral is initiated by the introduction of the notion of the integral of a bounded measurable function which is then generalized to a *summable nonnegative function* (which is unbounded but belongs to $L(E)$), the latter being defined as a measurable function f on E for which finite limit (19) exists. This limit is called, by definition, the Lebesgue integral of f on E.

Property 7 shows that the totality of all the functions thus defined exactly coincides with the collection of all unbounded nonnegative functions belonging to $L(E)$.

A measurable unbounded finite function f taking values of arbitrary sign is said to be *summable on E* if it is representable as a difference of two finite nonnegative summable functions or of bounded measurable functions. Each such difference obviously belongs to $L(E)$. It will be shown below that, conversely, every finite function f on E belonging to $L(E)$ can be represented in the form of a difference of two finite nonnegative functions belonging to $L(E)$, that is in the terminology mentioned above* each such function is either measurable and bounded or summable on E.

We shall say that an arbitrary function f defined almost everywhere on a measurable set E is *summable on E* if it is summable on the subset $E' \subset E$ at whose points it assumes finite values and if $\mu E' = \mu E$.

Let a function f be finite and measurable on E. Proceeding from f, let us construct the two nonnegative functions

$$\begin{aligned} f_+(x) &= \begin{cases} f(x) & \text{for} \quad f(x) > 0 \\ 0 & \text{for} \quad f(x) \leqslant 0 \end{cases} \\ f_-(x) &= \begin{cases} 0 & \text{for} \quad f(x) \geqslant 0 \\ -f(x) & \text{for} \quad f(x) < 0 \end{cases} \end{aligned} \tag{22}$$

which obviously are also finite and measurable on E. It is evident that

$$f(x) = f_+(x) - f_-(x) \quad \text{and} \quad |f(x)| = f_+(x) + f_-(x) \tag{23}$$

If $f_+, f_- \in L(E)$ then also $f \in L(E)$ (see (3)), and the equality

$$\int\limits_E f(x)\,dx = \int\limits_E f_+(x)\,dx - \int\limits_E f_-(x)\,dx \tag{24}$$

is fulfilled.

The converse is also true: if $f \in L(E)$ then f_+ and f_- also belong to $L(E)$. For (see the explanations below),

$$\int\limits_E f_+(x)\,dx = \int\limits_{f(x) > 0} f(x)\,dx + \int\limits_{f(\;) \leqslant 0} 0\,dx = \int\limits_{f(x) > 0} f(x)\,dx$$

The third member in these relations makes sense since the set $\{f > 0\} \subset E$ is measurable and the integrability of f on the set E implies the integrability of f on the former set (see property 4). The passage from the third member to the second one is trivial because the set $\{f \leqslant 0\}$ is measurable and the integral of the function identically equal to zero on that set is obviously equal to zero. Finally, the passage from the second member to the first one and the assertion that $f_+ \in L(E)$ follow from property 5. It is similarly proved that $f_- \in L(E)$.

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* By the way, when this terminology is used it is usually agreed that all the functions $f \in L(E)$, both bounded and unbounded, are termed summable.

8. *If F and Φ are two measurable functions on E (they may assume finite values and also the values equal to $+\infty$), $0 \leqslant F(x) \leqslant \Phi(x)$ and $\Phi \in L(E)$ then $F \in L(E)$.*

Indeed, as follows from the hypothesis, $(F)_N \leqslant (\Phi)_N$ on E for any N and, since $(F)_N$, $(\Phi)_N$ and Φ belong to $L(E)$, we have (see property 2)

$$\int_E (F)_N\,dx \leqslant \int_E (\Phi)_N\,dx \leqslant \int_E \Phi\,dx$$

Besides, the first member in these relations does not decrease as N increases indefinitely and hence it tends to a finite limit, whence, by virtue of (7), it follows that $F \in L(E)$.

9. *If $f \in L(E)$ then $|f| \in L(E)$ and*

$$\left|\int_E f\,dx\right| \leqslant \int_E |f|\,dx \tag{25}$$

The converse only holds if we state it as follows: *if a function f defined almost everywhere on E is measurable on E and $|f| \in L(E)$ then also $f \in L(E)$.*

Proof. Let $E' \subset E$ $(\mu E' = \mu E)$ be the greatest subset of E on which f is finite. As is known, on that set it is possible to construct the nonnegative functions f_+ and f_- belonging to $L(E')$ for which

$$|f(x)| = f_+(x) + f_-(x) \quad \text{and} \quad f(x) = f_+(x) - f_-(x)$$

It follows that $|f| \in L(E')$, and consequently we also have $|f| \in L(E)$. Besides, there hold the equalities

$$\int_{E'} |f|\,dx = \int_{E'} f_+\,dx + \int_{E'} f_-\,dx \quad \text{and} \quad \int_{E'} f\,dx = \int_{E'} f_+\,dx - \int_{E'} f_-\,dx$$

which directly imply inequality (25) with E replaced by E' since the integrals of f_+ and f_- are nonnegative numbers, whence it follows that this inequality holds for E as well.

Let a function f defined almost everywhere of E be measurable on E and let $|f| \in L(E)$. Then on the greatest subset E' of E where f is finite the nonnegative measurable functions f_+ and f_- make sense and the inequalities $|f(x)| \geqslant f_+(x)$ and $|f(x)| \geqslant f_-(x)$ are fulfilled, whence, by property (8), it follows that $f_+, f_- \in L(E')$ and therefore $f = f_+ - f_- \in L(E')$.

It should be noted that there exist sets nonmeasurable in Lebesgue's sense. Here we shall not dwell on such sets and shall limit ourselves to the following remark. Suppose that e is a nonmeasurable set belonging to a cube Δ, and let a function $\psi(x)$ be equal to 1 on e and to -1 on $\Delta - e$. It is evident that this function is not Lebesgue measurable and therefore it cannot belong to $L(\Delta)$. At the same time, $|\psi(x)| \equiv 1$ on Δ and $|\psi| \in L(\Delta)$.

10. *If $f \in L(E)$ and if φ is a bounded measurable function on E $(|\varphi(x)| \leqslant M)$ then $f\varphi \in L(E)$ and*

$$\int_E |\varphi f|\, dx \leqslant \int_E M|f|\, dx = M \int_E |f|\, dx \tag{26}$$

This follows from properties 2, 8 and from (15) for $B = 0$.

11. *If a function f belongs to $L(E)$ then for any $\varepsilon > 0$ there is $\delta > 0$ such that for any subset $e \subset E$ of measure $\mu e < \delta$ there holds the inequality*

$$\int_e |f|\, dx < \varepsilon \tag{27}$$

Proof. Let us begin with the case when f is nonnegative on E. On choosing an arbitrary $\varepsilon > 0$ we find N such that

$$\int_E (f-(f)_N)\, dx = \int_E f\, dx - \int_E (f)_N\, dx < \frac{\varepsilon}{2}$$

Now, taking into account that $f-(f)_N \geqslant 0$ on E, we obtain for an arbitrary sets $e \subset E$ with $\mu e < \frac{\varepsilon}{2N}$ the inequalities

$$\int_e f\, dx = \int_e (f)_N\, dx + \int_e (f-(f)_N)\, dx \leqslant \int_e (f)_N\, dx + \int_E (f-(f)_N)\, dx \leqslant$$

$$\leqslant N\mu e + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

In the general case when f belongs to $L(E)$, assumes values of arbitrary sign and is defined only almost everywhere on E, we consider the greatest subset $E' \subset E$ on which f is finite. The function $|f|$ is finite on E' and belongs to $L(E')$, and, by what has been proved, given any $\varepsilon > 0$, there is $\delta > 0$ such that for any set $e' \subset E'$ (which temporarily replaces e) with $\mu e' < \delta$ there holds inequality (27). Now it follows that the assertion of the theorem holds because if $e \subset E$ and $\mu e < \delta$ then $eE' \subset E'$ and $\mu(eE') = \mu e$.

Property 11 can be restated as follows: if $f \in L(E)$ then for any sequence of measurable sets e_k with $\mu e_k \to 0$ there holds the relation

$$\left|\int_{e_k} f\, dx\right| \leqslant \int_{e_k} |f|\, dx \to 0, \qquad k \to \infty \tag{28}$$

12. *If*

$$e = e_1 + e_2 + \ldots \subset E$$

where $e_1, e_2, \ldots$ are measurable sets, $\mu(e_k e_l) = 0$ for $k \neq l$ and $f \in L(E)$ then

$$\int_e f\, dx = \int_{e_1} f\, dx + \int_{e_2} f\, dx + \ldots$$

Indeed, the existence of the integrals on the right-hand side of this equality is implied by property 4. Further, by virtue of property 11,

$$\int_{e} f\,dx-\sum_{1}^{N}\int_{e_k} f\,dx = \int_{e-\sum_{1}^{N} e_k} f\,dx \to 0, \qquad N\to\infty$$

because (see § 19.1, Theorem 5)

$$\mu\left(e-\sum_{1}^{N} e_k\right) \to 0, \qquad N\to\infty$$

13. **Lebesgue's Theorem.** *If a sequence of functions $f_k \in L(E)$ is convergent almost everywhere on E to a function f and if the inequalities $|f_k(x)| \leqslant \Phi(x)$ $(k = 1, 2,\ldots)$ where $\Phi \in L(E)$ hold almost everywhere on E then $f \in L(E)$ and*

$$\lim_{k\to\infty} \int_{E} f_k(x)\,dx = \int_{E} f(x)\,dx \tag{29}$$

that is under the conditions stated it is allowable to pass to the limit under the integral sign.

In particular, *equality* (29) *holds for any bounded sequence of functions $f_k(x)$ convergent to $f(x)$.*

Proof. Let $E' \subset E$ be the greatest subset on which the functions Φ and f_k are finite, inequalities $|f_k| \leqslant \Phi$ are satisfied and $f_k(x) \to f(x)$. It is evident that $\mu E' = \mu E$, f is measurable on E' and $|f(x)| \leqslant \Phi(x)$ on E', and, since $\Phi \in L(E)$, we have $|f| \in L(E')$ and $f \in L(E)$. Let us put $E' = E'_k+E''_k$ where

$$E'_k = \{x:\ x\in E',\quad |f(x)-f_k(x)| > \delta\}$$

and

$$E''_k = \{x:\ x\in E',\quad |f(x)-f_k(x)| \leqslant \delta\}$$

Then $\mu E'_k \to 0$, $k\to\infty$ (see Theorem 3 in § 19.2) and we obtain

$$\left|\int_{E} f\,dx-\int_{E} f_k\,dx\right| = \left|\int_{E'} (f-f_k)\,dx\right| \leqslant \int_{E'_k} |f|\,dx+\int_{E'_k} |f_k|\,dx+\int_{E''_k} |f-f_k|\,dx \leqslant$$

$$\leqslant 2\int_{E'_k} \Phi\,dx+\delta\mu E < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon \qquad (k > k_0)$$

where, first, $\delta > 0$ is chosen so that $\delta\mu E < \frac{\varepsilon}{2}$, and then (see property 11), using the fact that $\mu E'_k \to 0$, the number k_0 is taken so large that $\int_{E'_k} \Phi\,dx < \frac{\varepsilon}{2}$ $(k > k_0)$.

14. *Let nonnegative functions $f_k \in L(E)$ (assuming finite values or equal to $+\infty$) defined on E form a nondecreasing sequence. Then the limiting function*

$$\lim_{k\to\infty} f_k(x) = f(x)$$

satisfies the relation

$$\lim_{k\to\infty}\int_E f_k(x)\,dx = \int_E f(x)\,dx \tag{30}$$

where the right-hand member is the number equal either to the Lebesgue integral of f on E if $f \in L(E)$ or to $+\infty$ if $f \notin L(E)$.

Proof. Let $f \in L(E)$. Then the function f and, together with it, the functions f_k are finite on a set $E' \subset E$ of measure $\mu E' = \mu E$, and for the set E' there holds Lebesgue's theorem (property 13) with $\Phi = f$. Therefore relation (30) holds if E is replaced in it by E', and consequently it also holds for the set E itself. Thus, there exists the limit on the left-hand side of (30) equal to the Lebesgue integral of f on E.

Conversely, let a finite limit

$$\lim_{k\to\infty}\int_E f_k\,dx = A < \infty$$

exist. Then, since $(f_k)_N \leqslant f_k$ on E, we have (see property 2)

$$\int_E (f_k)_N\,dx \leqslant \int_E f_k\,dx \leqslant A$$

that is $\int\limits_E (f_k)_N\,dx \leqslant A.$

Now, taking into account that $(f_k)_N \to (f)_N$ $(k \to \infty)$ on E, we obtain, on passing to the limit under the integral sign (see property 13), the inequality

$$\int_E (f)_N\,dx \leqslant A$$

holding for any $N > 0$, whence it follows (see property 7) that $f \in L(E)$.

15. *Let a function f be integrable in Riemann's improper sense on E. For such a function f to be absolutely (Riemann) integrable on E it is necessary and sufficient that $f \in L(E)$; in case $f \in L(E)$ the improper Riemann integral of f on E is equal to the Lebesgue integral of f on E.*

We shall confine ourselves to presenting the proof for the case when the improper Riemann integral of f has only one singularity on E at a point $x^0 \in E$.

On constructing for any natural number N the function

$$f_N(x) = \begin{cases} f(x) & \text{for} \quad x \in E - V_N \\ 0 & \text{for} \quad x \in EV_N \end{cases}$$

where V_N is the ball of radius $1/N$ with centre at x^0, we can write the integral in question as the limit

$$\lim_{N\to\infty}\int_E f_N\,dx = \lim_{N\to\infty}\int_{E-V_N} f\,dx = \int_E f\,dx$$

If it is known that f belongs to $L(E)$ then this integral can be regarded as a Lebesgue integral, which follows from Lebesgue's theorem (see property 13) because $f_N \to f$ and $|f_N| \leqslant |f| \in L(E)$. Besides, by virtue of property 11, we have

$$\int_{EV_N} |f|\,dx \to 0, \qquad N \to \infty$$

and consequently

$$\int_{E(V_N - V_{N'})} |f|\,dx \to 0, \qquad N, N' \to \infty,\ N < N'$$

which shows that the Riemann improper integral of f is absolutely convergent. Here it should also be taken into account that the integral $\int\limits_{E(V_N - V_{N'})} f\,dx$ can be regarded both in Lebesgue's and in Riemann's sense.

Conversely, the absolute convergence of the Riemann integral implies the existence of the limit

$$\lim_{N\to\infty} \int_E |f_N|\,dx = \lim_{N\to\infty} \int_{E-V_N} |f|\,dx$$

whence, taking into account that $|f_N| \leqslant |f_{N+1}|$, we conclude, by property 14, that $|f| \in L(E)$. Consequently, $f \in L(E)$ since f is Lebesgue measurable on E.

16. *If for a function $f \in L(E)$ nonnegative almost everywhere on E the equality*

$$\int_E f\,dx = 0 \tag{31}$$

is fulfilled then $f(x) = 0$ almost everywhere on E.

Indeed, let us suppose that there is a subset $e \subset E$ of positive measure on which $0 < f(x) < \infty$. Since the given function f is measurable on e, Theorem 5, § 19.2 guarantees the existence of a subset $e' \subset e$ of positive measure on which $f(x) \geqslant \lambda$ where λ is a positive constant. This inequality implies that

$$\int_E f\,dx \geqslant \int_{e'} f\,dx \geqslant \lambda\mu e' > 0$$

which contradicts equality (31).

17. *Let $f \in L(E)$. Then for any $\varepsilon > 0$ there is a simple function of the form*

$$\varphi(x) = \begin{cases} c_j & \text{for} \quad x \in F_j \subset E \qquad (j = 1, \ldots, N) \\ 0 & \text{for all the other values of } x \in E \end{cases} \tag{32}$$

with a finite number N of different values c_j where F_j $(j = 1, \ldots, N)$ are closed *pairwise disjoint sets, and therefore the inequality*

$$\int_E |f(x)-\varphi(x)|\,dx < \varepsilon \tag{33}$$

is fulfilled. Besides, if $f(x) \geqslant 0$ *then also* $\varphi(x) \geqslant 0$.

Proof. We take the greatest subset $E' \subset E$ $(\mu E' = \mu E)$ on which f is finite and construct the lower simple function

$$\varphi_1(x) = p_k, \quad x \in e_k = \{x\colon\ x \in E,\ p_k \leqslant f(x) < p_{k+1}\}$$

with $p_0 = 0$ and $\delta_R < \dfrac{\varepsilon}{3\mu E}$ (the case $\mu E = 0$ is trivial). Then

$$\int_E |f-\varphi_1|\,dx = \int_{E'} |f-\varphi_1|\,dx < \delta_R \mu E < \frac{\varepsilon}{3}$$

Next we introduce the function

$$\varphi_2(x) = \begin{cases} p_k & \text{for} \quad x \in e_k, \qquad |k| \leqslant N \\ 0 & \text{for all the other values of } x \in E \end{cases}$$

where N is chosen so large that

$$\int_E |\varphi_1-\varphi_2|\,dx = \sum_{|k|>N} |p_k|\,\mu e_k < \frac{\varepsilon}{3}$$

Finally, let us determine closed sets $F_k \subset e_k$ and the function

$$\varphi(x) = \begin{cases} p_k & \text{for} \quad x \in F_k, \quad |k| \leqslant N \\ 0 & \text{for all the other values of } x \end{cases}$$

so that the inequality

$$\int_E |\varphi_2-\varphi|\,dx = \sum_{|k|\leqslant N} |p_k|\,\mu(e_k-F_k) < \frac{\varepsilon}{3}$$

holds.

The function φ obviously satisfies the conditions of the theorem after p_k and F_k have been appropriately renumbered. Since we have put $p_0 = 0$, the function $\varphi(x)$ is nonnegative together with $f(x)$.

18. *Let* $f \in L(G)$ *where G is a bounded open set. Then for any* $\varepsilon > 0$ *there is a simple function*

$$\psi(x) = \begin{cases} a_i & \textit{for} \quad x \in \Delta_i, \qquad i = 1, \ldots, m \\ 0 & \textit{for all the other values of } x \end{cases}$$

finite in G (where $\Delta_i \subset G$ *are pairwise disjoint cubes with faces parallel to the* *coordinate axes) such that*

$$\int_G |f(x)-\psi(x)|\,dx < \varepsilon$$

Proof. To begin with, we determine a simple function φ of type (32) satisfying inequality (33) (see property 17) where we should put $E = G$. Then we construct pairwise disjoint figures $\sigma_k \subset G$ covering F_k and a simple function

$$\psi_1(x) = \begin{cases} c_j & \text{for} \quad x \in \sigma_j \qquad (k = 1, \dots, N) \\ 0 & \text{for all the other values of } x \end{cases}$$

so that the condition

$$\int_G |\varphi(x)-\psi_1(x)|\,dx = \sum_1^N |c_j|\mu(\sigma_j - F_j) < \varepsilon$$

holds, which is possible since the pairwise disjoint sets F_j are bounded and closed and therefore measurable. Each figure σ_k can be regarded as a sum of finite number of cubes with no interior points in common.* Instead of σ_k we can determine figures $\sigma'_k \subset \sigma_k$ each of which is a sum of nonintersecting cubes and then construct a simple function

$$\psi(x) = \begin{cases} c_k & \text{for} \quad x \in \sigma'_k \\ 0 & \text{for all the other values of } x \end{cases}$$

such that the condition

$$\int_G |\psi_1 - \psi|\,dx = \sum_1^N |c_k|\,|\sigma_k - \sigma'_k| < \varepsilon$$

is fulfilled. It is obvious that

$$\int_G |f-\psi|\,dx \leqslant \int_G |f-\varphi|\,dx + \int_G |\varphi-\psi_1|\,dx + \int_G |\psi_1-\psi|\,dx < 3\varepsilon$$

which proves our assertion because the cubes forming all the figures σ'_k can be numbered in a consecutive order.

19. **Fubini's** Theorem.** *For a function* $f(x) = f(x_1, \dots, x_n) = f(x_1, y)$ $(y = (x_2, \dots, x_n))$ *measurable on a cube*

$$\Delta = \{0 \leqslant x_j \leqslant a, \qquad j = 1, \dots, n\}$$

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*This can be achieved by using a rectangular network S breaking up the space R_n into cubes with edges of length 2^{-N} (see § 12.2) for sufficiently large N and putting $\sigma_k = \tilde{\omega}(F_k)$.

** G. Fubini (1879- 1943),an Italian mathematician.

there holds the equality

$$\int_{\Delta} f(x)\,dx = \int_{0}^{a} dx_1 \int_{\Delta'} f(x_1, y)\,dy, \quad \Delta' = \{0 \leqslant x_j \leqslant a,\ j = 2, \ldots, n\} \tag{34}$$

which should be understood in the following sense.

If $f(x) \in L(\Delta)$ then for almost all x_1 there exists the Lebesgue integral

$$\int_{\Delta'} f(x_1, y)\,dy \tag{35}$$

which is a function of x_1 integrable (in Lebesgue's sense) over the closed interval $[0, a]$, and equality (34) holds. Besides, if the function $f(x)$ is measurable and nonnegative on Δ and if integral (35) exists for almost all $x_1 \in [0, a]$ and is in its turn integrable with respect to x_1 over $[0, a]$ then $f \in L(\Delta)$.

An alternative statement of the theorem reads: *let $E \subset R_n$ be a measurable set and let $E(x_1^0)$ be its section by the plane $x_1 = x_1^0$, that is $E(x_1^0)$ is the set of all y's for which $(x_1^0, y) \in E$.* Let $|E(x_1^0)|_*$ *denote the ((n−1)-dimensional) measure of the set $E(x_1^0)$ (provided that the latter is measurable). Finally, let G be the set of those values of x_1 for which $|E(x_1)|_* > 0$.*

Then there holds the equality

$$\int_{E} f(x)\,dx = \int_{G} dx_1 \int_{E(x_1)} f(x_1, y)\,dy \tag{34'}$$

which should be understood in the following sense. If $f \in L(E)$ then G is a measurable set on the line, for almost all $x_1 \in G$ there exists the inner integral on the right-hand side of (34') which is a function integrable with respect to $x_1 \in G$ and equality (34') holds. Besides, if $f(x)$ is a nonnegative measurable function on E (not necessarily everywhere finite) for which the iterated integral on the right-hand side of (34') exists (in the sense indicated above) then $f \in L(E)$.

Proof. The function

$$\varphi_E(x) = \begin{cases} 1 & \text{for} \quad x \in E \\ 0 & \text{for} \quad x \notin E \end{cases} \tag{36}$$

where E is an arbitrary set is called the *characteristic* (or *defining*) *function* of the set E. If $E \subset \Delta$ is a measurable set then obviously

$$\int_{\Delta} \varphi_E(x)\,dx = \int_{E} dx = \mu E$$

As has been said, $|E(x_1^0)|_*$ will denote the $(n-1)$-dimensional measure of the section of E by the plane $x_1 = x_1^0$.

The assertion of the theorem is quite obvious for the characteristic function of a set of the form $\sigma_N = \sum_1^N \Delta_k \subset \Delta \left(\mu\sigma_N = \sum_1^N |\Delta_k|\right)$ consisting of a finite number of cubes (with no interior points in common). In this case equality (34) reduces to the relation

$$\mu\sigma_N = \int_0^a dx_1 \int_{\sigma_N(x_1)} dy = \int_0^a |\sigma_N(x_1)|_* \, dx_1 \tag{37}$$

Let us prove the following lemma.

Lemma 1. *Let f and f_N ($N = 1, 2, \ldots$) be nonnegative functions on Δ (whose values are finite or equal to $+\infty$) such that*

$$f_N \in L(\Delta), \qquad N = 1, 2, \ldots \tag{38}$$

and

$$f_N \to f \tag{39}$$

where the sequence f_N tends to f monotonically. Let Fubini's theorem hold for all N:

$$\int_\Delta f_N \, dx = \int_0^a dx_1 \int_{\Delta'} f_N(x_1, y) \, dy \tag{40}$$

Then the existence of the Lebesgue integral on the left-hand side of (34) *implies the existence of the iterated integral on the right-hand side of* (34) *equal to the former integral and vice versa.*

The lemma is proved with the aid of the following relations (see the explanations below)

$$\int_\Delta f \, dx = \lim_{N\to\infty} \int_\Delta f_N \, dx = \lim_{N\to\infty} \int_0^a dx_1 \int_{\Delta'} f_N(x_1, y) \, dy =$$

$$= \int_0^a dx_1 \lim_{N\to\infty} \int_{\Delta'} f_N(x_1, y) \, dy = \int_0^a dx_1 \int_{\Delta'} f(x_1, y) \, dy \tag{41}$$

which hold under the assumption that any one of the members of (41) exists. The first equality (41) follows from (39) (in case the sequence $\{f_N\}$ is monotone decreasing this is implied by Lebesgue's theorem and if it is increasing then by property 14). The second equality follows from (40) and the third one is again implied by (39) because the integral

$$\int_{\Delta'} f_N(x_1, y) \, dy$$

is a monotone function of N; the fourth equality also follows from (39). Indeed, if the fourth member of (41) exists then the limit

$$\lim \int_{\Delta'} f_N(x_1, y) \, dy \tag{42}$$

exists for almost all $x_1 \in [0, a]$, and, by virtue of (39), it is equal to

$$\int\limits_{\Delta'} f(x_1, y)\, dy \tag{43}$$

Conversely, if the last member in (41) exists then integral (43) exists for almost all $x_1 \in [0, a]$, and therefore, by virtue of (39), it is equal to limit (42). The lemma has been proved.

Lemma 2. *Fubini's theorem holds for the characteristic functions $\varphi_G(x)$ and $\varphi_F(x)$ of an arbitrary bounded open set $G \subset \Delta$ and of an arbitrary bounded closed set $F \subset \Delta$.*

Proof. Indeed, let

$$G = \sum_1^{\infty} \Delta_k \left(|G| = \sum |\Delta_k|_h\right)$$

where Δ_k are closed cubes. As we know, Fubini's theorem holds for the characteristic functions $\varphi_{\sigma_N}(x)$ of the figures $\sigma_N = \sum\limits_1^N \Delta_k$, and consequently, by Lemma 1, it also holds for $\varphi_G(x)$ because $\varphi_{\sigma_N}, \varphi_G \in L(\Delta)$ and the sequence $\{\varphi_{\sigma_N}(x)\}$ is nondecreasing and tends to $\varphi_G(x)$.

It should be noted that the inner integral on the right-hand side of the equality

$$\int\limits_{\Delta} \varphi_G(x)\, dx = \int\limits_0^a dx_1 \int\limits_{\Delta'} \varphi_G(x_1, y)\, dy$$

we have proved exists *for all* $x_1 \in [0, a]$ since the section $G(x_1)$ of G is a bounded open set for any x_1 and therefore it is a measurable (in the $n-1$)-dimensional sense) set.

Now let us consider a closed set F. Let us embed F in an open cube Δ. Then $\Delta - F = G$ is an open set and $\varphi_F(x) = \varphi_\Delta(x) - \varphi_G(x)$. By the obvious additivity properties of the integrals entering into (34), the validity of Fubini's theorem for φ_F follows from its validity for φ_Δ and φ_G.

Lemma 3. *Fubini's theorem holds for the characteristic function $\varphi_e(x)$ of an arbitrary measurable set $e \subset \Delta$. In particular, if $|e| = 0$ then the sections $e(x_1)$ are of $(n-1)$-dimensional measure zero for almost all $x_1 \in [0, a]$. Conversely, if e is measurable and the sections $e(x_1)$ are of $(n-1)$-dimensional measure zero for almost all x_1 then $|e| = 0$.*

Proof. Indeed, let a set $e \subset \Delta$ be measurable. Let us take two sequences

$$G_1 \supset G_2 \supset \ldots \supset e \supset \ldots \supset F_2 \supset F_1$$

of open sets G_j and of closed sets F_j ($j = 1, 2, \ldots$) such that $|G_k|, |F_k| \to |e|$, $k \to \infty$, and put

$$\bigcap_1^{\infty} G_k = \bar{e} \supset e \supset \underline{e} = \bigcup_1^{\infty} F_k$$

where, obviously, $|G_k| \to |\bar{e}| = |e|$ and $|F_k| \to |\underline{e}| = |e|$.

Since, by Lemma 2, Fubini's theorem holds for the functions $\varphi_{G_N}(x)$ and $\varphi_{F_N}(x)$ for any $N = 1, 2, \ldots$, and since these functions are nonnegative and tend monotonically to $\varphi_{\bar{e}}(x)$ and $\varphi_{\underline{e}}(x)$ respectively, Lemma 1 implies that Fubini's theorem also holds for the functions $\varphi_{\bar{e}}(x)$ and $\varphi_{\underline{e}}(x)$ themselves:

$$\int_{\Delta} \varphi_e(x)\, dx = \int_{\Delta} \varphi_{\bar{e}}(x)\, dx = \int_0^a dx_1 \int_{\Delta'} \varphi_{\bar{e}}(x_1, y)\, dy \tag{44}$$

and

$$\int_{\Delta} \varphi_e(x)\, dx = \int_{\Delta} \varphi_{\underline{e}}(x)\, dx = \int_0^a dx_1 \int_{\Delta'} \varphi_{\underline{e}}(x_1, y)\, dy \tag{45}$$

Now, if $|e| = 0$ then all the members of relations (44) are also equal to zero, and therefore the inner integral on the right-hand side of (44) is equal to zero for almost all $x_1 \in [0, a]$. However, we have

$$\varphi_{\bar{e}}(x) \geqslant \varphi_e(x) \geqslant 0$$

and therefore for the indicated values of x_1 there obviously exists the integral $\int_{\Delta'} \varphi_e(x_1, y)\, dy$ which is equal to zero:

$$\int_{\Delta'} \varphi_e(x_1, y)\, dy = 0$$

Consequently, the function $\varphi_e(x_1, y)$ is measurable with respect to y for all such x_1, and the set $e(x_1) = \{y\colon\ \varphi_e(x, y) = 1\}$ is of $(n-1)$-dimensional measure zero: $|e(x_1)|_* = 0$.

Conversely, if e is measurable and if

$$|e(x_1)|_* = \int_{\Delta'} \varphi_e(x_1, y)\, dy = 0$$

for almost all $x_1 \in [0, a]$ then the iterated integral

$$\int_0^a dx_1 \int_{\Delta'} \varphi_e(x_1, y)\, dy$$

exists and is equal to zero, and, by virtue of the inequalities

$$\varphi_e(x) \geqslant \varphi_e(x) \geqslant 0$$

the iterated integral on the right-hand side of (45) also exists and is equal to zero. Therefore the integral on the left-hand side exists and is equal to zero, whence $|e| = 0$.

Now let e be an arbitrary measurable set. Then, by the additivity properties of the iterated integrals on the right-hand sides of (44) and (45), it is allowable to replace in them $\bar{e}$ and $\underline{e}$ by e because $|\bar{e}-e| = |e-\underline{e}| = 0$.

Further, if the iterated integral

$$\int_0^a dx_1 \int_{\Delta'} \varphi_e(x_1, y)\, dy$$

exists, the same argument shows that the integral on the right-hand side of (44) exists and is equal to the former. This completes the proof of Fubini's theorem for $\varphi_e(x)$ where e is an arbitrary measurable set.

Lemma 3 can also be expressed by the following equalities:

$$|e| = \int_0^a |e(x_1)|_* \, dx_1 = \int_\omega |e(x_1)|_* \, dx_1, \qquad \omega = \{x_1\colon\ |e(x_1)|_* > 0\} \tag{46}$$

where $e \subset \Delta$ is an arbitrary measurable set.

Lemma 4. *Fubini's theorem holds for any set e of Lebesgue measure zero and for any function f (finite or infinite or even not defined on e):*

$$\int_e f\, dx = \int_0^a dx_1 \int_{e(x_1)} f(x_1, y)\, dy \tag{47}$$

This lemma is a direct consequence of the foregoing lemma and of the fact that the integral over a set e of measure zero of any function is equal to zero. Indeed, the left-hand member of (47) is equal to zero and, by Lemma 3, we have $|\Delta(x_1)|_* = 0$ for almost all $x_1 \in [0, a]$, which shows that the right-hand member of (47) is also equal to zero.

Now we shall consecutively prove that Fubini's theorem holds in the following cases:

(a) f is a (finite) nonnegative simple function on Δ:

$$f(x) = c_k, \ x \in e_k, \qquad \sum_{k=1}^{\infty} e_k = \Delta, \ e_k e_s = 0 \qquad (k \neq s)$$

Indeed, by virtue of Lemma 3, the function

$$f_N(x) = \begin{cases} c_k & \text{for} \quad x \in e_k, \qquad k = 1, \ldots, N \\ 0 & \text{for all the other values of} \quad x \in \Delta \end{cases}$$

obviously satisfies the conditions of Lemma 1.

Thus, if the nonnegative simple function f belongs to $L(\Delta)$ then there exists the iterated integral on the right-hand side of (34) which is equal to the left-hand member of (34). Conversely, if the indicated iterated integral exists for a nonnegative simple function f then $f \in L(\Delta)$.

(b) f is a finite nonnegative measurable function on Δ. Indeed, let us construct a sequence of partitions R_N ($N = 1, 2, \ldots$) dividing the positive real axis with the aid of the points $p_k^{(N)} = k2^{-N}$ ($k = 0, 1, 2, \ldots$). The lower simple functions $f_N(x)$ corresponding to these partitions obviously satisfy the conditions of Lemma 1.

In the case under consideration (when f is nonnegative and finite) the fact that $f \in L(E)$ implies the existence of the iterated integral on the right-hand side of (34) for this function, and vice versa.

Now we shall prove the theorem for the general case. Let $f \subset L(\Delta)$ and let e be a set (of measure zero) on which f is infinite or is not defined. Let us put

$$f_1(x) = \begin{cases} 0 & \text{for} \quad x \in e \\ f(x) & \text{for} \quad x \in \Delta - e \end{cases} \quad \text{and} \quad f = f_{1+} - f_{1-} + f_2$$

where f_{1+} and f_{1-} are defined for f_1 as usual (see § 19.3 (22)) and f_2 is thus equal to zero almost everywhere on Δ. Taking into account the obvious additivity properties of the integrals entering into (34) and the fact that the theorem holds for f_{1+} and for f_{1-} (see (b) and Lemma 4) we conclude that it also holds for f, that is the iterated integral on the right-hand side of (34) exists and equality (34) holds.

Now let f be a nonnegative measurable function on Δ (generally speaking, not necessarily finite) and let the iterated integral on the right-hand side of (34) exist for f.

Since f is measurable (see the footnote at the beginning of § 19.2), the set e on which f assumes the value $+\infty$ is measurable. Let us put $f = f_1 + f_2$ where f_1 and f_2 have the sense indicated above, that is f_1 is nonnegative and finite on Δ and $f_2 = 0$ outside e. By the hypothesis, the integral $\int_{\Delta'} f(x_1, y)\, dy$ is finite for almost all $x_1 \in [0, a]$, and therefore the sections $e(x_1)$ corresponding to these x_1 are of $(n-1)$-dimensional measure zero. Therefore (see Lemma 3) we have $|e| = 0$, and Fubini's theorem holds for f_2. Consequently the iterated integral on the right-hand side of (34) exists for the function $f_1 = f - f_2$ because it exists for f and for f_2. It follows that (see (b)) $f_1 \in L(\Delta)$, and hence equality (34) holds for f_1; consequently $f \in L(\Delta)$, and equality (34) holds for f.

We have completed the proof of the theorem in its first formulation.

It is obvious that in the case $E = \Delta$ the second statement of the theorem implies the first one. The converse is also true. Indeed, let $f \in L(E)$. On putting

$$\bar{f}(x) = \begin{cases} f(x) & \text{for} \quad x \in E \\ 0 & \text{for} \quad x \in \Delta - E \end{cases} \tag{48}$$

we obtain (see the explanations below)

$$\int_E f(x)\, dx = \int_\Delta \bar{f}(x)\, dx = \int_0^a dx_1 \int_{\Delta'} \bar{f}(x_1, y)\, dy = \int_G dx_1 \int_{E(x_1)} f(x_1, y)\, dy \tag{49}$$

The first equality in (49) follows from the fact that $f \in L(E)$, which implies that E is measurable and $\bar{f} = 0$ on the measurable set $\Delta - E$. The second equality follows from (34). The third one follows from (48) and from the fact that G is a measurable set on the line (Lemma 3) and $E(x_1)$ is measurable in the $(n-1)$-dimensional sense for almost all $x_1 \subset G$. Arguing in the same manner and considering the equalities in (49) in the reverse order we arrive at the second part of the theorem concerning the case when $f \geqslant 0$.

It should be taken into account that the function f may be such that the iterated integral on the right-hand side of (34) makes sense for it even when f does not belong to $L(\Delta)$. Of course, such a function does not retain sign on Δ.

For example, the function

$$f(x, y) = \begin{cases} \dfrac{1}{y^3} & \text{for} \quad x \leqslant |y| \leqslant 1, \ 0 < x \leqslant 1 \\ 0 & \text{at all the other points of } \Delta \end{cases}$$

defined on the rectangle $\Delta = \{-1 \leqslant y \leqslant 1, 0 < x \leqslant 1\}$ possesses the property that its iterated integral (taken over Δ) standing on the right-hand side of (34) is equal to zero whereas $f \notin L(\Delta)$.

20. **Theorem (on the completeness of the space $L_p(E)$).** *Let a sequence of functions $f_k \in L_p(E)$ ($1 \leqslant p < \infty$*) satisfy the condition of Cauchy's criterion in the sense of $L_p(E)$, that is, given any $\varepsilon > 0$, there is $N > 0$ such that*

$$\int_E |f_k - f_l|^p \, dx < \varepsilon, \qquad k, \ l > N \tag{50}$$

Then there exists a function $f \in L_p(E)$ determined uniquely to within its values assumed on a set of Lebesgue measure zero such that

$$\int_E |f - f_k|^p \, dx \to 0, \qquad k \to \infty \tag{51}$$

Proof. It suffices to consider the case when all the functions f_k are finite on E since in the general case the set for which the assertion may not hold is of Lebesgue measure zero.

Let us choose numbers $\varepsilon_s > 0$ ($s = 1, 2, \ldots$) so that the series $\sum_1^\infty \varepsilon_s < \infty$ converges, and, using the conditions of the theorem, take a subsequence $k_1 < k_2 < \ldots$ of the sequence of natural numbers so that

$$\left(\int_E |f_{k_{s+1}} - f_{k_s}|^p \, dx\right)^{1/p} < \varepsilon_s, \qquad s = 1, 2, \ldots \tag{52}$$

The following relations hold under the assumptions stated (see the explanations below):

---

* In the case $p = \infty$ the expression $\left(\int_E |\varphi|^p \, dx\right)^{1/p}$ is regarded as being equal to $\sup_{x \in E} |\varphi(x)|$ (or to $\operatorname{ess\,sup}_{x \in E} |\varphi(x)|$; see p. 333, § 18.3). Then the theorem also holds (and is quite trivial).

$$\sum_N^{\infty} \varepsilon_s \geqslant \lim_{m\to\infty} \sum_N^{N+m} \left(\int_E |f_{k_{s+1}} - f_{k_s}|^p \, dx\right)^{1/p} \geqslant$$

$$\geqslant \lim_{m\to\infty} \left(\int_E \left(\sum_N^{N+m} |f_{k_{s+1}} - f_{k_s}|\right)^p dx\right)^{1/p} =$$

$$= \left(\int_E \left(\sum_N^{\infty} |f_{k_{s+1}} - f_{k_s}|\right)^p dx\right)^{1/p} \geqslant$$

$$\geqslant \left(\int_E \left|\sum_N^{\infty} (f_{k_{s+1}} - f_{k_s})\right|^p dx\right)^{1/p} = \left(\int_E |f - f_{k_N}|^p \, dx\right)^{1/p} \tag{53}$$

where

$$f(x) = \lim_{s\to\infty} f_{k_s}(x) \quad \text{almost everywhere on } E \tag{54}$$

The first relation (53) follows from (52) and the second from Minkowski's inequality (see § 14.2 (12)). The third relation (equality) follows from property 14 because the expression $\left(\sum_N^{N+m} |f_{k_{s+1}} - f_{k_s}|\right)^p$ is nonnegative and does not decrease as m increases, and therefore its limit

$$\left(\sum_N^{\infty} |f_{k_{s+1}} - f_{k_s}|\right)^p \in L(E)$$

is a function on E finite almost everywhere and integrable. The fourth relation (53) (inequality) follows from the inequality

$$\sum_N^{\infty} |f_{k_{s+1}} - f_{k_s}| \geqslant \left|\sum_N^{\infty} (f_{k_{s+1}} - f_{k_s})\right| = \lim_{m\to\infty} \left|\sum_N^{N+m} (f_{k_{s+1}} - f_{k_s})\right| =$$

$$= \lim_{m\to\infty} (f_{k_{N+m}} - f_{k_N}) = f - f_{k_N} \tag{55}$$

where the series under the sign | | in the second member is convergent almost everywhere on E. This guarantees the existence of limit (54) almost everywhere on E, and thus the last relation (equality) in (55) justifies the last relation (equality) in (53).

We have proved the existence of a function f which obviously belongs to $L_p(E)$ for which

$$\left(\int_E |f - f_{k_N}|^p \, dx\right)^{1/p} < \sum_N^{\infty} \varepsilon_s \to 0, \qquad N \to \infty \tag{56}$$

From (50) it also follows that

$$\left(\int_E |f-f_m|^p\,dx\right)^{1/p} \leqslant \left(\int_E |f-f_{k_m}|^p\,dx\right)^{1/p} + \left(\int_E |f_{k_m}-f_m|^p\,dx\right)^{1/p} \to 0$$

$$m \to \infty$$

Finally, if we assume that there is another function $f_* \in L_p(E)$ for which $\int_E |f_*-f_m|^p\,dx \to 0,\ m \to \infty$, then

$$\left(\int_E |f-f_*|^p\,dx\right)^{1/p} \leqslant \left(\int_E |f-f_m|^p\,dx\right)^{1/p} + \left(\int_E |f_m-f_*|^p\,dx\right)^{1/p} \to 0, \quad m \to \infty$$

whence $\int_E |f-f_*|\,dx = 0$, which shows that (see property 16) $f(x) = f_*(x)$ almost everywhere on E.

21. *The relation*

$$\int_E \Big|f- \underset{k\to\infty}{f_k}\Big|^p\,dx \to 0, \qquad f,\ f_k \in L_p(E) \tag{57}$$

implies the existence of a sequence $k_1, k_2, \ldots$ *of natural numbers for which*

$$\lim_{s\to\infty} f_{k_s}(x) = f(x) \text{ almost everywhere on } E. \tag{58}$$

Proof. Since the expression

$$\|f\|_{L_p(E)} = \left(\int_E |f|^p\,dx\right)^{1/p}$$

is the norm in the normed linear space $L_p(E)$ (see § 14.2), (57) implies that $\|f-f_k\|_{L_p(E)} \to 0,\ k \to \infty$. Therefore the condition of Cauchy's criterion holds: given any $\varepsilon > 0$ there is N such that

$$\|f_k-f_l\|_{L_p(E)} \to 0, \qquad k, l > N$$

that is the conditions of property 20 hold. In the proof of this property it was shown that there exists a sequence $\{k_s\}$ of the indicated type for which (54) holds (it should be taken into account that the function f mentioned in property 20 is determined uniquely).

22. *Let there be fulfilled the requirements of Theorem 1,* § 12.16 *on change of variables in a multiple integral on condition that now Ω is an arbitrary bounded domain which thus is Lebesgue measurable (but not necessarily Jordan measurable).*

Then any Lebesgue measurable subset $e \subset \Omega$ is mapped under the operator A (see formula (1′) in § 12.1) *onto a set $e' = Ae \subset \Omega'$ which is also measurable, and there holds the inequality*

$$|e'| \leqslant \varkappa |e| \tag{59}$$

where $\varkappa$ is a constant independent of e.

There also holds the equality

$$\int\limits_{\Omega'} f(x')\,dx' = \int\limits_{\Omega} F(x)\,|D(x)|\,dx \tag{60}$$

(the integrals are understood in Lebesgue's sense) where

$$F(x) = f(Ax),\ D(x) = \frac{D(x_1', \dots, x_n')}{D(x_1, \dots, x_n)}$$

Equality (60) *is valid if at least one of these integrals exists. In particular, if*

$$e_0 = \{x\colon\ D(x) = 0\} \tag{61}$$

then e_0' is of measure zero since

$$\int\limits_{e_0} |D(x)|\,dx = |e_0'| = 0 \tag{62}$$

and, if $E \subset \Omega$ is an arbitrary measurable set, then

$$\int\limits_{E'} f(x')\,dx' = \int\limits_{E} F(x)\,|D(x)|\,dx \tag{63}$$

on condition that at least one of the integrals in (63) *exists.*

Proof. According to § 12.16 (6), we have

$$|\Delta'| = |D(x)|\,|\Delta| + O(\omega(h)\,|\Delta|) \tag{64}$$

where $\Delta \subset \Omega$ is an arbitrary cube, h is the length of its edge, $x \in \Delta$ and $\omega(h)$ is a continuous function of $h \geqslant 0$ such that $\omega(h) \to 0$ $(h \to 0)$, the constant entering into the definition of $O(\omega(h)\,|\Delta|)$ being independent of Δ and x. Since $D(x)$ is bounded on Ω, relation (64) implies the inequality $|\Delta'| \leqslant \varkappa |\Delta|$ where $\varkappa$ is a constant independent of $\Delta \subset \Omega$. Therefore, if $G \subset \Omega$ is an open set, then, on representing it in the form of a countable sum $G = \sum \Delta_k$ $(|G| = \sum |\Delta_k|)$ of cubes Δ_k, we see that the measure $|G'|$ of its image G' under A is

$$|G'| = \sum |\Delta_k'| \leqslant \varkappa \sum |\Delta_k| = \varkappa |G|$$

If $e \subset \Omega$ is measurable then, given any $\varepsilon > 0$, there are closed set F and open set G such that $F \subset e \subset G \subset \Omega$ and $|G-F| < \varepsilon$. Since the set $G-F$ is open we have

$$|G'-F'| = |(G-F)'| \leqslant \varkappa\varepsilon$$

and since F' is closed* while G' is open and $F' \subset e' \subset G'$, we see that e' is measurable. We also see that

$$|e'| = \inf_{e' \subset G'} |G'| \leqslant \varkappa \inf_{e \subset G} |G| = \varkappa |e|$$

and thus relation (59) has been proved.

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* For the case when $D(x) \neq 0$ on Ω see § 7.18; for the general case see § 12.20, Theorem 2.

Now let $f \in L(\Omega')$. There exists (see § 18.2, (iv)) a sequence of continuous functions $f_p(x')$ ($p = 1, 2, \ldots$) finite in Ω' (in the sense that they have compact supports in G) such that

$$\int_{\Omega'} |f(x')-f_p(x')|\,dx' \to 0, \qquad p \to \infty \tag{65}$$

By virtue of Theorem 1, § 12.16, we have the relations

$$\int_{\Omega'} f_p(x')\,dx' = \int_{\Omega} f_p(Ax)\,|D(x)|\,dx \qquad (p = 1, 2, \ldots) \tag{66}$$

and

$$\int_{\Omega} \big|f_p(Ax)\,|D(x)|-f_q(Ax)\,|D(x)|\big|\,dx = \int_{\Omega} |f_p(Ax)-f_q(Ax)|\,|D(x)|\,dx =$$
$$= \int_{\Omega'} |f_p(x')-f_q(x')|\,dx' \to 0 \qquad (p,\ q \to \infty) \tag{67}$$

which imply (see property 20) the existence of a function $\Phi(x) \in L(\Omega)$ such that

$$\int_{\Omega} \big|f_p(Ax)\,|D(x)|-\Phi(x)\big|\,dx \to 0, \qquad p \to \infty \tag{68}$$

Therefore (65), (66) and (67) imply that

$$\int_{\Omega'} f(x')\,dx' = \int_{\Omega} \Phi(x)\,dx \tag{69}$$

It also follows from (65) and (68) that there exists a subsequence of values of p which we renumber in a consecutive order such that the relations

$$\lim_{p\to\infty} f_p(x') = f(x') \quad \text{almost everywhere on } \Omega'$$

and

$$\lim_{p\to\infty} f_p(Ax)\,|D(x)| = \Phi(x) \quad \text{almost everywhere on } \Omega$$

hold.

Let us prove that

$$\Phi(x) = f(Ax)\,|D(x)| \quad \text{almost everywhere on } \Omega \tag{70}$$

This will imply the desired equality (60).

Indeed, let us represent Ω in the form of the sum of three nonintersecting measurable sets e_0, e_1 and e_2:

$$\Omega = e_0+e_1+e_2$$

where the set e_0 has already been defined (see (61)), e_1 consists of those points x for which $D(x) \neq 0$ and $f_p(Ax)\,|D(x)|$ converges to $\Phi(x)$ as $p \to \infty$ while e_2 consists of those points x for which $D(x) \neq 0$ and the convergence mentioned does not take place (this indicates that $|e_2| = 0$).

Equality (70) is quite trivial for the set e_0. On the set e_1 we have $f_p(Ax) \to \to \frac{\Phi(x)}{|D(x)|}$ and, at the same time,

$$f_p(Ax) = f_p(x') \to f(x') = f(Ax)$$

which means that (70) holds. We have thus proved (60) under the assumption that $f(x') \in L(\Omega')$.

If a set $E \subset \Omega$ is measurable then, by virtue of (59), so is the set $E' \subset \Omega'$ $(E' = AE)$, and therefore, if the function $f(x')$ belongs to $L(E')$ then its extension to Ω' determined by the equality $f(x') = 0$, $x' \notin E'$, belongs to $L(\Omega')$ and for the extended function, by what has already been proved, we have

$$\int_{E'} f(x')\,dx' = \int_{\Omega'} f(x')\,dx' = \int_{\Omega} f(Ax)|D(x)|\,dx = \int_E f(Ax)|D(x)|\,dx.$$

Thus, (63) holds.

In particular, for the function $f(Ax)$ which serves as the characteristic function of the set e_0 (that is it is equal to 1 on e_0 and to 0 outside e_0) we obtain from (61) and (63) relation (62).

Now let us prove equality (60) under the assumption that

$$F(x)|D(x)| \subset L(\Omega)$$

Indeed, we have the equality

$$\int_{e_0'} f(x')\,dx' = \int_{e_0} f(x)|D(x)|\,dx \tag{71}$$

because e_0 is measurable and the right-hand side of (71) is equal to zero, and, by virtue of (62), $|e_0'| = 0$, and therefore the left-hand side of (71) is also equal to zero. Let $e_k = \{x: |D(x)| > 1/k\}$ $(k = 1, 2, \dots)$ and $\sum_1^\infty e_k = \Omega_1$. By what has already been proved, the equality

$$\int_{e_k'} |f(x')|\,dx' = \int_{e_k} |f(Ax)\,D(x)|\,dx$$

holds for any $k = 1, 2, \dots$ because the inverse operator A^{-1} $(A^{-1}(x') = x)$ considered on the open set e_k' satisfies the conditions of Theorem 1, § 12.16 (where x and x' should be interchanged).

Since $f(Ax)D(x) \in L(\Omega)$, there exists the limit

$$\int_{\Omega_1} |f(Ax)D(x)|\,dx = \lim_{k\to\infty} \int_{e_k} |f(Ax)\,D(x)|\,dx =$$

$$= \lim_{k\to\infty} \int_{e_k'} |f(x')|\,dx' = \int_{\Omega_1'} |f(x')|\,dx'$$

equal to the integral of $|f(x')|$ over Ω_1' $(|f(x')| \geqslant 0)$, that is $f \in L(\Omega_1')$ and therefore $f \in L(\Omega')$.

§ 19.4. Lebesgue Integral on Unbounded Set

In this section we shall generalize the notion of the Lebesgue integral introduced in the foregoing section. We shall say that a function f defined on an arbitrary set E is *Lebesgue integrable on* E and write $f \in L(E)$ if $f \in L(VE)$ (in the sense of the previous definition stated in the foregoing section) for any ball $V \in R_n$ and if the limit

$$\lim_{N\to\infty} \int\limits_{EV_N} |f(x)|\,dx = \int\limits_{E} |f(x)|\,dx \tag{1}$$

exists for an arbitrary sequence of balls $V_N \subset R_n$ of radius N with centre at the origin.

It is obviously allowable to regard the balls V in this definition both as open and as closed; in both cases the definition stated expresses one and the same notion. It is also obvious that the existence of limit (1) for one of the indicated sequences of balls V_N implies its existence for the other ones and the coincidence of these limits.

The symbol on the right-hand side of (1) is the notation of the Lebesgue integral of the function $|f|$ on E. The integral of the function f on E is defined as the limit

$$\lim_{N\to\infty} \int\limits_{EV_N} f(x)\,dx = \int\limits_{E} f(x)\,dx \tag{2}$$

The existence of (1) implies that of (2) since it follows from (1) that, given any $\varepsilon > 0$, there is N such that for any $n' > n > N$ there holds

$$\left| \int\limits_{EV_{n'}} f\,dx - \int\limits_{EV_n} f\,dx \right| = \left| \int\limits_{E(V_{n'}-V_n)} f\,dx \right| \leqslant \int\limits_{E(V_{n'}-V_n)} |f|\,dx =$$

$$= \left| \int\limits_{EV_{n'}} |f|\,dx - \int\limits_{EV_n} |f|\,dx \right| < \varepsilon$$

These considerations are completely analogous to those establishing the convergence of an absolutely convergent improper Riemann integral.

It is quite clear that if E is a bounded set the definition of a function $f \in L(E)$ stated here coincides with the definition of a Lebesgue integrable function stated in the foregoing section. A more general type of integral appears when E is an unbounded set such that the set EV is measurable for any ball V. Examples of such sets are arbitrary closed and open sets because the intersection of any closed set with a closed ball is a closed (measurable) set and the intersection of an open set with an open ball is a (measurable) open set.

Properties 1-21 of the Lebesgue integral proved in the foregoing section continue to hold for the more general integral defined here. The corresponding assertions can be strengthened in the sense that in view of the new definition a set E of measure zero can be interpreted as one for which the set EV is of measure zero in the previous sense for any ball V (e.g. see § 19.3, properties 1, 2, 4 and 5). Accordingly, in the conditions of the corresponding assertions the expression "a measurable set E" should be replaced by "a set E such that EV is measurable for any ball V" and "a function f measurable on E" by "a function f measurable on EV for any ball V".

A function f measurable on EV for any ball V is called *locally measurable* on E.

The proof of each of the indicated properties reduces to its proof for the sets EV_N for any N (that is for any ball of radius N) and to the demonstration that the property in question continues to hold after the passage to the limit as $N \to \infty$. By the way, property 6 in § 19.3 does not concern the generalizations under consideration.

Below we present some of these proofs as examples.

Property 4. *If $E_1 \subset E$, VE_1 is measurable for any ball V and $f \in L(E)$ then $f \in L(E_1)$.*

Indeed, it follows that $VE_1 \subset VE$, VE_1 is measurable and $f \in L(VE)$, and therefore, by virtue of property 4, § 19.3, we have $f \in L(VE_1)$ for any V. Besides,

$$\lim_{N\to\infty} \int_{V_N E} |f|\, dx < \infty$$

and hence, by virtue of the inequalities

$$\int_{V_N E_1} |f|\, dx \leqslant \int_{V_N E} |f|\, dx \leqslant \int_{E} |f|\, dx$$

and, since the leftmost member of these inequalities does not decrease as N increases, there exists the limit

$$\lim_{N\to\infty} \int_{V_N E_1} |f|\, dx < \infty$$

and hence $f \in L(E_1)$.

Property 8. *If F is measurable on VE for any ball V, $0 \leqslant F(x) \leqslant \Phi(x)$ and $\Phi \in L(E)$ then $F \in L(E)$.*

Proof. By virtue of property 8 in § 19.3, the assertion holds if E is replaced by VE. Consequently, $\Phi \in L(VE)$ for any V. We also have

$$\int_{EV_N} F(x)\, dx \leqslant \int_{EV_N} \Phi(x)\, dx \tag{3}$$

for any N, and, since, by the hypothesis, the limit of the right-hand member of (3) is finite for $N \to \infty$, so is the limit of the left-hand member (because $F(x) \geqslant 0$), whence it follows that $F \in L(E)$.

Property 13. *There holds Lebesgue's theorem* (for the statement of the theorem see property 13 in § 19.3).

Proof. By what was already proved in § 19.3 (property 13) for a measurable set, we obviously have

$$\lim_{k\to\infty} \int_{EV} f_k(x)\,dx = \int_{EV} f(x)\,dx \tag{4}$$

and $f \in L(VE)$ for any ball V; consequently $f \in L(E)$ because $|f| \leqslant \Phi$ on E and $\Phi \in L(E)$.

Let us take an arbitrary $\varepsilon > 0$ and choose $V = V_N$ so that $\int_{E-V} \Phi\,dx < \varepsilon/2$. Then, by virtue of (4), we obtain

$$\left|\int_E f\,dx - \int_E f_k\,dx\right| \leqslant \int_{E-V} |f|\,dx + \int_{E-V} |f_k|\,dx + \left|\int_{EV} (f_k - f)\,dx\right| <$$

$$< 2\int_{E-V} |\Phi|\,dx + \frac{\varepsilon}{2} < \frac{3\varepsilon}{2}, \qquad k > k_0,$$

where k_0 is sufficiently large.

Property 17. For the statement of property 17 see § 19.3.

Proof. Let us choose V so that

$$\int_{E-V} |f|\,dx < \varepsilon$$

and take the function

$$f_1(x) = \begin{cases} 0 & \text{for} \quad x \in E-V \\ f(x) & \text{for} \quad x \in EV \end{cases}$$

for which $\int_{E-V} |f - f_1|\,dx < \varepsilon$. For the latter function there holds property 17 in § 19.3.

Property 18. In the case under consideration the statement of property 18 is the same as in § 19.3 with the only distinction that now G can be an arbitrary open set, not necessarily bounded. The proof is carried out as in the case of property 17.

Property 19 (Fubini's theorem). *Under the requirements stated in property 19, § 19.3 in which f should be considered as a locally measurable function on R_n there holds the equality*

$$\int_{R_n} f\,dx = \int_{-\infty}^{\infty} dx_1 \int_{R'} f(x_1, y)\,dy \tag{5}$$

where R' is the $(n-1)$-dimensional space of the points $y = (x_2, \ldots, x_n)$.

Proof. Let us begin with the case when $f(x) \geqslant 0$ and put

$$\Delta_N = \{|x_j| < N; \quad j = 1, \ldots, n\}$$
$$\Delta'_N = \{|x_j| < N; \quad j = 2, \ldots, n\}, \qquad N = 1, 2, \ldots$$

and

$$f_N(x) = \begin{cases} f(x) & \text{for} \quad x \in \Delta_N \\ 0 & \text{for all the other values of } x \end{cases}$$

By virtue of property 19, § 19.3, for f_N there holds Fubini's theorem:

$$\int_{R_n} f_N\,dx = \int_{\Delta_N} f\,dx = \int_{-N}^{N} dx_1 \int_{\Delta'_N} f(x_1, y)\,dy = \int_{-\infty}^{\infty} dx_1 \int_{R'} f_N(x_1, y)\,dy$$

(the last relation can be understood in both senses indicated in § 19.3). Besides, we have

(i) $f_N \in L(R_n)$, $\quad N = 1, 2, \ldots$
(ii) $f_N \leqslant f$ on R_n,
(iii) the sequence f_N is nondecreasing and converges to f on R_n.

It follows that there holds equality (5). This is implied by Lemma 1, property 19 in § 19.3 in which Δ and Δ' should be replaced by R_n and R' respectively, the course of the proof of the lemma remaining quite analogous.

The case when f assumes values of arbitrary sign is considered as in § 19.3, property 19 by putting $f = f_+ - f_-$.

Property 20. *There holds the theorem on the completeness of the space* $L_p(E)$. The statement of the theorem is the same as in § 19.3, property 20.

Proof. Let $V \subset R_n$ be an arbitrary ball and let $\varepsilon > 0$ be an arbitrary number. By the condition of the theorem, there is $N > 0$ such that

$$\varepsilon > \int_E |f_k - f_l|^p\,dx \geqslant \int_{EV} |f_k - f_l|^p\,dx, \qquad k, l > N$$

By property 20, § 19.3 it follows that for any ball V there is a function f defined on EV and determined uniquely (to within the equivalence in the space R_n) such that

$$\int_{EV} |f_k(x) - f(x)|^p\,dx \to 0$$

Since the ball V is quite arbitrary, we can consider the function f as being defined and determined uniquely (to within the equivalence) throughout R_n. For this function f we have $\varepsilon \geqslant \int_{EV} |f_k - f|^p\,dx$, $k > N$ for any V. It follows that $|f_k - f|^p \in L(E)$,

$$\varepsilon \geqslant \int_E |f_k - f|^p \in dx, \qquad k > N$$

and $f \in L_p(E)$ since $f_k \in L_p(E)$ and $f_k - f \in L_p(E)$.

§ 19.5. Sobolev's Generalized Derivative

Let $G \subset R_n$ be a domain. By D we shall denote the collection of all infinitely differentiable functions φ finite (having compact support) in G.

We shall say that a function f is *locally integrable* on G if it is defined on G and is (Lebesgue) integrable on any closed ball V belonging to G.

According to (Sobolev's) definition, a function F possesses on G *the generalized partial derivative f with respect to the variable x_1* $\left(\text{denoted } f = \frac{\partial F}{\partial x_1}\right)$ if both functions F and f are locally integrable on G and if the condition

$$\int F \frac{\partial \varphi}{\partial x_1} dx = -\int f\varphi \, dx \qquad \text{for all} \quad \varphi \in D \tag{1}$$

holds.

Here we can regard the integrals as taken over the support of the function φ (which is a bounded and closed set) or over the domain G or even over the whole space R_n on condition that F and f are extended to R_n in some way, for instance, by putting $f = F = 0$ on $R_n - G$.

If f is the generalized derivative of F with respect to x_1 on G and if functions f_1 and F_1 are equivalent to f and F respectively, then f_1 is the generalized derivative of F_1 on G with respect to x_1 since a modification (redefinition) of f and F on a set of measure zero does not violate equality (1). The set G can be unbounded.

To justify the definition stated it suffices to indicate that if a function F is continuous on G together with its partial derivative $\frac{\partial F}{\partial x_1} = f$ (understood in the ordinary sense), equality (1) obviously holds for F and, consequently, in this case f is not only the (ordinary) partial derivative but also the generalized partial derivative of F with respect to x_1 on G. For, if φ is a function belonging to D, its support is a bounded closed set belonging to G and it can be covered by a figure $\sigma \subset G$ (consisting of a finite number of cubes). Further, using the classical theorems on the reduction of a multiple (Riemann) integral to an iterated integral, integrating by parts and taking into account that $\varphi = 0$ on the boundary of σ we obtain

$$\int F \frac{\partial \varphi}{\partial x_1} dx = \int\limits_{\sigma} F \frac{\partial \varphi}{\partial x_1} dx = \int dy \int F \frac{\partial \varphi}{\partial x_1} dx_1 = -\int dy \int \frac{\partial F}{\partial x_1} \varphi \, dx_1$$

where $x = (x_1, y) = (x_1, x_2, \ldots, x_n)$. The integrals in the third and fourth members of these equalities are written without the limits of integration in order to simplify the formulas.

Theorem 1. *Let f be the generalized derivative of F with respect to x_1 on G. Then the regularization F_ε* (see § 18.2) *of the function F possesses the properties*

$$\|F_\varepsilon - F\|_{L(e)} \to 0 \quad \text{and} \quad \left\|\frac{\partial}{\partial x} F_\varepsilon - f\right\|_{L(e)} \to 0 \qquad (\varepsilon \to 0) \tag{2}$$

for any bounded and closed set $e \subset G$.

Proof. The first property (2) expresses the ordinary property of a regularization (see § 18.2 (5)):

$$F_\varepsilon(x) = \int \varphi_\varepsilon(x-t)F(t)\,dt$$

Let G_ε be the set of those points x belonging to G which are at a distance not less than $\varepsilon > 0$ from the boundary of G, the number ε being so small that $G_\varepsilon \supset e$. Then for $x \subset e$ we have

$$F'_\varepsilon = \frac{\partial}{\partial x_1}F_\varepsilon(x) = \int \frac{\partial}{\partial x_1}\varphi_\varepsilon(x-t)F(t)\,dx = -\int \frac{\partial}{\partial t_1}\varphi_\varepsilon(x-t)F(t)\,dt =$$

$$= \int \varphi_\varepsilon(x-t)f(t)\,dt = f_\varepsilon(x)$$

where the fifth equality holds since f is the generalized derivative of F with respect to x_1 and since $\varphi_\varepsilon(x-t)$ as function of t for a fixed $x \in e \subset G_\varepsilon$ belongs to the class D. Therefore the second property (2) also holds.

Theorem 1 directly implies the following corollary.

Corollary. *A function F can only have a uniquely determined (to within the equivalence) generalized partial derivative $f = \frac{\partial F}{\partial x_1}$ on G.*

Indeed, we have $f = \lim\limits_{\varepsilon\to 0} F'_\varepsilon$ in the sense of $L(\Delta)$ for any closed cube $\Delta \subset G$.

Theorem 2. *Let F and f be locally integrable functions on a domain G. For f to be the generalized partial derivative $\frac{\partial F}{\partial x_1}$ of the function F on G it is necessary and sufficient that there should exist a sequence of functions Φ_k ($k = 1, 2, \ldots$) continuous on G together with their partial derivatives $\frac{\partial \Phi_k}{\partial x_1}$ and such that for any closed cube $\Delta \subset G$ or, which is the same, for any bounded closed set belonging to G the relations*

$$\|F-\Phi_k\|_{L(\Delta)} \to 0 \quad \text{and} \quad \left\|f-\frac{\partial \Phi_k}{\partial x_1}\right\|_{L(\Delta)} \to 0 \qquad (k \to \infty) \tag{3}$$

hold.

Proof. Let $f = \frac{\partial F}{\partial x_1}$ on G. By virtue of Theorem 1, relations (3) will hold if we put $\Phi_k = F_{1/k}$ where F_ε is the ε-regularization of F.

Conversely, let us suppose that there is a sequence of functions Φ_k continuous on G together with their partial derivatives $\frac{\partial \Phi_k}{\partial x_1}$ such that relations (3) hold for any closed cubes $\Delta \subset G$. Then, obviously, these relations continue to hold when Δ is an arbitrary closed set belonging to G because the latter can be covered by a figure consisting of a finite number of closed cubes belonging to G.

Let us take an arbitrary function $\varphi \in D$. As is already known for the functions Φ_k we have the equalities

$$\int_\sigma \Phi_k \frac{\partial \varphi}{\partial x_1}\, dx = -\int_\sigma \frac{\partial \Phi_k}{\partial x} \varphi \, dx \qquad (k = 1, 2, \ldots) \tag{4}$$

where σ is a figure covering the support of φ. On passing to the limit as $k \to \infty$ in equality (4) we obtain (see the explanations below)

$$\int F \frac{\partial \varphi}{\partial x_1}\, dx = -\int f\varphi \, dx \tag{5}$$

(the domain of integration σ under the integral signs is omitted here). This proves that f is the generalized derivative of F with respect to x_1 on G.

The integrals in (5) exist in Lebesgue's sense because $F, f \in L(\sigma)$ and the functions φ and $\frac{\partial \varphi}{\partial x_1}$ are continuous and bounded $\left(|\varphi|, \left|\frac{\partial \varphi}{\partial x_1}\right| \leqslant M\right)$. Besides, by virtue of (3) (where Δ should be replaced by σ), we have

$$\left|\int \Phi_k \frac{\partial \varphi}{\partial x_1} dx - \int F \frac{\partial \varphi}{\partial x_1} dx\right| \leqslant M \int_\sigma |\Phi_k - F| \, dx \to 0 \qquad (k \to \infty)$$

and

$$\left|\int \frac{\partial \Phi_k}{\partial x_1} \varphi \, dx - \int f \frac{\partial \varphi}{\partial x_1} dx\right| \leqslant M \int_\sigma \left|\frac{\partial \Phi_k}{\partial x_1} - f\right| dx \to 0 \qquad (k \to \infty)$$

and therefore (4) implies (5).

Theorem 3. *Let f* and *F be locally integrable functions on a domain G. For f to be the generalized derivative of F with respect to x_1 on G it is necessary and sufficient that for any closed rectangle $\Delta = \{a_i \leqslant x_i \leqslant b_i;\ i = 1, \ldots, n\}$ belonging to G it should be possible to redefine F, if necessary, on a set of n-dimensional measure zero so that the modified function F is representable in the form*

$$F(x) = F(x_1, y) = \lambda(y) + \int_{a_1}^{x_1} f(t, y)\, dt \tag{6}$$

for almost all $y = (x_2, \ldots, x_n) \in \Delta'$ (in the sense of the $(n-1)$-dimensional measure) where $\Delta' = \{a_i \leqslant x_i \leqslant b_i;\ i = 2, \ldots, n\}$ is the projection of Δ into the coordinate subspace $(x_2, x_3, \ldots, x_n)$ and for any $x_1 \in [a_1, b_1)$, and $\lambda \in L(\Delta')$ is a uniquely determined[*] *function of y.*

In the one-dimensional case when f and F are locally integrable on $(c, d) = G$, for the function f to be the generalized derivative $f(x) = F'(x)$ on (c, d) it is necessary and sufficient that for any closed interval $[a, b] \subset (c, d)$

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[*] If equality (6) holds almost everywhere (in the sense of the n-dimensional measure) for two functions $\lambda_1(y)$ and $\lambda_2(y)$ of the type of $\lambda(y)$, then $\lambda_1(y) = \lambda_2(y)$ almost everywhere in the n-dimensional sense and, consequently, in the $(n-1)$-dimensional sense as well.

there should be a representation

$$F(x) = A + \int_a^x f(t)\,dt, \qquad f \in L(a, b) \tag{6'}$$

where A is a constant.

Proof. Suppose that f is the generalized derivative of F with respect to x_1 on G. Then, by virtue of the foregoing theorem, there is a sequence of functions F_k continuous together with their partial derivatives $\frac{\partial F_k}{\partial x_1}$ such that

$$\|F - F_k\|_{L(\Delta)} \to 0 \quad \text{and} \quad \left\| f - \frac{\partial F_k}{\partial x_1} \right\|_{L(\Delta)} \to 0 \qquad (k \to \infty) \tag{7}$$

for any rectangle Δ of the type indicated in the statement of the theorem.

Let us write the classical equality

$$F_k(x, y) = F_k(a_1, y) + \int_{a_1}^{x_1} \frac{\partial F_k}{\partial x_1}(t, y)\,dt \tag{8}$$

which takes the form

$$F_k(x) = A + \int_a^x F_k'(t)\,dt \tag{8'}$$

in the one-dimensional case. Equalities (8) and (8′) imply (6) and (6′) respectively after the passage to the limit with respect to the metric of $L(\Delta)$ as $k \to \infty$.

To prove what has been said we note that, since, by the hypothesis, the integral $\int_{\Delta'} dy \int_{a_1}^{b_1} |f(t, y)|\,dt = \|f\|_{L(\Delta)}$ exists, the integral

$$\int_{\Delta'} dy \int_{a_1}^{x_1} |f(t, y)|\,dt$$

also exists (see property 4 in § 19.3) and, besides, it is a continuous function of x_1 (see property 11, § 19.3). Therefore, by Fubini's theorem, the three-fold multiple integral

$$\int_{a_1}^{b_1} dx_1 \int_{\Delta_1} dy \int_{a_1}^{x_1} |f(t, y)|\,dt$$

exists and, together with it, the three-fold integral

$$\int_{a_1}^{b_1} dx_1 \int_{\Delta_1} dy \int_{a_1}^{x_1} f(t, y)\,dt$$

also exists. Consequently, by Fubini's theorem,

$$\Phi(x_1, y) = \int_{a_1}^{x_1} f(t, y)\, dt \in L(\Delta)$$

Since the integral $\int_{a_1}^{b_1} f(t, y)\, dt$ exists for almost all $y \in \Delta'$ the function $\Phi(x_1, y)$ exists for these y and for any $x_1 \in [a_1, b_1]$.

In the one-dimensional case the argument is simplified. Namely, the function

$$\Phi(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$ and consequently it belongs to $L(a, b)$.

We have

$$\left\| \int_{a_1}^{x_1} \frac{\partial F_k}{\partial x_1}(t, y)\, dt - \int_{a_1}^{x_1} f(t, y)\, dt \right\|_{L(\Delta)} \leq \int_{a_1}^{b_1} dx_1 \int_{\Delta'} dy \int_{a_1}^{b_1} \left| \frac{\partial F_k}{\partial x_1} - f \right| dt =$$
$$= (b_1 - a_1) \left\| \frac{\partial F_k}{\partial x_1} - f \right\|_{L(\Delta)} \to 0, \qquad k \to \infty \qquad (9)$$

Besides, by the hypothesis, $\|F_k - F\|_{L(\Delta)} \to 0$, $k \to \infty$. Therefore (see (8)), the sequence of the functions $F_k(a_1, y)$ is also convergent in the metric of $L(\Delta)$ and consequently in the metric of $L(\Delta')$ as well, to a function $\lambda(y) \in L(\Delta')$ which results in equality (6) holding for almost all $x \in \Delta$. The right-hand side of (6) is defined for almost all $y \in \Delta'$ and for *any* $x_1 \in [a_1, b_1]$.

In the one-dimensional case we have

$$\left\| \int_a^x F_k'(t)\, dt - \int_a^x f_k(t)\, dt \right\|_{L(a, b)} \leq (b - a) \int_a^b |F_k'(t) - f(t)|\, dt \to 0 \qquad (k \to \infty)$$

and $\lambda(y)$ turns into a constant A, which leads to (6').

We have completed the proof of the necessity.

Now we proceed to the proof of the sufficiency.

Suppose that functions f and F locally integrable on G are such that after F has been appropriately redefined, if necessary, on a set of n-dimensional measure zero, representation (6) holds on any indicated rectangle Δ for almost all $y \in \Delta'$ and for any $x_1 \in [a_1, b_1]$ (in the one-dimensional case the same is assumed for (6')).

Let us determine a sequence of functions $\lambda_k(y)$ continuous on Δ' (in the one-dimensional case $\lambda_k = A$) and another sequence of functions $f_k(x_1, y)$ continuous on Δ such that

$$\|\lambda - \lambda_k\|_{L(\Delta')} \to 0 \quad \text{and} \quad \|f - f_k\|_{L(\Delta)} \to 0 \qquad (k \to \infty) \qquad (10)$$

Let us put

$$F_k(x_1, y) = \lambda_k(y)+\int\limits_a^{x_1} f_k(t, y)\,dt \qquad (k = 1, 2, \dots)$$

It is evident that the functions F_k are continuous on Δ and possess on Δ continuous partial derivatives $\frac{\partial F_k}{\partial x_1} = f_k$, and properties (7) hold (see (9) and (10)). Consequently, by the foregoing theorem, since $\Delta \subset G$ is arbitrary, the function f is the generalized derivative of F with respect to x_1 on G. This completes the proof of the theorem.

A function $\Phi(x)$ of one variable x is said to be *absolutely continuous* on a closed interval $[a, b]$ if it is representable in the form

$$\Phi(x) = A+\int\limits_c^{x} f(t)\,dx, \qquad x \in [a, b] \tag{11}$$

where c is a point belonging to the interval $[a, b]$, A is a constant and $f \in L(a, b)$.

It is clear that such a function Φ is continuous on the interval $[a, b]$ because for $x, x+h \in [a, b]$ we have

$$\Phi(x+h)-\Phi(x) = \int\limits_x^{x+h} f(t)\,dt \to 0 \qquad (h \to 0)$$

(see § 19.3, property 11). The meaning of the constant A is quite simple:

$$A = \Phi(c)$$

because $\int\limits_c^c f\,dt = 0$. If Φ has been represented in the form

$$\Phi(x) = \Phi(c)+\int\limits_c^{x} f(t)\,dt \tag{12}$$

for a given point $c \in [a, b]$, the corresponding representation for another point $c_1 \in [a, b]$ is of the form

$$\Phi(x) = \Phi(c_1)+\int\limits_{c_1}^{x} f(t)\,dt$$

because, by virtue of (12), there must be

$$\Phi(c_1)+\int\limits_{c_1}^{x} f\,dt = \Phi(c)+\int\limits_c^{c_1} f\,dt+\int\limits_{c_1}^{x} f\,dt = \Phi(c)+\int\limits_c^{x} f\,dt$$

identically for all $x \in [a, b]$.

A function f defined on an open set G on the line will be called *locally absolutely continuous* if it is absolutely continuous on every closed interval belonging to G.

If a function $\Phi(x)$ is locally absolutely continuous on an interval (a, b) it can obviously be represented on that interval in the form

$$\Phi(x) = A+\int_c^x f(t)\,dt, \qquad c \in (a, b) \tag{13}$$

where f is locally integrable on (a, b) but not necessarily integrable on the open interval (a, b) itself (see Theorem 3 (6′)).

In the one-dimensional case Theorem 3 can be restated as follows.

Theorem 4. *For a function $f(x)$ to possess the generalized derivative on an open one-dimensional set G (in particular, on an open interval) it is necessary and sufficient that it should be possible to redefine the function $f(x)$, if necessary, on a set of measure zero so that it becomes locally absolutely continuous on G.*

In representation (13) of a locally absolutely continuous function $\Phi(x)$ on an open interval not only the constant A is determined uniquely but also the function $f \in L(a, b)$ (at least to within the equivalence). For, as follows from Theorem 3, the function f is the generalized derivative of $F(F'(x)=f(x))$ and therefore it is determined uniquely to within the equivalence (see the corollary of Theorem 1).

We already noted that a function Φ absolutely continuous on a closed interval $[a, b]$ is continuous on it. An absolutely continuous function on a closed interval $[a, b]$ does in fact possess a stronger property (which, generally speaking, does not hold in the case of an open interval (a, b)!). Namely, given any $\varepsilon > 0$, there is $\delta > 0$ such that for any set

$$\Omega = \bigcup_{j=1}^{N} (x_i, x_i')$$

belonging to $[a, b]$ and consisting of any number of nonintersecting open intervals (x_i, x_i') the sum of whose lengths does not exceed δ, that is

$$\sum_1^N (x_j'-x_j) < \delta$$

there holds the inequality

$$\sum_{j=1}^{N} |\Phi(x_j')-\Phi(x_j)| < \varepsilon$$

Taking into account that (11) implies

$$\sum_{j=1}^{N} |\Phi(x_j')-\Phi(x_j)| \leqslant \sum_{j=1}^{N} \int_{x_j}^{x_j'} |f(t)|\,dt = \int_{\Omega} |f(t)|\,dt$$

we see that this property is a direct consequence of property 11, § 19.3.

It is important to note that this is a characteristic property of an absolutely continuous function in the sense that it can be proved that, conversely, the assumption that a function Φ defined on $[a, b]$ possesses the indicated property implies that Φ is representable on $[a, b]$ by formula (11) in which A is a constant and f is a function defined almost everywhere on $[a, b]$ and belonging to $L(a, b)$.

Thus, we have two equivalent definitions of an absolutely continuous function. Here we shall not present the proof of their equivalence referring the reader to comprehensive courses in the theory of functions of a real variable. The first definition is more frequently used in practice.

In courses in the theory of functions of a real variable it is also proved that an absolutely continuous function Φ determined by formula (11) posseses, for almost all $x \in [a, b]$ an ordinary derivative, equal to $f(x)$:

$$\Phi'(x) = \lim_{h\to 0} \frac{\Phi(x+h)-\Phi(x)}{h} = f(x) \tag{14}$$

This shows that *the generalized derivative of an absolutely continuous function is equivalent to its ordinary derivative.*

Here we do not present the proof of this assertion either and limit ourselves to indicating that equality (14) becomes trivial when x is a point of continuity of the function $f \in L(a, b)$. Indeed, for such a point and for any $\varepsilon > 0$ there is $\delta > 0$ such that $|f(t)-f(x)| < \varepsilon$ when $|t-x| < \delta$ and therefore

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} f(x)\,dt-f(x)\right| = \left|\frac{1}{h}\int_x^{x+h} |f(t)-f(x)|\,dt\right| < \varepsilon$$

for all h with $|h| < \delta$.

Theorem 5. *For absolutely continuous functions F and Φ defined on a closed interval $[a, b]$ there holds the formula for integration by parts:*

$$\int_a^b F(x)\Phi'(x)\,dx = F(b)\Phi(b)-F(a)\Phi(a)-\int_a^b F'(x)\Phi(x)\,dx \tag{15}$$

Proof. Let us put $F' = f$ and $\Phi' = \varphi$ and construct sequences of continuously differentiable functions f_N and φ_N on $[a, b]$ such that (see § 18.2. property (ii))

$$\|f-f_N\|_{L(a,b)} \quad \text{and} \quad \|\varphi-\varphi_N\|_{L(a,b)} \to 0 \qquad N \to \infty$$

By the hypothesis, we have

$$F(x) = A+\int_a^x f(t)\,dt \quad \text{and} \quad \Phi(x) = B+\int_a^x \varphi(t)\,dt, \qquad x \in [a, b]$$

where A and B are constants. Let us put

$$F_N(x) = A+\int_a^x f_N(t)\,dt, \quad \Phi_N(x) = B+\int_a^x \varphi_N(t)\,dt \qquad x \in [a, b]$$

Then

$$|F(x)-F_N(x)| \leqslant \int_a^b |f-f_N|\,dt \to 0, \qquad N \to \infty$$

and

$$|\Phi(x)-\Phi_N(x)| \leqslant \int_a^b |\varphi-\varphi_N|\,dt \to 0, \quad N \to \infty, \qquad x \in [a, b]$$

Consequently F_N and Φ_N tend uniformly on $[a, b]$ to F and Φ respectively as N increases indefinitely. Since f_N and φ_N are the continuous derivatives of F_N and Φ_N respectively, we can use the classical formulas for integration by parts

$$\int_a^b F_N\varphi_N\,dx = F_N(b)\,\Phi_N(b)-F_N(a)\,\Phi_N(a)-\int_a^b f_N\Phi_N\,dx$$

from which, on passing to the limit as $N \to \infty$, we obtain (15). For instance,

$$\left|\int_a^b F\varphi\,dx-\int_a^b F_N\varphi_N\,dx\right| \leqslant \int_a^b |F-F_N|\,|\varphi|\,dx+\int_a^b |F_N|\,|\varphi-\varphi_N|\,dx \leqslant$$

$$\leqslant \int_a^b |\varphi|\,dx \max_x |F(x)-F_N(x)|+K\int_a^b |\varphi-\varphi_N|\,dx \to 0, \qquad N \to \infty$$

Here K is a constant exceeding $|F_N|$ for all N and all x, which is sure to exist because the sequence of the continuous functions F_N converges uniformly.

Theorem 6. *Let f and F be locally integrable functions on a domain G. For f to be the generalized derivative of F with respect to x_1 on G it is necessary and sufficient that the function F, after it has been redefined (if necessary) on a set of n-dimensional measure zero, should be locally absolutely continuous with respect to x_1 on G for almost all points $y = (x_2, \ldots, x_n)$ belonging to the projection G' of the domain G on the plane $x_1 = 0$ and that for the indicated points y the derivative $\frac{\partial F}{\partial x_1}$ should be equal to $f(x_1, y)$ almost everywhere in the one-dimensional sense.*

Proof. Let us make the following convention on the notation: if a set e belongs to G then its projection on the plane $x_1 = 0$ will be denoted e' and its intersection with the straight line parallel to the x_1-axis and passing through the point (x_1, y) will be noted e_y. Further, by a figure σ we shall mean a set consisting of a finite number of rectangles with edges parallel to the coordinate axes.

We shall begin with the proof of the sufficiency of the conditions of the theorem. Let the function F be modified (redefined), if necessary, as is indicated in the statement of the theorem and let $\varphi \in D$. The support of φ is a bounded closed set belonging to G and it can be covered by a figure $\sigma \subset G$. Therefore (see the explanations below) we have

$$\int F \frac{\partial \varphi}{\partial x_1}\, dx = \int_{\sigma'} dy \int_{\sigma_y} F \frac{\partial \varphi}{\partial x_1}\, dx_1 = -\int_{\sigma'} dy \int_{\sigma_y} \frac{\partial F}{\partial x_1} \varphi\, dx_1 =$$

$$= -\int_{\sigma'} dy \int_{\sigma_y} f(x_1, y)\varphi\, dx_1 = -\int f\varphi\, dx \qquad (16)$$

In the first equality (16) we have applied Fubini's theorem to the function $F \frac{\partial \varphi}{\partial x_1} \in L(\sigma)$ (which is equal to zero outside σ). The inner integral with respect to x_1 (considered for only those y for which $F(x_1, y)$ is absolutely continuous) is representable as the sum of the corresponding integrals taken over the closed intervals of which σ_y consists. To the latter integrals we apply integration by parts, which is legitimate by virtue of the foregoing theorem.

The third equality follows from the condition of the theorem according to which $\frac{\partial F}{\partial x_1} = f$ for almost all y and for almost all x_1 (in the one-dimensional sense). Finally, the fourth equality expressing the passage from the consecutive integration to the multiple integration holds by virtue of Fubini's theorem since $f \in L(\sigma)$.

Since (16) holds for any $\varphi \in D$, the sufficiency of the condition of the theorem has been proved.

Now we pass to the proof of the necessity of the condition of the theorem. Let f be the generalized derivative of F with respect to x_1 on G.

Let G be represented in the form

$$G = \sum_1^{\infty} \Delta_k, \quad |G| = \sum_1^{\infty} |\Delta_k|, \quad \Delta_k = \{a_i^{(k)} \leqslant x_i \leqslant b_i^{(k)}, \quad i = 1, \ldots, n\}$$

as a countable sum of cubes with no interior points in common.

By Theorem 3, the function F can be redefined, if necessary, on a set of n-dimensional measure zero so that the modified function can be represented with the aid of equalities

$$\left.\begin{array}{l} F(x) = F(x_1, y) = \lambda_k(y) + \displaystyle\int_{a_1^{(k)}}^{x_1} f(t, y)\, dt \qquad (k = 1, 2, \ldots) \\ x = (x_1, y) \in \Delta_k, \quad y \in \Delta_k' - e_k, \qquad |e_k|_* = 0 \end{array}\right\} \qquad (17)$$

where $\lambda_k(y) \in L(\Delta'_k)$ $(k = 1, 2, \ldots)$ is a function assuming finite values and determined uniquely to within a set of $(n-1)$-dimensional measure zero. Since the ($(n-1)$-dimensional) measure of the set $\sum\limits_1^\infty e_k$ is equal to zero, equalities (17) determine the (modified) function F almost everywhere on G. It is readily seen that F is absolutely continuous with respect to x_1 for every indicated point y within each cube Δ_k, separately. Let us show that the sets e_k can be extended with the retention of the property $|e_k|_* = 0$ so that the function F determined by equalities (17) will be locally absolutely continuous with respect to x_1 on G for all $y \in G'-e$ where $e = \sum\limits_1^\infty e_k$ $(|e|_* = 0)$.

Let us consider two cubes Δ_k and Δ_l with common boundary in the direction of the x_1-axis which is a rectangle of the form

$$\Delta' = \{\alpha_j \leqslant x_j \leqslant \beta_j; \quad j = 2, \ldots, n\}$$

Let us suppose that Δ_l lies to the right of Δ_k. We shall be interested in those parts $\Delta_* \subset \Delta_k$ and $\Delta_{**} \subset \Delta_l$ which have the common projection Δ' indicated above. Thus, let

$$\Delta_* = [\alpha, \beta]\times\Delta', \quad \Delta_{**} = [\beta, \gamma]\times\Delta'$$

and

$$\alpha = a_1^{(k)}, \ \beta = b_1^{(k)} = a_1^{(l)}, \ \gamma = b_1^{(l)}, \quad \alpha < \beta < \gamma$$

We have

$$F(x_1, y) = \left\{ \begin{array}{lll} \lambda_k(y) + \int\limits_\alpha^{x_1} f(t, y)\, dt & \text{on } \Delta_*, & y \in \Delta' - e_k \\ \lambda_l(y) + \int\limits_\beta^{x_1} f(t, y)\, dt & \text{on } \Delta_{**}, & y \in \Delta' - e_l \end{array} \right\} \tag{18}$$

We can also apply Theorem 3 to the rectangle $\Delta = \Delta_* + \Delta_{**}$ and thus establish the existence of a function F_0 coinciding with F almost everywhere on Δ and of a function $\lambda(y) \in L(\Delta')$ such that

$$F_0(x_1, y) = \lambda(y) + \int\limits_\alpha^{x_1} f(t, y)\, dt \quad \text{on } \Delta, \qquad y \in \Delta' - e_{kl} \tag{19}$$

$$|e_{kl}|_* = 0$$

Since $F = F_0$ almost everywhere on Δ_*, we have $\lambda_k(y) = \lambda(y)$ almost everywhere on Δ' in the sense of the $(n-1)$-dimensional measure, and, since $F_0 = F$ almost everywhere on Δ_{**}, the equality

$$\lambda_l(y) = \lambda(y) + \int\limits_\alpha^\beta f(t, y)\, dt$$

holds almost everywhere on Δ' in the sense of the $(n-1)$-dimensional measure. This shows that the sets e_{kl} can be widened with the retention of their measures $[e_{kl}]_*$ so that the identities $F \equiv F_0$ and $F \equiv F_0$ hold for almost all $y \in \Delta'-e_{kl}$ on Δ_* and on Δ_{**} respectively, and hence the function F_0 and, together with it, the function F, are absolutely continuous on Δ with respect to x_1 for the indicated y.

Now let

$$e = \sum e_{kl}, \quad |e|_* = 0$$

where the sum is extended over all the possible pairs (k, l) for which Δ_l has a common boundary with Δ_k on the right of Δ_k. It can readily be seen that the function F is locally absolutely continuous with respect to x_1 on G for all $y \in G'-e$.

It should be noted that, since Theorems 2, 3 and 6 provide necessary and sufficient conditions for a function f to be the generalized derivative of F with respect to x_1 on G, the conditions of these theorems can be taken as the original definitions of the generalized derivative.

We have defined the generalized derivative $\frac{\partial F}{\partial x_1}$ with respect to x_1. The generalized derivatives $\frac{\partial F}{\partial x_j}$ $(j = 2, \ldots, n)$ are defined by analogy with $\frac{\partial F}{\partial x_1}$.

The generalized derivatives of higher order are defined by induction. For instance, the second generalized derivative

$$\frac{\partial^2 F}{\partial x_1\,\partial x_2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x_2}\right)$$

is understood as the generalized derivative with respect to x_1 of the generalized derivative $\frac{\partial F}{\partial x_2}$.

It should be mentioned that if a given function F possesses on G the generalized derivatives $\frac{\partial F}{\partial x_1}$, $\frac{\partial F}{\partial x_2}$ and $\frac{\partial^2 F}{\partial x_1\,\partial x_2}$ then it follows automatically that the generalized derivative $\frac{\partial^2 F}{\partial x_2\,\partial x_1}$ also exists and is equal almost everywhere to $\frac{\partial^2 F}{\partial x_1\,\partial x_2}$. For, taking an arbitrary function $\varphi \in D$ we can write

$$\int \frac{\partial^2 F}{\partial x_1\,\partial x_2}\varphi\,dx = -\int \frac{\partial F}{\partial x_2}\frac{\partial \varphi}{\partial x_1}dx = \int F\frac{\partial^2 \varphi}{\partial x_2\,\partial x_1}dx = \int F\frac{\partial^2 \varphi}{\partial x_1\,\partial x_2} =$$
$$= -\int \frac{\partial F}{\partial x_1}\frac{\partial \varphi}{\partial x_2}dx$$

which shows that the locally integrable functions $\frac{\partial^2 F}{\partial x_1\,\partial x_2}$ and $\frac{\partial F}{\partial x_1}$ satisfy the equality

$$\int \frac{\partial^2 F}{\partial x_1\,\partial x_2}\varphi\,dx = -\int \frac{\partial F}{\partial x_1}\frac{\partial \varphi}{\partial x_2}dx$$

This exactly means that the former is the generalized derivative with respect to x_2 of the latter, that is $\frac{\partial^2 F}{\partial x_1\,\partial x_2} = \frac{\partial^2 F}{\partial x_2\,\partial x_1}$ almost everywhere on G.

These properties can easily be generalized to mixed partial derivatives of higher order.

§ 19.6. Space D' of Generalized Functions

In this section we shall again deal with the space D of (real or complex) infinitely differentiable functions φ finite (i.e. having compact support) in a domain $\Omega \subset R_n$. Our aim will be to define the space D' of the generalized functions, that is continuous linear functionals, on D by analogy with the definition of the space S' corresponding to S studied in § 16.7.

We shall say that a sequence $\{\varphi_k\}$ of function $\varphi_k \in D$ is *convergent to* $\varphi \in D$ *in the sense of* D (or, simply $\{\varphi_k\}$ *is* (D)*-convergent to* φ) and write

$$\varphi_k \to \varphi \quad (D)$$

if the supports of the functions φ_k belong to one and the same compact (i.e. a bounded closed set) $F \subset \Omega$ and if the limiting relation

$$\lim_{k\to\infty} \varphi_k^{(s)}(x) = \varphi^{(s)}(x), \qquad \varphi^{(0)} = \varphi$$

holds uniformly on R_n for any partial derivative $\varphi^{(s)}$ of order $s = (s_1, \ldots, s_n)$.

We shall refer to the set D as the *space* D (because its definition includes the notion of convergence or, as we say, a topology is defined on D).

A *generalized function* $f \in D'$ is defined as a continuous linear functional (f, φ) on D, that is a functional for which the following two conditions are fulfilled:

(i) $f(\alpha\varphi+\beta\psi) = \alpha f(\varphi)+\beta f(\psi)$ for any (real or complex) numbers α and β and for any functions $\varphi, \psi \in D$ $\big(f(\varphi) = (f, \varphi)\big)$.

(ii) $f(\varphi_k) \to f(\varphi)$, $\varphi_k \to \varphi$ (D).

It is obvious that $D \subset S$ where S is the space of infinitely differentiable functions on R_n (see its definition in § 16.6) tending to zero at infinity together with their partial derivatives faster than any power of $|x|$. Besides, the convergence $\varphi_k \to \varphi\,(D)$ is at the same time the convergence $\varphi_k \to \varphi\,(S)$. Therefore, a continuous linear functional f on S when regarded as defined only for $\varphi \in D$ is automatically a continuous linear functional on D. In this connection we say that the former functional on S *induces* the latter on D. In this sense it is allowable to consider S' as a part of D', i.e. $S' \subset D'$.

For example, the δ-function defined as the linear functional $(\delta, \varphi) = \varphi(0)$, $\varphi \in S$, induces the functional $(\delta, \varphi) = \varphi(0)$, $\varphi \in D$, which is also called the δ-function (defined on D).

The derivative of $f \in D'$ of order $k = (k_1, \ldots, k_n)$ is defined as the functional $(f^{(k)}, \varphi) = (-1)^{|k|}(f, \varphi^{(k)})$, $\varphi \in D$. But the same equality, when regarded for all $\varphi \in S$, determines the derivative $f^{(k)}$ of $f \in S'$. Consequently, if $f \in S'$ then the derivative $f^{(k)}$ in the sense of the space D can be defined as a functional induced on D by the functional $f^{(k)}$ defined in the sense of the space S.

If $f(x)$ is a locally integrable function on Ω it *represents the (regular) generalized function* $f \in D'$ by means of the integral

$$(f, \varphi) = \int f(x)\varphi(x)\,dx, \qquad \varphi \in D$$

The continuity of this functional follows from the fact that if $\varphi_k \to \varphi\,(D)$ then $\varphi_k \to \varphi$ uniformly on R_n, and there exists a compact $F \subset \Omega$ containing the supports of φ and of all the functions φ_k, and therefore

$$|(f, \varphi)-(f, \varphi_k)| \leqslant \int\limits_F |f(x)|\,|\varphi(x)-\varphi_k(x)|\,dx \leqslant$$
$$\leqslant \max_x |\varphi(x)-\varphi_k(x)| \int\limits_F |f|\,dx \to 0, \qquad k \to \infty$$

Now we proceed to the following important theorem.

Theorem 1. *Two locally integrable functions f_1 and f_2 on Ω represent one and the same functional if and only if they are equal to each other almost everywhere on Ω.*

Proof. Let us put $f(x) = f_1(x)-f_2(x)$. It is evident that the proof of the theorem reduces to showing that for the equality

$$\int f(x)\varphi(x)\,dx = 0 \quad \text{for all} \quad \varphi \in D \tag{1}$$

to be fulfilled it is necessary and sufficient that

$$f(x) \equiv 0 \quad \text{for almost all} \ \ x \in \Omega \tag{2}$$

The derivation of (1) from (2) is quite trivial. To prove the converse we shall use the following lemma.

Lemma. *If a function $\psi(x)$ is locally measurable* and bounded on an open set G, then there exists a sequence of infinitely differentiable functions $\varphi_k \in D$ finite in G satisfying the conditions*

$$\lim_{k\to\infty} \varphi_k(x) = \psi(x) \quad \textit{almost everywhere on } G \tag{3}$$

and

$$|\varphi_k(x)| \leqslant \sup_{x\in G} |\psi(x)| \tag{4}$$

Indeed, let us denote by G_N the intersection of G with the ball $|x| \leqslant N$ and let $\{\eta_N\}$ be a sequence of positive numbers tending to zero. Since $\psi \in L(G_N)$, there is a function ψ_N infinitely differentiable and finite in G_N (and, consequently, in G as well) such that (see § 18.2, property (iv))

$$||\psi-\psi_N||_{L(G_N)} \leqslant \eta_N \tag{5}$$

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* This means that $\psi(x)$ is measurable on VG where V is an arbitrary ball.

and

$$|\psi_N(x)| \leqslant \sup_{x\in G} |\psi(x)| \tag{6}$$

It follows from (5) and (6) (see § 19.3, property 21)* that the sequence $\{\psi_N\}$ contains a subsequence for which the assertions of the lemma are fulfilled.

Now let us suppose that (1) holds. Let us put

$$\psi(x) = \operatorname{sgn} f(x) \tag{7}$$

and let $G \subset \bar{G} \subset \Omega$ be an arbitrary bounded open set. Let us choose for ψ a sequence of infinitely differentiable functions φ_k finite in G such that conditions (3) and (4) of the lemma are fulfilled. Since $\varphi_k \in D$, we see that, by virtue of (1), $\int f(x)\varphi_k(x)\,dx = 0$ $(k = 1, 2, \ldots)$, and therefore (see the explanations below)

$$0 = \lim_{k\to\infty} \int_G f(x)\varphi_k(x)\,dx = \int_G f(x)\psi(x)\,dx = \int_G |f(x)|\,dx \tag{8}$$

that is $f(x) = 0$ almost everywhere on G and, by the arbitrariness of G, almost everywhere on Ω as well. We have thus proved (1).

In the second member of (8) it is allowable to integrate only over G because the support of φ_k belongs to G. The passage to the third member is carried out on the basis of Lebesgue's theorem which is applicable because, by virtue of (4) and (7) and by the local integrability of f, we have

$$|f(x)\varphi_k(x)| \leqslant |f(x)| \leqslant L(G)$$

In conclusion we state the following theorem.

Theorem 2. *Let f and F be two locally integrable functions on Ω (and hence $f, F \in D'$). If*

$$f(x) = \frac{\partial F}{\partial x_1} \quad \textit{on } \Omega \textit{ in Sobolev's sense} \tag{9}$$

then

$$f = \frac{\partial F}{\partial x_1} \quad \textit{in the sense of the space } D \tag{10}$$

and vice versa.

---

* Property 21, § 19.3 implies directly that, since $\|\psi-\psi_N\|_{L(G_1)} < \eta_N$, there is a (first) subsequence $N_1^1, N_2^1, \ldots$, such that $\psi_{N_k^1} \to \psi$ almost everywhere on Ω_1. Further, we have $\left\|\psi-\psi_{N_k^1}\right\|_{L(G_2)} < \eta_{N_k^1}$, and the first subsequence contains a second subsequence N_1^2, $N_2^2, \ldots$ for which $\psi_{N_k^2} \to \psi$ almost everywhere on G_2. Continuing this process indefinitely we arrive at the collection of the subsequences $N_1^k, N_2^{(k)}, \ldots$ $(k = 1, 2, \ldots)$ each of which is a subsequence of the preceding subsequence. It is readily seen that for the diagonal subsequence $N_1^1, N_2^2, \ldots$ we have $\psi_{N_k^k} \to \psi$ almost everywhere on G.

Indeed, for the locally integrable functions F and f in question defined on Ω both equalities (9) and (10) are equivalent to the following relation:

$$\int f(x)\varphi(x)\,dx = -\int F(x)\,\frac{\partial\varphi}{\partial x_1}\,dx \qquad \text{for all } \varphi \in D$$

§ 19.7. Incompleteness of Space L'_p

Let us number the rational points belonging to the open interval (0, 1) as $(r_1, r_2, \ldots)$, cover the kth point r_k $(k = 1, 2, 3, \ldots)$ of this sequence by an open interval σ_k, with centre at r_k, belonging to (0, 1) and having the length less than $\varepsilon 2^{-k}$ $(0 < \varepsilon < 1)$. Let G denote the sum $\sum_1^\infty \sigma_k$ which is an open set belonging to (0, 1) and let

$$f(x) = \begin{cases} 1 & \text{for } x \in G \\ 0 & \text{for } x \in [0, 1]-G = F \end{cases}$$

that is let f coincide with the characteristic function f_G of the set G. Further, let f_1 be a function equivalent to f (i.e. equal to f almost everywhere).

The Lebesgue measure of the open set G satisfies the inequality

$$\mu G \leqslant \sum_1^\infty |\sigma_k| < \varepsilon < 1$$

Consequently, the complement F of G (with respect to (0, 1)) which is a closed set is of a positive Lebesgue measure. Let us denote by e the set (of Lebesgue measure zero) on which f and f_1 differ and put $G' = G-e$ and $F' = F-e$. If $x \in F'$ then $f_1(x) = 0$. Any neighbourhood σ of x contains a rational point r_k which is covered by the open interval σ_k. The intersection $\sigma\sigma_k$ ($|\sigma\sigma_k| > 0$!) is sure to contain points of G' at which $f_1 = 1$. This indicates that f_1 is discontinuous on the set F' of positive measure, and therefore, by Lebesgue's theorem (see § 12.10), f_1 is Riemann nonintegrable on [0, 1]. The function f_1 does not belong to L' (0, 1) either since not only the bounded (Riemann integrable) functions belonging to L' (0, 1) but also the unbounded functions belonging to this class are continuous almost everywhere on [0, 1]. This readily follows from the fact that if a function $f_1 \in L'$ (0, 1) is unbounded, there is only a finite number of points $x_0, \ldots, x_N$ on [0, 1] such that on any open interval (a, b) belonging to [0, 1] and not containing any of the points $x_0, \ldots, x_N$ the function f_1 is Riemann integrable (in the proper sense).

We see that *the function f in question can serve as an example of a function bounded on* [0, 1], *nonintegrable in Riemann's sense and possessing the property that any function f_1 equivalent to it does not belong to L'* (0, 1).

It should also be noted that since $f = f_G$ is the characteristic function of the set G *the latter is an example of a bounded open set which is Jordan nonmeasurable and F can obviously serve as an example of a bounded closed set nonmeasurable in Jordan's sense.*

But these sets G and F are of course Lebesgue measurable since any open or closed bounded set possesses this property. Consequently the function f is Lebesgue integrable on (0, 1) and its integral is equal to μG:

$$\int_G f\,dx = \int_G f_1\,dx = \mu G$$

Let $f_k(x)$ $(k = 1, 2, \ldots)$ be the characteristic function of the set $\omega_k = \sigma_1 + \ldots + \sigma_k$. It is apparent that

$$\int_0^1 |f(x)-f_k(x)|\,dx = \int_{G-\omega_k} |f(x)|\,dx = \mu(G-\omega_k) \to 0 \qquad (k \to \infty)$$

Hence, the condition of Cauchy's criterion holds in this case, that is given any $\eta > 0$, there is N such that

$$\int_0^1 |f_k(x)-f_l(x)|\,dx < \eta \qquad k,\ l > N \tag{1}$$

All the integrals above were understood in Lebesgue's sense. Integrals (1) can also be understood in Riemann's sense because the functions $f_k(x)$ are Riemann integrable on [0, 1].

Thus, the situation is as follows: the sequence of the functions $f_k \in L'(0, 1)$ satisfies the condition of Cauchy's criterion in the sense of the space $L(0, 1)$ and at the same time there is no function $\varphi \in L'(0, 1)$ to which f_k tends in the sense of $L(0, 1)$ as $k \to \infty$.

For, if such a function φ existed, it would belong to the space $L(0, 1)$ as well (see § 19.3, property 20) and therefore, by virtue of the uniqueness (to within the equivalence) of the function f for which

$$\int_0^1 |f-f_k|\,dx \to 0, \qquad k \to \infty$$

(the integral is understood in Lebesgue's sense), the function φ should be equal to one of the functions f_1 equivalent to f. We have thus arrived at a a contradiction because $f_1 \notin L'(0, 1)$.

Hence, we have proved that *the space* $L'(0, 1)$ *is not complete*. Let the reader prove, using the same function f and arguing in like manner, that *the space* $L'_p(0, 1)$ *is not complete either.*

§ 19.8. Generalization of Jordan Measure

We shall deal with (bounded) half-open rectangles (rectangular parallelepipeds)

$$\Delta = \{c_j \leq \xi < d_j; \qquad j = 1, \ldots, n\}$$

and with (half-open) figures representable as finite sums

$$\sigma = \sum_{j=1}^{N} \Delta^j$$

of pairwise disjoint half-open rectangles.

We shall sometimes simply speak of figures σ meaning that they are half-open. In the cases when we shall have to consider closed or open figures (rectangles) we shall always make the necessary stipulations.

Let

$$G = \{a_j \leqslant \xi_j < b_j; \qquad j = 1, \ldots, n\}$$

be a fixed rectangle and let to each rectangle $\Delta \subset G$ (including the rectangle G itself) there correspond a nonnegative number $\alpha(\Delta)$ so that $\alpha(\Delta)$ is an additive function of Δ in the sense that if

$$\Delta = \sum_{j=1}^{N} \Delta^j$$

where Δ^j are pairwise nonintersecting rectangles, then

$$\alpha(\Delta) = \sum_{j=1}^{N} \alpha(\Delta^j)$$

We shall consider the function $\alpha(\Delta)$ as being extended to arbitrary $\Delta \subset R_n$ by putting

$$\alpha(\Delta) = \alpha(G\Delta) \tag{1}$$

In other words, we put $\alpha(\Delta) = 0$ for any rectangle Δ not intersecting G. The extended function $\alpha(\Delta)$ is obviously nonnegative and additive.

Let us extend $\alpha(\Delta)$ to all the figures $\sigma \subset G$ by putting

$$\alpha(\sigma) = \sum_{j=1}^{N} \alpha(\Delta^j) \qquad \text{for} \quad \sigma = \sum_{j=1}^{n} \Delta^j$$

where Δ^j are pairwise disjoint rectangles.

It is readily seen that $\alpha(\sigma)$ is a nonnegative additive function of $\sigma \subset G$. Thus,

$$\alpha(\sigma') + \alpha(\sigma'') = \alpha(\sigma' + \sigma'')$$

for any nonintersecting figures $\sigma'\sigma'' \subset G$.

An empty set O will also be regarded as a figure, and we shall put $\alpha(O) = 0$. Then for any figure $\sigma \subset G$ we have

$$\alpha(\sigma + O) = \alpha(\sigma) = \alpha(\sigma) + \alpha(O)$$

If σ and σ' are some figures then obviously $\sigma' - \sigma$ is also a figure, and if $\sigma \subset \sigma'$ then

$$0 \leqslant \alpha(\sigma' - \sigma) = \alpha(\sigma') - \alpha(\sigma)$$

whence $\alpha(\sigma) \leqslant \alpha(\sigma')$.

Let us agree that the relation $A \subset\subset B$ means that $\bar{A} \subset \mathring{B}$ where $\mathring{B}$ is the interior of B, that is the collection of all interior points of B. Thus, if $\bar{A} \subset \mathring{B}$ then the boundaries of A and B do not intersect.

Let $\Omega \subset G$ be an arbitrary set.

We shall define the *interior (inner) measure of* Ω *(with respect to* $\alpha(\Delta)$ *in Jordan's sense)* as the supremum of $\alpha(\sigma)$ taken over all $\sigma \subset\subset \Omega$:

$$m_i\Omega = \sup_{\sigma \subset\subset \Omega} \alpha(\sigma)$$

It should be noted that in the case the ordinary Jordan measure ($\alpha(\Delta) = |\Delta|$) we have the equality

$$\sup_{\sigma \subset \Omega} |\sigma| = \sup_{\sigma \subset\subset \Omega} |\sigma|$$

which holds even if σ are understood as closed figures (instead of being half-open as it was assumed in the definition of the Jordan interior measure; see § 12.5). In the general case of an arbitrary measure $\alpha(\Delta)$ this equality may no longer hold (see Examples 1 and 2 below).

Since $0 \leqslant \alpha(\sigma) \leqslant \alpha(G)$ for all $\sigma \subset\subset \Omega$, the (finite) number $m_i\Omega$ exists and is nonnegative.

Of course, if Ω does not contain any rectangle except the empty set then $m_i\Omega = 0$.

By the *exterior (outer) measure of* Ω *(with respect to* $\alpha(\Delta)$ *in Jordan's sense)* we shall mean the infimum of $\alpha(\sigma)$ taken over all $\sigma \subset\subset \Omega$:

$$m_e\Omega = \inf_{\sigma \subset\subset \Omega} \alpha(\sigma)$$

If the closure $\bar{\Omega}$ of Ω has points in common with the boundary of G, then, to make this definition correct, we agree that $\alpha(\Delta)$ is extended to all $\Delta \subset R_n$ as was indicated in (1).

Since $\alpha(\sigma) \geqslant 0$ for any σ, the number $m_e\Omega$ exists for any $\Omega \subset G$ and is nonnegative.

It is obvious that $m_i\Omega \leqslant m_e\Omega$. In the case when this inequality becomes an exact equality we shall say that the set Ω is *measurable (with respect to* $\alpha(\Delta)$ *in Jordan's sense)* and call the number

$$m\Omega = m_i\Omega = m_e\Omega$$

the *measure* of Ω.

The measure thus defined turns into the ordinary Jordan measure when $\alpha(\Delta)$ is equal to the n-dimensional volume $|\Delta|$ of Δ. The basic properties of the generalized Jordan measure are completely analogous to the properties of the ordinary measure and their proofs are also quite analogous (see § 12.5). We shall enumerate these properties below.

If Ω_1 *and* Ω_2 *are measurable then the sets* $\Omega_1 \pm \Omega_2$ *and* $\Omega_1\Omega_2$ *are also measurable; if* Ω_1 *and* Ω_2 *do not intersect then* $m(\Omega_1 + \Omega_2) = m\Omega_1 + m\Omega_2$*; thus, if* $\Omega_1 \supset \Omega_2$ *then* $m(\Omega_1 - \Omega_2) = m\Omega_1 - m\Omega_2$.

A set Ω *is measurable if and only if its boundary is of (Jordan) measure zero (with respect to* $\alpha(\Delta)$*).*

We see that, like in the case of the ordinary Jordan measure, in order to find whether a given set is measurable it is sufficient to establish that its boundary Γ is of measure zero in the sense of $\alpha(\Delta)$. In the case of the ordinary Jordan measure the answer to this question is completely determined by the geometrical properties of Γ. In the general case this is not always so.

Example 1. Suppose that a mass equal to 2 (units of mass) is distributed over the closed interval [0, 1] in the following way. Half the mass (equal to 1) is distributed uniformly along [0, 1] while the other half is concentrated at the point $x = 1/2$. Let the function $\alpha(\Delta)$ defined for all $\Delta \subset [0, 1]$ (where Δ is a half-open interval) be equal to the mass carried by Δ. In other words,

$$\alpha(\Delta) = \begin{cases} |\Delta| & \text{if } \frac{1}{2} \notin \Delta \\ |\Delta| + 1 & \text{if } \frac{1}{2} \in \Delta \end{cases}$$

where $|\Delta|$ is the length of Δ.

It can easily be seen that any point $x \in [0, 1]$ (regarded as a single-point set) distinct from the point $x = 1/2$ is of measure zero in the sense of $\alpha(\Delta)$ whereas the set e consisting of that single point $x = 1/2$ is nonmeasurable (relative to $\alpha(\Delta)$) because $m_i e = 0$ and $m_e e = 1$. It is evident that the interior measure of the closed interval [1/2, 1] is equal to $m_i\,[1/2, 1] = \sup\limits_{\Delta \subset\subset [1/2, 1]} |\Delta| = 1$ while $\sup\limits_{\Delta \subset [1/2, 1]} \alpha(\Delta) = \alpha([1/2, 1]) = 3/2$.

Example 2. Let a mass of 3 units be distributed over the square $\bar{G} = \{0 \leqslant x, y \leqslant 1\}$ as is described below. Let unit mass be distributed uniformly over G and another unit mass be uniformly distributed over the line segment γ of the straight line $x = 1/2$ belonging to G. Finally, let the third unit mass be concentrated at the point (1/2, 1/2). We shall define the function $\alpha(\Delta)$ as assuming the value equal to the mass carried by Δ. In other words,

$$\alpha(\Delta) = \\ = \begin{cases} |\Delta| & \text{if } \Delta\gamma = 0 \\ |\Delta| + |\Delta'| & \text{if } \Delta\gamma \neq 0 \text{ and } \Delta \text{ does not contain the point } \left(\frac{1}{2}, \frac{1}{2}\right) \\ |\Delta| + |\Delta'| + 1 & \text{in all the other cases} \end{cases}$$

Here $|\Delta|$ denotes the area of Δ and $|\Delta'|$ denotes the length of the projection of Δ on the y-axis.

In this example every single-point set except the point (1/2, 1/2) is of measure zero (with respect to $\alpha(\Delta)$). Further, any line segment belonging to γ is nonmeasurable. On the other hand, as is readily seen, any continuous curve Γ described by an equation $y = \varphi(x)$ $(0 \leqslant x \leqslant 1)$ where $\varphi(x)$ satisfies the conditions $0 \leqslant \varphi(x) \leqslant 1$ and $\varphi(1/2) \neq 1/2$ is of measure zero (relative to $\alpha(\Delta)$) despite the fact that it intersects γ.

These examples show that a nonnegative additive function $\alpha(\Delta)$ can be such that G contains separate points and curves which are nonmeasurable sets relative to $\alpha(\Delta)$ in Jordan's sense.

For an arbitrary but fixed value of i ($i = 1, \ldots, n$) let us consider the rectangle

$$\Delta_t^i = \{\xi a_i \leqslant \xi_i < t; \quad a_j \leqslant \xi_j < b_j, \qquad j \neq i\}$$

and the function

$$\beta_i(t) = \alpha(\Delta_t^i), \qquad a_i < t \leqslant b_i, \ \beta_i(a_i) = 0$$

Let us fix $t \in (a_i, b_i)$ and denote by $G_i(t)$ the section of G by the plane $x_i = t$. Let $a_i < t' < t < t'' < b_i$. If we choose a rectangle $\Delta \supset\supset G$, the figure $\sigma = (\Delta - G) + (\Delta_{t''}^i - \Delta_{t'}^i)$ will contain $G_i(t)$ strictly inside itself, and, according to the convention on the extension of $\alpha(\Delta)$ (see (1)), we shall have $\alpha(\Delta - G) = 0$. Therefore

$$\alpha(\sigma) = \alpha(\Delta_{t''}^i) - \alpha(\Delta_{t'}^i) = \beta_i(t'') - \beta_i(t') \tag{2}$$

The limit of the left-hand side of (2) and, together with it, the limit of the right-hand side of (2) for $t'' - t' \to 0$ is equal to zero if and only if the measure of the section $G_i(t)$ is $m(G_i(t)) = 0$ or, which is the same, if the function β_i is continuous at the point t. In this case we shall say that $G_i(t)$ is a *regular* section of G ($a_i < t < b_i$!).

A finite system of regular sections (generally speaking, taken for different $i = 1, \ldots, n$) determines a partition of G into pairwise nonintersecting rectangles which we shall call a regular partition.

Since the functions $\beta_i(t)$ are nondecreasing and therefore have at most countable number of points of discontinuity (provided there are such), for any $\varepsilon > 0$ there is a regular partition of G with cells (rectangles) of diameter less than ε.

In the one-dimensional case when $G = \{a \leqslant \xi < b\}$ we have

$$\Delta_x = \{a \leqslant \xi < x\}, \quad \beta(x) = \alpha(\Delta_x), \quad a < x \leqslant b, \quad \beta(a) = 0 \tag{3}$$

Therefore, if $\Delta = \{x \leqslant \xi < y\} \subset G$ is an arbitrary half-open interval, then

$$\alpha(\Delta) = \beta(y) - \beta(x) \tag{4}$$

Thus, *every nonnegative additive function* $\alpha(\Delta)$, $\Delta \subset G$ *specifies, with the aid of equalities* (3), *an ordinary nondecreasing, nonnegative function* $\beta(x)$ *on* $[a, b]$ *such that for any* $\Delta \subset G$ *there holds relation* (4).

Conversely, it is evident that to any nondecreasing function $\beta(x)$ with $\beta(a) = 0$ there corresponds the nonnegative additive function $\alpha(\Delta)$, $\Delta \subset G$, determined by equalities (3) and (4).

In the two-dimensional case we have $G = \{a \leqslant \xi < b, \ c \leqslant \eta < d\}$. In that case it is expedient to consider half-open rectangles

$$\Delta_{xy} = \{a \leqslant \xi < x, \qquad c \leqslant \eta < y\}$$

and the function $F(x, y)$ specified by the relations

$$F(x, y) = \alpha(\Delta_{xy}), \quad a < x \leqslant b, \quad c < y \leqslant d \tag{5}$$

and

$$F(a, y) = F(x, c) = 0, \quad a \leqslant x \leqslant b, \quad c \leqslant y \leqslant d \tag{5'}$$

This function obviously possesses the property

$$0 \leqslant F(x, y) \leqslant F(x', y'), \quad x \leqslant x', \quad y \leqslant y' \tag{6}$$

Besides, if

$$\Delta = \{x \leqslant \xi < x', \quad y \leqslant \eta < y'\}$$
$$\Delta_1 = \{x \leqslant \xi < x', \quad 0 \leqslant \eta < y\}$$

and

$$\Delta_2 = \{0 \leqslant \xi < x, \quad y \leqslant \eta < y'\}$$

(where $x > a$ and $y > c$) then

$$\Delta_{x'y'} = \Delta_{xy} + \Delta_1 + \Delta_2 + \Delta$$

where the rectangles on the right-hand side are pairwise disjoint, and therefore

$$\alpha(\Delta_{x'y'}) = \alpha(\Delta_{xy} + \Delta_1) + \alpha(\Delta_{xy} + \Delta_2) - \alpha(\Delta_{xy}) + \alpha(\Delta)$$

It follows that we have the equality

$$\alpha(\Delta) = F(x', y') - F(x', y) - F(x, y') + F(x, y) \tag{7}$$

which holds, by virtue of (5′), for $x = a, c \leqslant y \leqslant d$ and for $y = c, a \leqslant x \leqslant b$.

Consequently,

$$F(x', y') - F(x', y) - F(x, y') + F(x, y) \geqslant 0 \tag{8}$$

(where $x \leqslant x'$ and $y \leqslant y'$).

Thus, *a nonnegative additive function* $\alpha(\Delta)$ *specifies, with the aid of equalities* (5), (5′) *and* (7), *a function* $F(x, y)$ *satisfying conditions* (6) *and* (8). *Conversely, it is obvious that a function* $F(x, y)$ *possessing properties* (5′), (6) *and* (8) *determines, with the aid of* (5) *and* (7), *a nonnegative additive function* $\alpha(\Delta)$.

If F is continuous at points (x, y), (x', y), (x, y') and (x', y'), then for any $\varepsilon > 0$ there are Δ' and Δ'', $\Delta' \subset\subset \Delta \subset\subset \Delta''$ such that $\alpha(\Delta'' - \alpha(\Delta') < \varepsilon$ and then the rectangle Δ (half-open or closed or open) is a measurable set in the sense of $\alpha(\Delta)$.

We note that if F has the continuous partial derivatives F'_x, F'_y and F''_{xy} on $\bar{G}$, then the right-hand side of (8) can be written (see § 7.7 (3)) in the form

$$F''_{xy}(x' - x)(y' - y)$$

where the second derivative F''_{xy} is taken at some interior point of the open rectangle $\{x < \xi < x', y < \eta < y'\}$.

These considerations can be extended by analogy to the n-dimensional case when the role of expression (8) is played by the nth finite difference $\Delta^n F = \Delta_1 \Delta_2 \dots \Delta_n F$ which results from the consecutive application to $F(x)$ of the operations Δ_i of forming the first finite difference with respect to the variables x_i with steps $x_i' - x_i$. In this case instead of F''_{xy} we shall have the mixed partial derivative $F^{(n)}_{x_1 \dots x_n}$.

§ 19.9. Riemann-Stieltjes Integral

Let $\Omega \subset G$ be a measurable set (with respect to $\alpha(\Delta)$, in Jordan's sense) and let $f(x)$ be a function defined on that set. With each partition ϱ of the set Ω $\left(\Omega = \sum\limits_{j=0}^{N-1} \Omega_j\right)$ into a finite number of measurable parts having no interior points in common (i.e. these parts can only intersect along some parts of their boundaries) we shall associate the corresponding (Riemann-Stieltjes) sum

$$S_\varrho(f) = \sum_{j=0}^{N-1} f(\xi^j) m\Omega_j, \qquad \xi^j \in \Omega_j \tag{1}$$

By the *Riemann-Stieltjes integral of f on Ω (with respect to $\alpha(\Delta)$)* is meant the limit

$$\lim_{\max d(\Omega_j) \to 0} S_\varrho(f) = \int_\Omega f(x)\, dm \tag{2}$$

when the maximum diameter $d(\Omega_j)$ tends to zero.

As usual, it is supposed that this limit is independent of the choice of the sequence of partitions and of the points $\xi^j \in \Omega_j$.

If $\alpha(\Delta)$ is the n-dimensional volume of Δ, that is $\alpha(\Delta) = |\Delta|$, then (2) becomes the Riemann integral of f on Ω.

Generally speaking, an unbounded function may be integrable in the Riemann-Stieltjes sense. *If for any $\varepsilon > 0$ there is a partition of Ω into parts of positive measure and diameter less than ε then, as in the case of the Riemann integral* (see Theorem 1 in § 12.11), *the integrability of $f(x)$ on Ω (with respect to $\alpha(\Delta)$) implies the boundedness of the function $f(x)$ on Ω.*

The basic properties of the Riemann-Stieltjes integral are completely analogous to the corresponding properties of the Riemann integral. As in the case of the latter, we can construct for a bounded function $f(x)$ its *upper* and *lower integral sums*

$$\bar{S}_\varrho = \sum_0^{N-1} M_j m\Omega_j \quad \text{and} \quad \underline{S}_\varrho = \sum_0^{N-1} m_j m\Omega_j$$

where

$$m_j = \inf_{x \in \Omega_j} f(x) \quad \text{and} \quad M_j = \sup_{x \in \Omega_j} f(x)$$

and, proceeding from them, construct the *upper* and the *lower* integrals $\overline{J}$ and $\underline{J}$. The corresponding theorems are extended to the general case without any changes in their proofs.

There also holds Lebesgue's theorem (see §§ 12.8 and 12.10): *for a bounded function f defined on a closed measurable set (with respect to $\alpha(\Delta)$, in Jordan's sense) Ω to be integrable in the Riemann-Stieltjes sense it is necessary and sufficient that its points of discontinuity should form a set e of Lebesgue measure zero (with respect to $\alpha(\Delta)$), that is* $\mu e = 0$ (see § 19.12).

The notion of the Riemann-Stieltjes integral can also be generalized to the so-called functions $\gamma(\Delta)$ of *bounded variation.* Each such function is a *difference of two nonnegative additive functions $\alpha_1(\Delta)$ and $\alpha_2(\Delta)$ of $\Delta \subset G$*: $\gamma(\Delta) = \alpha_1(\Delta) - \alpha_2(\Delta)$.

If m_1 and m_2 are the measures (in Jordan's sense) generated by the nonnegative additive functions $\alpha_1(\Delta)$ and $\alpha_2(\Delta)$ respectively, then the *Riemann-Stieltjes integral with respect to $\gamma(\Delta)$* is defined as the difference of the corresponding integrals:

$$\int_\Omega f(x)\,d\gamma = \int_\Omega f(x)\,dm_1 - \int_\Omega f(x)\,dm_2 \tag{3}$$

Here, of course, by Ω is meant a set measurable (in Jordan's sense) both with respect to $\alpha_1(\Delta)$ and to $\alpha_2(\Delta)$.

The Stieltjes integral (see below) can be generalized in the same manner to functions $\gamma(\Delta)$ of bounded variation (see § 19.10 where m_1, m_2 and Ω should be replaced by α_1, α_2 and G respectively). The same applies to the Lebesgue-Stieltjes integral (see § 19.12 where m_1 and m_2 should be replaced by μ_1 and μ_2 respectively).

§ 19.10 Stieltjes Integral

In this section we shall define the classical Stieltjes integral and investigate its connection with the Riemann-Stieltjes integral. The domain G for which the Stieltjes integral will be defined is a rectangle.

Let a function $f(x)$ be defined on the closure $\overline{G}$ of a half-open rectangle

$$G = \{a_i \leqslant x_i < b_i; \quad i = 1, \ldots, n\} \tag{1}$$

where a nonnegative additive function $\alpha(\Delta)$ of $\Delta \subset G$ is defined (Δ are half-open rectangles).

An arbitrary rectangular network splits G into half-open rectangles which we shall number in a consecutive order by one index: $\Delta_1, \ldots, \Delta_{N-1}$.

The Stieltjes integral of the function $f(x)$ on G is defined as the limit

$$\lim_{\max d(\Delta^j)\to 0} \sum_0^{N-1} f(\xi^j)\,\alpha(\Delta_j) = \int_G f(x)\,d\alpha, \qquad \xi^j \in \overline{\Delta}_j \tag{2}$$

This definition differs from that of the Riemann-Stieltjes integral; here the set $\Omega = G$ and the cells Δ_j entering into its partitions are rectangles not necessarily measurable (with respect to $\alpha(\Delta)$, in Jordan's sense).

The expressions $\alpha(\Delta_i)$ in (2) can of course be written in terms of an ordinary function generating the given nonnegative additive function $\alpha(\Delta)$. For instance, in the one-dimensional case we have

$$\left.\begin{array}{ll}\alpha(\Delta) = \beta(y)-\beta(x), & \Delta = [x, y),\ a \leqslant x < y \leqslant b \\ \beta(x) = \alpha(\Delta_x), & \Delta_x = [a, x),\ \beta(a) = 0\end{array}\right\} \qquad (3)$$

where $\beta(x)$ is a nonnegative and nondecreasing function defined on $\bar{G} = [a, b]$. If we put $a = x_0 < x_1 < \ldots < x_N = b$, $\Delta_j = [x_j, x_{j+1})$ then limit (2) can be written in the following equivalent form:

$$\lim_{\max(x_{j+1}-x_j)\to 0} \sum_{0}^{N-1} f(\xi^j)\,[\beta(x_{j+1})-\beta(x_j)] = \int_a^b f(x)\,d\beta(x) \qquad (4)$$

By the way, form (4) is used more frequently than (2) and was introduced by Stieltjes himself.

In the two-dimensional case the corresponding equivalent expression for integral (2) is written in terms of the function

$$\left.\begin{array}{ll}F(x, y) = \alpha(\Delta_{xy}), & \Delta_{xy} = \{a \leqslant \xi < x,\ c \leqslant \eta < y\} \\ F(x, c) = F(a, y) = 0, & (x, y) \in \bar{G}\end{array}\right\} \qquad (5)$$

possessing the property

$$\alpha(\Delta) = F(x', y')-F(x, y')-F(x', y)+F(x, y) \geqslant 0$$
$$\Delta = \{x \leqslant \xi < x', y \leqslant \eta < y'\}$$

Thus, if

$$a = x_0 < x_1 < \ldots < x_M = b$$
$$c = y_0 < y_1 < \ldots < y_N = d$$
$$\Delta_{ij} = \{x_i \leqslant \xi < x_{i+1},\ y_j \leqslant \eta < y_{j+1}\}, \qquad (\xi_i, \eta_j) \in \bar{\Delta}_{ij}$$

and

$$\alpha(\Delta_{ij}) = F(x_{i+1}, y_{j+1})-F(x_i, y_{j+1})-F(x_{i+1}, y_j)+F(x_i, y_j)$$

hen limit (2) can be written in the form

$$\lim_{\max[(x_{i+1}-x_i)+(y_{j+1}-y_j)]\to 0} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} f(\xi_i, \eta_j)\,\alpha(\Delta_{ij}) = \int_a^b \int_c^d f(x, y)\,d^2F(x, y) \qquad (6)$$

Let us consider a bounded function $f(\mathbf{x})$ on $\bar{G} \subset R_n$ and take an arbitrary partition ϱ of the form

$$G = \sum_{0}^{N-1} \Delta_j$$

where Δ s are half-open pairwise nonintersecting rectangles. For this partition we form the *Stieltjes sum*

$$S_\varrho = \sum_{i=0}^{N-1} f(\xi^i)\alpha(\Delta_i) \qquad (\xi^i \in \bar{\Delta}_i)$$

the lower and upper sums

$$\underline{S}_\varrho = \sum_{i=0}^{N-1} m_i\alpha(\Delta_i) \quad \text{and} \quad \bar{S}_\varrho = \sum_{i=0}^{N-1} M_i\alpha(\Delta_i)$$

and the lower and upper integrals

$$\underline{J} = \sup_\varrho \underline{S}_\varrho \quad \text{and} \quad \bar{J} = \inf_\varrho \bar{S}_\varrho$$

Theorem 1. *For a bounded function* $f(x)$ $(|f(x)| \leq K)$ *defined on a rectangle* $\bar{G}$ *the following properties are equivalent.*

(ii′) *For any* $\varepsilon > 0$ *there is a regular partition* ϱ_* *of the rectangle G (into pairwise disjoint half-open rectangles* Δ_j^**) such that*

$$\bar{S}_{\varrho_*} - \underline{S}_{\varrho_*} < \varepsilon$$

(iii′) *For any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that for all the partitions (both regular and irregular) such that* $d(\Delta_j) < \delta$ *there holds*

$$\bar{S}_\varrho - \underline{S}_\varrho < \varepsilon$$

(iv′) *Stieltjes integral* (2) *exists.*

Proof. The implication (ii′) → (iii′). Let Γ_* be the union of the boundaries of all Δ^*'s from which the boundary of $\bar{G}$ is deleted. Since ϱ_* is a regular partition, we have $m\Gamma_* = 0$, and therefore there are figures $\varrho \supset\supset \varrho' \supset\supset \Gamma_*$ such that $d(\sigma) < \frac{\varepsilon}{2k}$. Let δ be the distance between the boundaries of σ and σ' and let ϱ be a partition of G into cells (rectangles) ω of diameter less than δ (we do not supply these rectangles ω with indices). We have

$$\bar{S}_\varrho - \underline{S}_\varrho = \sum{}'(M-m)\alpha(\omega) + \sum{}''(M-m)\alpha(\omega)$$

where the sum $\sum'$ is extended over all those ω's each of which has a point in common with Γ_* and the sum $\sum''$ is extended over the remaining ω's.

The final part of the proof (the derivation of the estimation $\bar{S}_\varrho - \underline{S}_\varrho < \varepsilon$) is similar to what was said in Theorem 1, § 12.7 in the proof of the implication (ii) → (iii) where ω should be replaced by $\alpha(\omega)$.

The implication (iii′) → (ii′) is trivial.

The implication (iv′) → (iii′). Here the argument is analogous to that used in the proof of the implication (iv) → (iii) in Theorem 1, § 12.7; Ω_j should be replaced by Δ_j in that argument and J should be understood as integral (2).

The implication (iii′) → (iv′). The existence of the Stieltjes integral implies the inequality

$$|J - S_\varrho| < \frac{\varepsilon}{2}$$

that is

$$J-\frac{\varepsilon}{2} < S_\varrho < J+\frac{\varepsilon}{2}$$

for all ϱ with $d(\Delta_i) < \delta$. Therefore we also have $J-\frac{\varepsilon}{2} \leq \underline{S}_\varrho \leq \bar{S}_\varrho \leq J+\frac{\varepsilon}{2}$ whence $\bar{S}_\varrho - \underline{S}_\varrho < \varepsilon$.

Remark. Theorem 1 no longer holds if the word "regular" is deleted from (ii′) (see Example 1 below). This shows that it is impossible to introduce in the statement of Theorem 1 property (i′) which expresses the fact that $\underline{J} = \bar{J}$ (see the proof of the implications (i) $\rightleftarrows$ (ii) in Theorem 1, § 12.7). From (ii′) follows (i′) but (i′) does not imply (ii′).

Theorem 2. *For a function $f(x)$ continuous on $\bar{G}$ there exists its Stieltjes integral* (2).

Indeed, let $\omega(t)$ be the modulus of continuity of $f(x)$ on $\bar{G}$. Then if $\delta = \max_i d(\Delta_i)$ where $G = \sum \Delta_i$ is a partition ϱ of the rectangle G, then

$$\bar{S}_\varrho - \underline{S}_\varrho = \sum (M_i - m_i)\,\alpha(\Delta_i) \leq \omega(\delta) \sum \alpha(\Delta_i) = \omega(\delta)\,\alpha(G) < \varepsilon, \qquad \delta < \delta_0$$

provided that δ_0 is sufficiently small. Therefore the existence of Stieltjes integral (2) follows from Theorem 1.

Theorem 3. *If x^0 is a point of discontinuity of a bounded function $f(x)$ on $\bar{G}$ and the single-point set v consisting of that point is of positive outer measure* ($m_e(v) > 0$) *then Stieltjes integral* (2) *does not exist.*

Indeed, the oscillation $\omega(x^0)$ of the function f at the point x^0 is positive (see Theorem 5, § 7.11). Let ϱ be a partition of G into rectangles Δ_j such that x^0 is strictly inside some Δ_i. Then

$$\bar{S}_\varrho - \underline{S}_\varrho = \sum_j (M_j - m_j)\,\alpha(\Delta_j) \geq (M_i - m_i)\,\alpha(\Delta_i) \geq \omega(x^0)\, m_e(v)$$

where the right-hand side is positive and is independent of the indicated partition, and, since the latter can consist of rectangles Δ_i of arbitrarily small diameter, Theorem 1 shows that integral (2) does not exist.

Example 1. Let

$$f(x) = \begin{cases} 0 & \text{for} \quad -1 \leq x < 0 \\ 1 & \text{for} \quad 0 \leq x \leq 1 \end{cases} \quad \text{and} \quad \beta(x) = \begin{cases} 0 & \text{for} \quad -1 \leq x \leq 0 \\ 1 & \text{for} \quad 0 < x \leq 1 \end{cases}$$

Let us denote by ϱ the (irregular) partition $-1 < 0 < 1$ of the closed interval $\bar{G} = [-1, +1]$. For this partition we have $\underline{S}_\varrho = \bar{S}_\varrho = 1$, and therefore $\underline{J} = \bar{J}$. However, the Stieltjes integral $\int_{-1}^{1} f(x)\, d\beta(x)$ does not exist because $x = 0$ is a point of discontinuity of both f and β (see Theorem 4).

The implication (ii′) → (iii′) in Theorem 1 can also be expressed as follows.

Theorem 4. *If for a sequence of regular partitions ϱ_k there exists the limit*

$$J=\lim_{k\to\infty} S_{\varrho_k}=\lim_{k\to\infty}\sum_{j=1}^{N_k} f(\xi_k^j)\,\alpha(\Delta_j^k) \tag{7}$$

for any choice of $\xi_k^j \in \bar{\Delta}_j^k$ then there exists Stieltjes integral (2) *equal to J* (see § 12.7, Theorem 2).

Indeed, (7) implies that $\bar{S}_{\varrho_k}-\underline{S}_{\varrho_k}\to 0$ $(k\to\infty)$, i.e. (ii′). But we have the implication (ii′) → (iii′), and consequently integral (2) exists and is obviously equal to J.

Theorem 5. *Stieltjes integral* (2) *and Riemann-Stieltjes integral* (*see* (2) *n the foregoing section*) *of a bounded function $f(x)$ over a measurable rectangle G exist simultaneously and are equal to each other.*

Indeed, let us take a sequence of regular partitions ϱ_k of the rectangle G into cells (rectangles) Δ_j^k with $\max_j d(\Delta_j^k)\to 0$ $(k\to\infty)$. All Δ_j^k's, even those adjoining the boundary of G, are measurable now since the rectangle G is measurable.

The existence of limit (7) is necessary and sufficient for the existence of both Stieltjes integral (Theorem 4) and Riemann-Stieltjes integral (by virtue of the analogue of Theorem 2, § 12.7).

Hence, for a measurable rectangle G the properties of the Riemann-Stieltjes integral discussed in § 19.9 are extended automatically to the Stieltjes integral. In particular, for a measurable rectangle G there holds the following theorem.

Theorem 6 (Lebesgue). *For Stieltjes integral* (2) *of a bounded function $f(x)$ defined on $\bar{G}$ to exist it is necessary and sufficient that the Lebesgue measure* (*in the sense of $\alpha(\Delta)$*) *of the set e of the points of discontinuity of f be equal to zero, that is $\mu_e=0$.*

Theorem 6 does in fact hold for a nonmeasurable G as well. It implies as special cases Theorems 2 and 3. Let us prove this proposition.

To this end we take an arbitrary measurable rectangle $G'\supset\supset G$ and extend the function $\alpha(\Delta)$ to G' as was indicated in (1), § 19.8.

We shall also extend the function f to G' by using the following construction. Let ξ' be a point belonging to $G'-G$. The line segment joining this point with the centre of G meets the boundary Γ of the rectangle G at a point which we shall denote by ξ. Let us put $f(\xi')=f(\xi)$. The function f thus extended is constant on the segments (belonging to $G'-G$) of the rays issued from the centre of G.

The extended function f is obviously bounded on G'. Besides, the function f is continuous at a point ξ belonging to the boundary of G if it is continuous at that point relative to $\bar{G}$. In other words, if for any $\varepsilon>0$ there is $\delta>0$ such that the inequality $|f(x)-f(\xi)|<\varepsilon$ holds for all $x\in\bar{G}$ satisfying the condition $|x-\xi|<\delta$ then there is $\delta_1>0$ such that $|x-\xi|<\delta_1$ implies $|f(x)-f(\xi)|<\varepsilon$ for all such x. If $\Delta'\subset\bar{G}'$ is a rectangle intersecting Γ and

$\xi' \in \bar{\Delta}'$ then there is a point ξ belonging to $\bar{\Delta}$ where $\Delta = G\Delta'$ such that

$$f(\xi')\alpha(\Delta') = f(\xi)\alpha(\Delta) \tag{8}$$

Indeed, we have $\alpha(\Delta') = \alpha(\Delta)$ (see (1) § 19.8). If $\xi' \in \Delta$ we put $\xi = \xi'$, and if $\xi' \overline{\in} \Delta$ then we take as ξ the point of intersection with Γ of the line segment connecting ξ' with the centre of G.

If ϱ' is an arbitrary regular partition of the rectangle G', it induces a regular partition ϱ of the rectangle G, and with each integral sum $S_{\varrho'}$ we can associate an integral sum S_ϱ equal to the former, that is

$$S_{\varrho'} = \sum f(\xi')\alpha(\Delta') = \sum f(\xi)\alpha(\Delta) = S_\varrho$$

by using the following construction. If Δ' does not intersect G we delete the corresponding summands entering into $S_{\varrho'}$ (they are equal to zero). If Δ' intersects G then we apply equality (8) to the corresponding summands.

Conversely, given a regular partition ϱ of the rectangle G, we extend the regular sections of G (lying in the $(n-1)$-dimensional planes) of which ϱ consists and delete the faces of G, which results in a regular partition ϱ' of the rectangle G'. Repeating this argument in the reverse order (even without moving the points ξ) we conclude that to each integral sum S_ϱ there corresponds a sum $S_{\varrho'}$ equal to the former.

What has been said implies the equality of the Stieltjes integrals over G and over G':

$$\int_G f(x)\,d\alpha = \int_{G'} f(x)\,d\alpha \tag{9}$$

(the existence of one of them implies that of the other).

The second integral in (9) is simultaneously a Riemann-Stieltjes integral (since G' is measurable) and therefore it exists if and only if the set e' of the points of discontinuity of f on $\bar{G}'$ is of Lebesgue measure zero (with respect to $\alpha(\Delta)$), that is

$$\mu e' = 0 \tag{10}$$

or, which is the same, if the set e of the points of discontinuity of f on $\bar{G}$ is of Lebesgue measure zero (relative to $\alpha(\Delta)$):

$$\mu e = 0 \tag{11}$$

Indeed, the existence of the integral on the left-hand side of (9) implies the existence of that on the right-hand side and equality (10) and, consequently, it also implies (11) since $e \subset e'$. Conversely, (11) implies that for any $\varepsilon > 0$ there is a countable system of rectangles $\Delta_1, \Delta_2, \ldots$ such that $\sum \Delta_j \supset e$ and $\sum \alpha(\Delta_j) < \varepsilon$. Let us add to this system the (finite) system of half-open rectangles forming the set $G'-G$. For the resultant system whose rectangles we renumber as $\Delta_1', \Delta_2', \ldots$ we have $\sum \Delta_j' \supset e$ and $\sum \alpha(\Delta_j') < \varepsilon$ because $\alpha(G'-G) = 0$, that is (8) will hold. Therefore the integral on the right-hand side of (9) exists and, together with it, the integral on the left-hand side also exists.

Theorem 7. *Let two nonnegative additive functions* $\alpha_1(\Delta_1)$ *and* $\alpha_2(\Delta_2)$ *be defined for the half-open intervals belonging to half-open intervals* $[a, b)$ *and* $[c, d)$ *respectively, and let them generate the function*

$$\begin{aligned}\alpha(\Delta) &= \alpha_1(\Delta_1)\cdot\alpha_2(\Delta_2)\\ \Delta_1 &= \{\lambda \leqslant x \leqslant \mu\},\ \Delta_2 = \{\nu \leqslant y \leqslant \omega\}\\ \Delta &= \{\lambda \leqslant x \leqslant \mu,\ \nu \leqslant y \leqslant \omega\} = \Delta_1\times\Delta_2 \subset G\\ G &= [a, b)\times[c, d)\end{aligned}$$

Then for a bounded function $f(x, y)$ *there holds the equality*

$$\iint_G f(x, y)\, d\alpha = \int_c^d d\alpha_2 \int_a^b f(x, y)\, d\alpha_1$$

under the assumption that the integral on the left-hand side exists. Therefore the iterated integral on the right-hand side automatically exists (the inner integral with respect to x *is understood, for every* $y \in [c, d]$*, either in the ordinary sense as the Stieltjes integral, provided it exists, or as a number lying between the lower and upper Stieltjes integrals of* $f(x, y)$ *with respect to* α *on* $[a, b]$*).*

The proof of this theorem is completely analogous to that of Theorem 1, § 12.12.

To the Stieltjes integral apply the theorems analogous to those proved for the Riemann integral and also what was said in connection with formula (3), § 19.9.

Exercises

1. Show that

$$\int_a^b f(x)\, d\beta(x) = \int_a^b f(x)\,\beta'(x)\, dx$$

where $\beta(x)$ possesses the continuous derivative $\beta'(x) \geqslant 0$ on $[a, b]$.

2. Let $f(x)$ be continuous on $[a, b]$ and let $c_0, c_1, \ldots$ be a sequence of nonnegative numbers such that $\sum c_j < \infty$. Also, let $x_j \in [a, b]$ $(j = 1, 2, \ldots)$ and

$$\beta(x) = \sum_{x_j\in[a, x)} c_j \tag{12}$$

Show that

$$\int_a^b f(x)\, d\beta(x) = \sum f(x_j)c_j$$

3. Show that the function defined by formula (12) where c_j are numbers of arbitrary sign but such that $\sum |c_j| < \infty$ is a function of bounded variation, that is it is representable as the difference of two nondecreasing functions on $[a, b]$.

4. Let $f(x)$ and $\varphi(x)$ be continuous nondecreasing functions (or functions of bounded variation). Prove the formula

$$\int_a^b f(x)\,d\varphi(x)+\int_a^b \varphi(x)\,df(x) = f(b)\,\varphi(b)-f(a)\,\varphi(a)$$

5. A function $f(x)$ defined on a rectangle G is sometimes called integrable with respect to $\alpha(\varDelta)$ when its upper integral $\bar{J}$ (with respect to $\alpha(\varDelta)$) is equal to the lower integral $\underline{J}$. In this case the *generalized Stieltjes integral* (see Example 1 above) J is defined as the number

$$J = \underline{J} = \bar{J}$$

denoted by the symbol

$$J = J(f) = \int_G f(x)\,d\alpha$$

To elucidate this new notion we indicate that for a bounded function f the following assertions are equivalent.

(i*) $\underline{J} = \bar{J} = J$;

(ii*) there is a sequence of partitions ϱ_k $\left(G = \sum_j \varDelta_k^j\right)$ for which

$$\bar{S}_{\varrho_k}-\underline{S}_{\varrho_k} \to 0, \qquad k \to \infty$$

(iii*) there is a sequence of partitions ϱ_k such that

$$\lim_{k\to\infty} S_{\varrho_k} = J \quad \text{for any } \xi_k^j \in \varDelta_k^j$$

Indeed, from (ii*) and from the inequalities $\underline{S}_{\varrho_k} \leqslant \underline{J} \leqslant J \leqslant \bar{S}_{\varrho_k}$ and $\underline{S}_{\varrho_k} \leqslant S_{\varrho_k} \leqslant \bar{S}_{\varrho_k}$ follows (iii*) (for the same ϱ_k's). Further, (iii*) implies that

$$\lim \underline{S}_{\varrho_k} = \lim \bar{S}_{\varrho_k} = J$$

and, if ϱ is an arbitrary partition and $\varrho_k^* = \varrho+\varrho_k$, then $\underline{S}_\varrho \leqslant \underline{S}_{\varrho^*} \leqslant \bar{S}_{\varrho^*} \leqslant \bar{S}_{\varrho_k} \to J$, $k \to \infty$, that is $\underline{S}_\varrho \leqslant J$ and $\underline{J} = J$. It is similarly proved that (iii*) implies $\bar{J} = J$. We have proved the implication (iii*) → (i*).

If $\varepsilon_k > 0$, $\varepsilon_k \to 0$ then (i*) implies the existence of partitions ϱ_k' and ϱ_k'' such that $J-\varepsilon_k < \underline{S}_{\varrho_k'}$ and $\bar{S}_{\varrho_k''} < J+\varepsilon_k$ and therefore for $\varrho_k = \varrho_k'+\varrho_k''$ we have $J-\varepsilon_k < \underline{S}_{\varrho_k'} \leqslant \underline{S}_{\varrho_k} \leqslant S_{\varrho_k} \leqslant S_{\varrho_k''} < J+\varepsilon_k$, which proves (ii*).

Thus, the generalized Stieltjes integral can be defined as the number J for which (ii*) or (iii*) holds for some sequence of partitions ϱ_k.

If it is possible to choose a sequence of partitions of the indicated type consisting of regular partitions ϱ_k, then, according to Theorem 4, the number J is an ordinary Stieltjes integral. If otherwise, J is the generalized Stieltjes integral. The notion of the generalized Stieltjes integral may prove useful for functions not satisfying Lebesgue's condition in the sense of $\alpha(\varDelta)$ (see Theorem 6).

Let the reader use (ii*) and (iii*) to prove (see Theorems 1, 2 and 4, § 12.11) that if f and φ are integrable in the indicated sense then $Af+B\varphi$, $|f|$, $f\varphi$

and φ^{-1} $(|\varphi(x)| > a > 0)$ are also integrable and there hold the relations

$$J(Af+B\varphi) = AJ(f)+BJ(\varphi)$$

and

$$|J(f)| \leqslant K \sup_{x\in G} |f(x)|$$

characterizing the linearity and the continuity properties of the functional $J(f)$. Here A and B are arbitrary numbers and K is a constant dependent on the generalized measure $\alpha(\Delta)$ introduced in G.

In the one-dimensional case we have

$$K = \beta(b)-\beta(a)$$

and in the two-dimensional case

$$K = F(c, d)-F(a, d)-F(b, c)+F(a, b)$$

(see (3) and (5)).

We also suggest that the reader should prove the equality

$$\int_G f(x)\,d\alpha = \sum_{j=1}^{N} \int_{\Delta^j} f(x)\,dx$$

$\left(\text{where } G = \sum_{j=1}^{n} \Delta^j \text{ and } \Delta^j \text{ are pairwise nonintersecting rectangles}\right)$ on condition that the integral on the left-hand side exists.

§ 19.11. Generalization of Lebesgue Integral

The notion of the Lebesgue integral can be generalized as follows.

Let X be a set of elements x of arbitrary nature and let $\mathfrak{M}$ be a system of subsets e of X, $(e \subset X)$ forming a *ring*. This means that if two subsets e_1 and e_2 of X belong to $\mathfrak{M}$ then their sum e_1+e_2, difference e_1-e_2 and intersection e_1e_2 also belong to $\mathfrak{M}$. It is readily proved by induction that if $e_1, \ldots, e_n$ belong to $\mathfrak{M}$ then the finite sum $\sum_{j=1}^{n} e_j$ also belongs to $\mathfrak{M}$:

$$e = \sum_{j=1}^{n} e_j \in \mathfrak{M}$$

A ring $\mathfrak{M}$ must also satisfy the additional requirement that if $e_k \in \mathfrak{M}$ for any $k = 1, 2, \ldots$ and the countable sum

$$e = \sum_{k=1}^{\infty} e_k$$

belongs to some set $\mathcal{E} \in \mathfrak{M}$ $(e \subset \mathcal{E})$, then $e \in \mathfrak{M}$.

Now let us suppose that to every set $e \in \mathfrak{M}$ there corresponds a (finite) nonnegative number μe possessing the property that if a set $e \in \mathfrak{M}$ is representable as a finite or countable sum

$$e = \sum e_k$$

of pairwise disjoint sets $e_k \in \mathfrak{M}$ then

$$\mu e = \sum \mu e_k \tag{1}$$

The nonnegative set function μe ($e \in \mathfrak{M}$) thus defined on $\mathfrak{M}$ is called *Lebesgue's measure*. We shall also call it the *generalized Lebesgue measure* in order to distinguish between the general case treated here and the special case which was studied in Chapter 19 where the role of X was played by the n-dimensional space R_n and the role of $\mathfrak{M}$ by the corresponding totality of Lebesgue measurable sets. The sets $e \in \mathfrak{M}$ will be called *measurable*. If a subset $\mathcal{E} \subset X$ is a part of a measurabe set $e \in \mathfrak{M}$ ($\mathcal{E} \subset e$) we shall say that the set $\mathcal{E}$ is *bounded*.

According to this terminology, Theorems 3-9, § 19.1 apply to the generalized Lebesgue measure. Theorems 3, 4 and 5 hold for them according to the definition (see equality (1); as to Theorems 6-9, they are proved without any changes).

Proceeding from the measure thus defined we can define the notion of a *measurable function* following from the course of Chapter 19. Generally, all that was said in § 19.2 can be extended without any changes to the case of the generalized Lebesgue measure with the exception of Theorem 4 and the proposition (see page 350) on continuous functions. Since we did not define a topology on X (that is a system of neighbourhoods, etc.) it does not make sense to speak of a continuous function on X.

The *Lebesgue integral of a function* $f(x)$ on a set E where f is a measurable function and E is a measurable set (with respect to the measure μ) is defined as the limit

$$\lim_{\delta_k \to 0} \sum_{-\infty}^{\infty} \varrho_k m e_k = \int_E f(x)\, \mu(dx)$$

where ϱ_k, e_k and δ_k are defined by analogy with § 19.3 (see (2), (3) and (4) in § 19.3), and the measure is understood in the generalized sense.

The properties of the Lebesgue integral enumerated in § 19.3, with a possible exception of property 19 (Fubini's theorem) and property 22, are extended without any changes in their proofs to the general case under consideration.

An important example of the generalized Lebesgue integral is the so-called *Lebesgue-Stieltjes integral* which will be treated in the following section.

§ 19.12. Lebesgue-Stieltjes Integral

Let us consider a half-open rectangle

$$G = \{a_i \leqslant x_i < b_i;\ i = 1, \ldots, n\}$$

and let a nonnegative additive function $\alpha(\Delta)$ be defined for the half-open rectangles Δ belonging to G. In § 19.8 we used such a function $\alpha(\Delta)$ as the basis for constructing the generalized Jordan measure. In the present section, proceeding form $\alpha(\Delta)$, we shall construct a generalized Lebesgue measure (see § 19.11) for some sets $\mathcal{E} \subset G$. We shall call it the *measure generated by* $\alpha(\Delta)$ or, briefly, the *measure* and will denote it as $\mu\mathcal{E}$ (the same notation was used for the Lebesgue measure).

In our presentation we shall use the notions of the Jordan interior measure $m_i\mathcal{E}$, exterior measure $m_e\mathcal{E}$ and the Jordan measure $m\mathcal{E}$ introduced in § 19.8 (with respect to $\alpha(\Delta)$).

When calling $\mathcal{E}$ a set we shall mean that $\mathcal{E} \subset G$, and, as usual, by F and G we shall denote closed and open sets respectively. We shall also use the notation $A \subset\subset B$ expressing the fact that $\bar{A} \subset \mathring{B}$ where $\bar{A}$ is the closure of A and $\mathring{B}$ is the interior (the collection of all the interior points) of B.

For an open set g we put by definition (see the explanations below)

$$\mu g = m_i g = \sup_{\sigma \subset\subset g} \alpha(\sigma) = \sum_{k=1}^{\infty} \alpha(\Delta_k) \tag{1}$$

where

$$g = \sum_{1}^{\infty} \Delta_k, \qquad \Delta_k \subset\subset g \tag{2}$$

is a representation of g as a sum of half-open pairwise nonintersecting rectangles Δ_k each of which has a boundary with no points in common with the boundary of g.

A representation of g in the form of sum (2) can, for instance, be obtained as follows. We cut the space R_n into half-open cubes with edges of length 2^{-N} by means of a rectangular network and number in a consecutive order as $\Delta_1, \ldots, \Delta_s$ those of the cubes which possess the property $\Delta_k \subset\subset g$. Then we refine the network taken and pass to a new network with cubes whose edges are of length 2^{N+1} and supply with indices $s+1, s+2, \ldots, s+q$ those (half-open) cubes $\Delta_{s+1}, \Delta_{s+2}, \ldots, \Delta_{s+q}$ of the new network which possess the property $\Delta \subset\subset g$ and do not intersect with those former cubes which have already been numbered. Continuing this process indefinitely for $N+2$, $N+3, \ldots$ we arrive at sum (2) of the sets Δ_k consisting of an infinite (countable) number of terms.

To explain the last equality (1) we note that for each half-open figure $\sigma \subset\subset g$ there is N such that $\sigma \subset\subset \sum_{1}^{N} \Delta_k$, whence $\alpha(\sigma) \leqslant \sum_{1}^{N} \alpha(\Delta_k) \leqslant \alpha(G)$ and $\mu g \leqslant \sum_{1}^{\infty} \alpha(\Delta_k)$. But the last relation is in fact an exact equality because the sum $\sum_{1}^{N} \Delta_k$ is a figure σ' belonging to g for any N, and therefore $\sum_{1}^{N} \alpha(\Delta_k) = \alpha(\sigma') \leqslant \mu g$, that is $\sum_{1}^{\infty} \alpha(\Delta_k) \leqslant \mu g$.

It should also be mentioned that the interior $\tilde{\sigma}$ of a figure σ can be represented as a sum $\tilde{\sigma} = \sum_1^\infty \Delta_k$ of pairwise disjoint cubes $\Delta_k \subset\subset \sigma$, and, since $\sum_1^N \Delta_k \subset \sigma$, we have $\sum_1^N \alpha(\Delta_k) \leqslant \alpha(\sigma)$ for any N. Consequently,

$$\mu(\tilde{\sigma}) = \sum_1^\infty \alpha(\Delta_k) \leqslant \alpha(\sigma) \tag{3}$$

For a closed set F we put, by definition (see the explanations below),

$$\mu F = \alpha_e F = \inf_{\sigma \subset\subset F} \alpha(\sigma) = \inf_{g \subset F} \mu g \tag{4}$$

To elucidate the last equality let us denote its left-hand member by J' and its right-hand member by J''. Since $\sigma \supset\supset F$, we have $\tilde{\sigma} \supset\supset F$. But $\tilde{\sigma}$ is an open set, and therefore

$$\alpha(\sigma) \geqslant \mu(\tilde{\sigma}) \geqslant J'', \qquad \text{that is } J' \geqslant J''$$

On the other hand, if any open set $g \supset F$ is represented in the form of sum (2), then, since F is closed, there is N such that

$$\sigma = \sum_1^N \Delta_k \supset\supset F$$

whence

$$\mu(g) \geqslant \alpha(\sigma) \geqslant J', \qquad \text{that is } J'' \geqslant J'$$

Consequently, $J' = J''$.

Let us prove that for any half-open figure σ we have the inequality

$$\alpha(\sigma) \leqslant \alpha(\bar{\sigma}) \tag{5}$$

Let $g \supset \bar{\sigma}$ be an arbitrary open set and let it be represented in form (2). By the closedness of $\bar{\sigma}$, there is N such that

$$\sigma \subset \bar{\sigma} \subset \sum_1^N \Delta_k \subset g$$

Therefore

$$\alpha(\sigma) \leqslant \sum_1^N \alpha(\Delta_k) \leqslant \mu g$$

and, since the infimum of all the measures μg in question is equal to $\mu(\bar{\sigma})$, we arrive at (5).

The definitions stated readily imply that if $F \subset F'$ and $g \subset g'$ then $\mu F \leqslant \mu F'$ and $\mu g \leqslant \mu g'$. Besides, since $F \subset F$ and (4) holds, we obtain

$$\mu F = \sup_{F' \subset F} \mu F' = \inf_{g \supset F} \mu g \tag{6}$$

We also have the equalities

$$\mu g = \sup_{F \subset g} \mu F = \inf_{g' \supset g} \mu g' \tag{7}$$

The second equality in (7) is obvious since $g \supset g$. To justify the first equality let us denote the second member of (7) by J. Since $\mu F \leqslant \mu g$ (see (4)), we have $J \leqslant \mu g$. On the other hand, we can represent g in form (2) and put $\sigma_N = \sum_1^N \Delta_k$. Then for any $\varepsilon > 0$ there is N such that

$$\mu g - \varepsilon < \sum_1^N \alpha(\Delta_k) = \alpha(\sigma_N) \leqslant \mu(\bar{\sigma}_N) \leqslant J \qquad (\bar{\sigma}_N \subset g)$$

and, since ε is arbitrary, we obtain $\mu g \leqslant J$. Consequently, $\mu g = J$.

By the *interior (inner) measure of a set* $\mathcal{E}$ *(with respect to* $\alpha(\Delta)$*)* we shall mean the number

$$\mu_i \mathcal{E} = \sup_{F \subset \mathcal{E}} \mu F$$

and by the *exterior (outer) measure of* $\mathcal{E}$ *(with respect to* $\alpha(\Delta)$*)* we shall mean the number

$$\mu_e \mathcal{E} = \inf_{g \supset \mathcal{E}} \mu g$$

If $\mu_i \mathcal{E} = \mu_e \mathcal{E}$ we shall say that *the set* $\mathcal{E}$ *is measurable (in the sense of the measure generated by* $\alpha(\Delta)$*)* and the number $\mu \mathcal{E} = \mu_i \mathcal{E} = \mu_e \mathcal{E}$ will be called the *generalized measure of* $\mathcal{E}$ or the *measure of* $\mathcal{E}$ *with respect to* $\alpha(\Delta)$.

It follows from equalities (6) and (7) that *the closed and open sets* (belonging to *G) are measurable* (in the indicated sense).

The basic properties of measurable (in the generalized sense) sets defined here are analogous to the corresponding properties of the classical Lebesgue measurable sets. We enumerate them below.

If $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *are measurable sets, its sum, difference and intersection are also measurable and, if* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *do not intersect then*

$$\mu(\mathcal{E}_1 + \mathcal{E}_2) = \mu \mathcal{E}_1 + \mu \mathcal{E}_2$$

The sum $\mathcal{E}$ *of a countable number of measurable sets* $e_1, e_2, \ldots$ *(belonging to G) is measurable and, if these sets are pairwise nonintersecting, then*

$$\mu e = \sum_1^\infty \mu e_k$$

The intersection of a countable number of measurable sets is measurable.

In the case of the ordinary Lebesgue measure the properties enumerated here and the other properties following from them turn into Theorems 1-9 proved in § 19.1. In the general case their proofs are quite analogous.

These theorems were based on Lemmas 1-5, § 19.1. They can also be proved quite analogously for the general case. To this end we should replace in their proofs $|\Delta_k|$ and $|\sigma_k|$ by $\alpha(\Delta_k)$ and $\alpha(\sigma_k)$ respectively and also take into account the following remarks. The statement of Lemma 1 should be replaced by the following: (3) implies (4) where σ_k and σ_k' are half-open figures, the figures σ_k being pairwise disjoint. In the statement of Lemma 2 the last assertion saying that the inequality turns into an equality should be deleted.

We also mention the following inequalities analogous to (14) in § 19.1:

$$m_i\mathscr{E} \leqslant \mu_i\mathscr{E} \leqslant \mu_e\mathscr{E} \leqslant m_e\mathscr{E} \tag{8}$$

They follow from the relations (see (3) and (5))

$$\left.\begin{aligned} \sup_{\sigma\subset\subset\mathscr{E}} \alpha(\sigma) &\leqslant \sup_{\bar{\sigma}\subset\subset\mathscr{E}} \mu\bar{\sigma} \leqslant \sup_{F\subset\mathscr{E}} \mu F, \\ \inf_{\sigma\supset\supset\mathscr{E}} \alpha(\sigma) &\geqslant \inf_{\tilde{\sigma}\supset\supset\mathscr{E}} \mu\tilde{\sigma} = \inf_{g\supset\mathscr{E}} \mu g \end{aligned}\right\}$$

It follows from (8) that *if a set $\mathscr{E}$ is Jordan measurable (in the sense of $\alpha(\Delta)$) it is also Lebesgue measurable.*

In the theory of the Jordan measure (with respect to $\alpha(\Delta)$) we found that some rectangles may be nonmeasurable. In the theory presented here we have no such setback because all the closed and open rectangles are measurable since they are closed and open sets respectively. An arbitrary half-open rectangle

$$\Delta = \{\lambda_i \leqslant x < \mu_i; \qquad i = 1, \ldots, n\}$$

is also measurable since it is the difference of two closed sets. But the measure $\mu\Delta$ must not necessarily be equal to $\alpha(\Delta)$ (see Exercises 1 and 2 below).

A single-point set ω has the interior measure $m_i\omega = 0$ (with respect to $\alpha(\Delta)$) in the Jordan theory and therefore it is measurable if and only if $m_e\omega = 0$. In the present theory such a set ω is measurable for any $\alpha(\Delta)$ because it is closed. Its measure coincides with its Jordan exterior measure: $\mu\omega = m_e\omega$.

In the classical theory of the Lebesgue integral the Lebesgue measure serves as the basis for the definition of the notion of a measurable function. The notion of a *function measurable with respect to the measure generated by* $\alpha(\Delta)$ is introduced in just the same way. All that was said in this connection in § 19.2 is extended without any changes to the general case, and only the classical Lebesgue measure should be replaced by the Lebesgue measure generated by $\alpha(\Delta)$ and the expression "Riemann integrable function" in Theorem 4 should be replaced by "Riemann-Stieltjes integrable function".

Further, the notion of the *Lebesgue-Stieltjes integral with respect to an arbitrary set function* $\alpha(\Delta)$ is introduced using the notion of a function measurable with respect to the measure generated by $\alpha(\Delta)$ by analogy with the definition of the (ordinary) Lebesgue integral of a (Lebesgue) measurable function.

Thus, the Lebesgue-Stieltjes integral can be defined as one of the limits

$$\int_{\mathscr{E}} f(x)\,d\mu = \lim_{\delta_k\to 0} \underline{S}_R(f) = \lim_{\delta_k\to 0} \bar{S}_R(f) \tag{9}$$

Here $f(x)$ is a measurable (with respect to the measure μ generated by $\alpha(\Delta)$) function on the set $\mathscr{E}$ and $\underline{S}_R(f)$ and $\bar{S}_R(f)$ are the lower and upper integral sums defined by a complete analogy with § 19.3 where these sums were defined for the classical Lebesgue integral; the only difference is that in the present case the sets e_k (see the beginning of § 19.3) are supposed to be measurable with respect to μ and their measures μe_k are understood in the sense of μ (with respect to $\alpha(\Delta)$).

Generally, all that was said in § 19.3 can be extended to the Lebesgue-Stieltjes integral.

Of course, in the case of the Lebesgue-Stieltjes integral, in all the integrals in § 19.3 we should replace dx by $d\mu$ and the expression "the Lebesgue integral" by "the Lebesgue-Stieltjes integral". In property 6 the expressions "f is Riemann integrable", "the Jordan measure" and "having no interior points in common" (or, which is the same, "can only intersect along some parts of their boundaries") should be replaced, respectively, by "f is Riemann-Stieltjes integrable", "the Jordan measure with respect to $\alpha(\Delta)$" and "pairwise disjoint". Property 15 establishing the equality of the absolutely convergent improper Riemann integral of f and the Lebesgue integral of f can also be extended to the general case if we define, by analogy, the notion of the improper Riemann-Stieltjes integral.

Property 19 (Fubini's theorem) can be immediately extended to the general case if we assume that there are two nonnegative additive functions $\alpha_1(\delta_1)$ and $\alpha'(\delta')$ defined for the half-open intervals $\delta_1 = \{0 \leqslant x_1 < a\}$ and for the half-open rectangles $\delta' = \{0 \leqslant x_i < a, \ i = 2, \ldots, n\}$ respectively which generate the function $\alpha(\delta) = \alpha(\delta_1)\alpha(\delta')$, $\delta = \delta_1 \times \delta' = \Delta$ (see property 19 in § 19.3). Then if we denote by μ, μ_1 and μ' the three measures generated by the functions α, α_1 and α' respectively, formula (34) will continue to hold together with the remarks to this formula (of course, dx, dx_1 and dy should be replaced by $d\mu$, $d\mu_1$ and $d\mu'$ respectively).

The proof of this generalized formula is quite analogous to that of (34), § 19.3.

We shall also discuss the following question concerning the relationship between the Lebesgue integral and the Lebesgue-Stieltjes integral.

Let us consider a nonnegative Lebesgue integrable function $\beta'(x)$ on a closed interval $[a, b]$. Let us put

$$\beta(x) = \int_a^x \beta'(t)\,dt, \qquad x \in [a, b]$$

As is known (see § 19.5 (11)), $\beta(x)$ is an absolutely continuous function and $\beta'(x)$ is its generalized derivative (on $[a, b]$). Besides, the condition $\beta'(x) \geqslant 0$ indicates that the function $\beta(x)$ is nondecreasing on $[a, b]$ and can therefore be regarded as the generating function for the nonnegative additive function

$$\alpha(\Delta) = \beta(d) - \beta(c), \quad \Delta = [c, d) \subset G = [a, b)$$

defined on the half-open intervals.

It should be noted that, conversely, we can proceed from a given nondecreasing absolutely continuous function $\beta(x)$ on $[a, b]$ and show that its generalized derivative $\beta'(x)$ is nonnegative almost everywhere on $[a, b]$.

The function $\alpha(\Delta)$ constructed above generates a measure.

If $g \subset [a, b]$ is an open set which, as we know, can be represented as a sum of not more than countable number of open intervals (c_k, d_k) $(k = 1, 2, \ldots)$, then it is measurable both in Lebesgue's classical sense and in the sense of the Lebesgue measure with respect to $\alpha(\Delta)$, its measure in the latter sense being

$$\mu g = \int_g d\mu = \sum_{k=1}^{\infty} [\beta(d_k)-\beta(c_k)] = \sum_{k=1}^{\infty} \int_{c_k}^{d_k} \beta'(t)\,dt = \int_g \beta'(t)\,dt$$

We have thus proved the formula

$$\int_g d\mu = \int_g \mu'(t)\,dt \tag{10}$$

Formula (10) can readily by generalized to any set $\mathcal{E}$ which is simultaneously measurable in the two senses indicated above:

$$\int_{\mathcal{E}} d\mu = \int_{\mathcal{E}} \mu'(t)\,dt \tag{11}$$

To prove (11) it suffices to take the infimum of both members of (10) over all $g \supset \mathcal{E}$.

For we have

$$\int_{\mathcal{E}} d\mu = \mu\mathcal{E} = \inf_{g\supset\mathcal{E}} \mu g = \inf_{g\supset\mathcal{E}} \int_g d\mu$$

and, on the other hand, since $\mu'(x)$ is nonnegative, we have

$$\int_g \mu'(t)dt \geqslant \int_{\mathcal{E}} \mu'(t)\,dt,\ g \supset \mathcal{E}$$

and, by the measurability of $\mathcal{E}$, for any $\varepsilon > 0$ there is g such that (see § 19.3, property 11)

$$\int_g \mu'(t)\,dt = \int_{\mathcal{E}} \mu'(t)\,dt + \int_{g-\mathcal{E}} \mu'(t)\,dt < \int_{\mathcal{E}} \mu'(t)\,dt + \mathcal{E}$$

Equality (11) admits of a further generalization.

Let μ be a measure possessing the property that if a set $e \subset [a, b]$ is measurable with respect to μ then it is also Lebesgue measurable. Then there holds the equality

$$\int_{\mathcal{E}} f(x)\,d\mu = \int_{\mathcal{E}} f(x)\,\mu'(x)\,dx \tag{12}$$

provided that the integral on the left-hand side exists.

Indeed, let the integral on the left-hand side of (12) exist. Then, by properties 17 and 21 in § 19.3 holding for the Lebesgue-Stieltjes integral with respect to $\alpha(\Delta)$, there is a sequence of simple functions $\sigma_p(x)$ such that

$$\sigma_p(x) \to f(x) \quad \text{almost everywhere on } [a, b]$$

and

$$\int_{\mathscr{E}} |f(x)-\sigma_p(x)|\, d\mu \to 0, \qquad p \to \infty \tag{13}$$

Therefore (see the explanations below)

$$\int_{\mathscr{E}} |\sigma_p(x)\, \mu'(x)-\sigma_q(x)\, \mu'(x)|\, dx = \int_{\mathscr{E}} |\sigma_p(x)-\sigma_q(x)|\, \mu'(x)\, dx =$$

$$= \int_{\mathscr{E}} |\sigma_p(x)-\sigma_q(x)|\, d\mu \to 0, \qquad p, q \to \infty$$

where the second relation (equality) holds by virtue of (11) because $|\sigma_p(x)-\sigma_q(x)|$ is a simple function and the third equality holds by virtue of (13). By property 20, § 19.3 (the completeness of the space $L(a, b)$), there is a function to which $\sigma_p(x)\mu'(x)$ tends in the sense of $L(a, b)$. Further, since $\sigma_p(x)\mu'(x) \to f(x)\mu'(x)$ almost everywhere, this limiting function must coincide with $f(x)\mu'(x)$ (see property 21, § 19.3). Thus,

$$\int_{\mathscr{E}} |\sigma_p(x)-f(x)|\, \mu'(x)\, dx \to 0, \qquad p \to \infty \tag{14}$$

From (13) and (14) and from the property of the simple functions $\sigma_p(x)$ expressed by the equality

$$\int_{\mathscr{E}} \sigma_p(x)\, d\mu = \int_{\mathscr{E}} \sigma_p(x)\, \mu'(x)\, dx$$

it follows that (12) holds.

Conversely, if $f(x)$ is a Lebesgue function measurable on $\mathscr{E}$ and finite almost everywhere we have $\mu'(x) \geqslant 0$, and any Lebesgue measurable set $e \subset [a, b]$ is measurable with respect to μ as well. Therefore the existence of the integral on the right-hand side of (12) implies the existence of the integral on the left-hand side and the validity of equality (12).

The latter assertion can be proved by repeating the above argument in the reverse order. The existence of the integral on the right-hand side of (12) and the inequality $\mu'(x) \geqslant 0$ imply the existence of a sequence of simple functions $\sigma_p(x)$ for which

$$\sigma_p(x) \to f(x) \quad \text{almost everywhere on } [a, b]$$

and

$$\int_{\mathscr{E}} |f(x)-\sigma_p(x)|\, \mu'(x)\, dx \to 0, \qquad p \to \infty$$

which can be proved by analogy with property 17, § 19.3 in whose proof the differential dx under the integral signs should be replaced everywhere by $\mu'(x)\,dx$.

We also mention the following proposition:

If a nondecreasing function $\beta(x)$ satisfies the Lipschitz condition

$$|\beta(x')-\beta(x)| \leqslant M|x'-x|, \qquad a \leqslant x, x' \leqslant b \tag{15}$$

(where M is independent of x and x') then the Lebesgue measurability of a set $e \subset [a, b]$ implies its measurability with respect to the measure μ_β generated by the function $\beta(x)$.

Conversely, if the inverse function $x = \beta^{-1}(y)$ of a function $y = \beta(x)$ defined on $[a, b]$ exists and satisfies a Lipschitz condition then the measurability of a set $e \subset [a, b]$ with respect to μ_β implies its measurability in Lebesgue's sense.

Indeed, an arbitrary open set $g \subset [a, b]$ is representable as a sum of at most countable number of pairwise disjoint open intervals (c_k, d_k), and therefore, denoting by μ the Lebesgue measure, we derive from (15) the relations

$$\mu_\beta g = \sum [\beta(d_k)-\beta(c_k)] \leqslant M\sum (d_k-c_k) = M\mu g \tag{16}$$

Further, if $\mathcal{E} \subset [a, b]$ is a measurable set then, given any $\varepsilon > 0$, there are closed and open sets F and g such that $F \subset \mathcal{E} \subset g$ and $\mu(g-F) < \varepsilon$. Now, since the set $g-F$ is open we can apply (16) to it and thus obtain

$$\mu_\beta(g-F) \leqslant M\mu(g-F) < M\varepsilon$$

Hence, the measure of $g-F$ in the sense of β can also be made arbitrarily small, and therefore the set $\mathcal{E}$ is measurable with respect to β.

The converse proposition is proved analogously.

In conclusion we remind the reader of what was said in connection with formula (3), § 19.9.

Exercises

1. Let $\beta(x)$ be a nondecreasing function on $[a, b]$ generating the non-negative additive function

$$\alpha(\Delta) = \beta(d)-\beta(c), \quad \Delta = [c; d) \subset G = [a, b)$$

Show that if $\beta(x)$ is extended by putting $\beta(x) = \beta(a)$ for $x < a$ and $\beta(x) = \beta(b)$ for $x > b$, then the extended function generates the extension of $\alpha(\Delta)$ specified by the rule given in § 19.8 (1).

For the Lebesgue measure μ generated by $\alpha(\Delta)$ prove the equalities

$$\mu(c, d) = \beta(d-0)-\beta(c+0)$$
$$\mu[c, d] = \beta(d+0)-\beta(c-0)$$
$$\mu[c, d) = \beta(d-0)-\beta(c-0)$$

and

$$\mu(c, d] = \beta(d+0)-\beta(c+0)$$

Thus, $\mu(\Delta) = \alpha(\Delta)$ if and only if $\beta(c) = \beta(c-0)$ and $\beta(\alpha) = \beta(d-0)$.

2. Let $G = \{a_i \leqslant x_i < b_i,\ i = 1, \ldots, n\}$ and

$\Delta = \{c \leqslant x_i < d_i;\ i = 1, \ldots, n\} \in G, \quad \Delta_\varepsilon = \{c_i - \varepsilon \leqslant x_i < d_i;\ i = 1, \ldots, n\}$

where $\varepsilon > 0$.

Prove the equalities

$$\mu\tilde{\Delta} = \lim_{\substack{\Delta' \subset\subset \Delta \\ \Delta' \to \Delta}} \alpha(\Delta')$$

and

$$\mu\bar{\Delta} = \lim_{\substack{\Delta' \supset\supset \Delta \\ \Delta' \to \Delta}} \alpha(\Delta')$$

§ 19.13. Extension of Functions. Weierstrass' Theorem

Let us denote the points of the n-dimensional space R_n as $v = (v_1, \ldots, v_n)$.

Suppose that $\Omega \subset R_n$ is a bounded domain with boundary Γ which is a smooth surface (a closed $(n-1)$-dimensional differentiable manifold; see § 17.1). We shall assume that Γ is a sum of a finite number of connected manifolds.

The points of Γ will be denoted as x. The normal to Γ at a point x consists of two rays N_1 and N_2 issued from x. One of the rays, say N_1, contains a sufficiently small open interval, with end point x, entirely belonging to Ω while the other ray N_2 contains an open interval, with end point x, having no points in common with Ω. The rays N_1 and N_2 are called the inner and outer normals to Γ at the point x, respectively (see § 17.2).

Let $\nu = \nu(x)$ be the unit vector of the outer normal to Γ issued from $x \in \Gamma$. It is a continuous function of x on Γ.

Indeed, Ω is a connected n-dimensional differentiable manifold whose closure belongs to the n-dimensional oriented manifold determined by the equations $v_1 = v_1, \ldots, v_n = v_n$ which is also connected. Therefore (see § 17.2 and, in particular, Theorem 4 and the following material) Γ is also an orientable manifold, and it is possible to state the coherence rule for the orientations of Ω and Γ which reads as follows.

For any point $x^0 \in \Gamma$ there is its (n-dimensional) neighbourhood W_{x^0} and functions

$$v_i = \varphi_i(u) = \varphi_i(u_1, \ldots, u_n), \quad i = 1, \ldots, n, \qquad u \in \Omega \tag{1}$$

defined on W_{x^0} providing a local representation of the oriented manifold R_n (see § 17.1 (1)) and possessing the following properties:

(i) The Jacobian of (1) (i.e. of the transformation from u to v) is positive.

(ii) For sufficiently small $\delta > 0$ the points v belong to the exterior or to the interior of Ω depending on whether $u_1 > 0$ or $u_1 < 0$ for them where $|u_1| < \delta$.

(iii) The equations

$$x_i = \varphi_i(0, u_2, \ldots, u_n) \qquad i = 1, \ldots, n \tag{2}$$

describe Γ (in the indicated neighbourhood).

The totality of all representations (2) of the manifold determines an orientation of Γ or, in other words, any two such representations of some intersecting parts σ, $\sigma' \subset \Gamma$ are such that the Jacobian of the transformation connecting their parameters $(u_2, \dots, u_n)$ and $(u_2', \dots, u_n')$ is positive on $\sigma\sigma'$.

Let us consider the unit normal vector $\nu = \nu(x)$ to Γ at the points $x \in \Gamma$ which is specified, within W_{x^0}, by formulas § 7.25 (3) in which we should replace $(u_1, \dots, u_{n-1})$ by $(u_2, \dots, u_n)$ and t by h. It is readily seen from these formulas that the vector $\nu(x)$ depends continuously on $x \in \Gamma$ within a neighbourhood W_{x^0} of an arbitrary point $x^0 \in \Gamma$. But it is also continuous throughout Γ, which is seen from the same formulas because the vector ν is one and the same on the intersection of two parts σ and σ' irrespective of whether it is computed in terms of the parameters $(u_2, \dots, u_n)$ describing the part σ or in terms of the parameters $(u_2', \dots, u_n')$ describing the part σ'. The fact that the Jacobian of the transformation from one group of the parameters to another group is positive plays an essential role here.

It should be noted that the vector ν is directed to the exterior of Ω because we have

$$0 < \frac{D(\varphi_1, \dots, \varphi_n)}{D(u_1, \dots, u_n)} = \sum_1^n \frac{\partial \varphi_i}{\partial u_1} A_i$$

on Γ, and the vector $\left(\frac{\partial \varphi_1}{\partial u_1}, \dots, \frac{\partial \varphi_n}{\partial n_1}\right)$ is sure to be directed to the exterior of Ω.

Now let us suppose that Γ is a manifold of class C^{r+1}, that is the functions φ_i describing it locally are $r+1$ times continuously differentiable. Then there is a sufficiently small $\delta > 0$ such that the equality

$$v = x + h\nu(x), \qquad x \in \Gamma, \ |h| \leqslant \delta \tag{3}$$

establishes a one-to-one correspondence

$$v \rightleftarrows (x, h), \qquad x \in \Gamma, \qquad |h| \leqslant \delta \tag{4}$$

continuously differentiable r times. This should be understood in the sense that if a local representation (3) is written, with the aid of substitution (2), in the form

$$v \rightleftarrows (u_2, \dots, u_n, h), \qquad |h| \leqslant \delta \tag{5}$$

then correspondence (5) thus obtained is continuously differentiable r times.

The fact that such a correspondence (5) does in fact exist in some neighbourhood of an arbitrary point $x^0 \in \Gamma$ was proved in § 7.25 (5). Using the Heine-Borel lemma we can choose a finite number of such neighbourhoods entirely covering the bounded and closed set Γ, and then for a sufficiently small $\delta > 0$ relation (4) will take place for all $x \in \Gamma$.

The set of the points v determined by relations (3) will be denoted $\Pi(\delta)$, and we shall put $\Omega^\delta = \Omega + \Pi(\delta)$.

Theorem 1. *Let Ω be a bounded set whose boundary is an $(n-1)$-dimensional differentiable manifold of class C^2. If $f(v)$ is a function continuous in $\bar{\Omega}$, it can* *be extended from* $\bar{\Omega}$ to R_n *so that the extended function* $\tilde{f}(v)$ *will be continuous on* R_n *and finite (having compact support) in* Ω^δ *and the inequality*

$$|\tilde{f}(v)| \leqslant \max_{v \in R_n} |f(v)| \tag{6}$$
$$\qquad\qquad\qquad v \in \bar{\Omega}$$

will hold.

Proof. The given function can be regarded as a continuous function of v and of (x, h) on $\Omega\Pi(\delta)$ $(f(v) = f(x, h),\ x \in \Gamma)$.

The function

$$f^*(v) = \begin{cases} f(v) & \text{for } v \in \Omega - \Pi(\delta) \\ f(v) = f(x, h) & \text{for } x \in \Gamma,\ -\delta \leqslant h \leqslant 0 \\ f(x, -h) & \text{for } x \in \Gamma,\ 0 \leqslant h \leqslant \delta \\ 0 & \text{for } v \notin \Omega^\delta \end{cases} \tag{7}$$

is obviously continuous on Ω^δ and serves as an extension of f from $\bar{\Omega}$ to R_n.

Now let us take a function $\psi(v)$ satisfying the inequalities $0 \leqslant \psi(v) \leqslant 1$ and equal to 1 on Ω whose support is compact and belongs to Ω^δ (see § 18.4, Lemma 1). It it quite evident that the assertion of the theorem holds for the function $\tilde{f}(v) = \psi(v)\, f^*(v)$.

Theorem 2 (Weierstrass). *Let a continuous function* $f(x)$ $(x \in \bar{\Omega})$ *be defined on the closure* $\bar{\Omega}$ *of a domain* Ω *satisfying the conditions of Theorem 1. Then for any* $\varepsilon > 0$ *there is a polynomial* (see § 7.3 (2))

$$P(x) = \sum_{0 \leqslant k_j \leqslant N} a_k x^k \qquad (k = k_1, \ldots, k_n,\ x^k = x_1^{k_1} \ldots x_n^{k_n})$$

such that the inequality

$$|f(x) - P(x)| < \varepsilon, \qquad x \in \bar{\Omega} \tag{8}$$

is fulfilled.

Proof. Let us consider an extension $\tilde{f}(x)$ of the function $f(x)$ from $\bar{\Omega}$ to R_n, $\tilde{f}(x)$ being continuous and finite in Ω^δ. Since the set Ω^δ is bounded there is $l > 0$ such that the cube $\Delta = \{|x_j| \leqslant l;\ j = 1, \ldots, n\}$ contains Ω^δ. On making the substitutions

$$x_j = l \cos t_j, \qquad 0 \leqslant t_j \leqslant \pi, \quad j = 1, \ldots, n \tag{9}$$

we obtain the function

$$F(t) = \tilde{f}(l \cos t_1, \ldots, l \cos t_n)$$

which can be regarded as defined for any $t = (t_1, \ldots, t_n)$. It is continuous in t, and is periodic with period 2π and even with respect to each variable t_j.

The Fejér sum of the function $F(t)$ is an even trigonometric polynomial

$$\sigma_N(t) = \sum_{0 \leqslant k_j \leqslant N} \alpha_k \cos k_1 t_1 \ldots \cos k_n t_n, \qquad k = (k_1, \ldots, k_n)$$

which converges uniformly to $F(t)$ as $N \to \infty$ (see § 15.11, Theorem 1). Consequently, given any $\varepsilon > 0$, there is N such that the inequality $|F(t) - \sigma_N(t)| < \varepsilon$ holds for all t.

If we pass back from t to x in this inequality (see (9)) this will result in the inequality

$$|\bar{f}(x)-P(x)| < \varepsilon$$

where $P(x)$ is the polynomial expressed as

$$P(x) = \sum_{0 \leq k_j \leq N} \alpha_k \cos k_1 \arccos \frac{x_1}{l} \dots \cos k_n \arccos \frac{x_n}{l}$$

(see § 15.12). Then, moreover, inequality (8) is fulfilled since $\bar{f}(x) = f(x)$ on $\bar{\Omega}$.

Theorem 3. *Let a domain Ω consisting of points $v = (v_1, \dots, v_n)$ satisfy the conditions of Theorem 1 under the additional assumption that its boundary is a manifold of class C^{r+1}. Let $f(v)$ be a function of class C^r defined on $\bar{\Omega}$, that is possessing on $\bar{\Omega}$ continuous partial derivatives up to the order r inclusive. Then $f(v)$ can be extended from $\bar{\Omega}$ to R_n so that the extended function $\bar{f}(v)$ belongs to $C^r(R_n)$, is finite in Ω^δ and satisfies the inequality*

$$|\bar{f}^{(k)}(v)|_{v \in R_n} \leq C \sum_{|s| \leq r} \max_{v \in \bar{\Omega}} |f^{(s)}(v)|, \qquad |k| \leq r$$

where C is a constant independent of f and v.

Proof. Let us construct the function

$$f_*(v) = \begin{cases} f(v) & \text{for } v \in \Omega - \Pi(\delta) \\ f(v) = f(x, h) & \text{for } -\delta \leq h \leq 0 \\ \sum\limits_{s=0}^{r} \lambda_s f\left(x, -\dfrac{h}{s+1}\right) & \text{for } 0 \leq h \leq \delta \\ 0 & \text{for } v \notin \Omega^\delta \end{cases} \tag{10}$$

defined in Ω^δ where the numbers λ_s simultaneously satisfy the equations

$$\sum_{s=0}^{r} \lambda_s \left(-\frac{1}{s+1}\right)^k = 1 \qquad (k = 0, 1, \dots, r)$$

It can easily be checked that $f_*(v)$ possesses continuous partial derivatives up to the order r inclusive on Ω^δ, and

$$\bar{f}(v) = \psi(v) f_*(v)$$

is a function for which the assertion of the theorem holds provided that $\psi(v)$ is a function continuously differentiable r times on R_n, finite in Ω^δ and that $\psi(v) = 1$ on $\bar{\Omega}$ and $0 \leq \psi(v) \leq 1$ on R_n (see § 18.4, Lemma 1).

Remark. Theorems 1-3 can be proved under some weaker conditions imposed on Ω but then the extension of $f_*(v)$ should be constructed by means of some other techniques different from those used in (7) and (10).

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Printed in the Union of Soviet Socialist Republics

A textbook for university students (physicists and mathematicians) with special supplementary material on mathematical physics. Based on the course read by the author at the Moscow Engineering Physics Institute.

Volume 2 contains multiple integrals, field theory, Fourier series and Fourier integral, differential manifolds and differential forms, and the Lebesgue integral.